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Geometry of $(\kappa,\mu)'$-almost Kenmotsu manifolds with divergence free Cotton tensor and vanishing Bach tensor

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ABSTRACT: In this paper, we prove that a non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold of dimension $(2n+1)$ has divergence free Cotton tensor if and only if it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. Finally, we show that a Bach flat non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold is 3-dimensional and is locally isometric to the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$.

Key Words: Bach tensor, Cotton tensor, almost Kenmotsu manifold, Weyl conformal curvature tensor.

Contents

1. Introduction

The notion of Bach tensor was first introduced by Rudolf Bach in 1921 [2] when studying so-called conformal relativity. That is, instead of using the Hilbert-Einstein functional, one considers the functional

$$\mathcal{W}(g)=\int_M |W_g|^2 dvol_g, \tag{1.1}$$

for 4-dimensional manifolds, where W denotes the Weyl conformal curvature tensor of type (0,4) defined by

$$\begin{aligned}W(X,Y)Z=R(X,Y)Z-\frac{1}{d-2}\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX\\ -g(X,Z)QY\}+\frac{r}{(d-1)(d-2)}\{g(Y,Z)X-g(X,Z)Y\},\end{aligned} \tag{1.2}$$

where R denotes curvature tensor, S is a Ricci tensor, Ricci operator Q defined by $g(QX,Y)=S(X,Y)$ and d is dimension of the manifold. Critical points of the functional (1.1) are characterized by the vanishing of certain symmetric 2-tensor $\mathcal{B}$. The tensor $\mathcal{B}$ is usually referred as Bach tensor and the metric is called Bach flat if $\mathcal{B}$ vanishes. On any Riemannian manifold (M,g) of dimension d, the Bach tensor $\mathcal{B}$ is defined by

$$\begin{aligned}\mathcal{B}(X,Y)=\frac{1}{d-3}\sum_{i,j=1}^{d}((\nabla_{e_i}\nabla_{e_j}W)(X,e_i)e_j,Y)\\ +\frac{1}{d-2}\sum_{i,j=1}^{d}S(e_i,e_j)W(X,e_i,e_j,Y),\end{aligned} \tag{1.3}$$

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where $\{e_i\}_{i=1}^d$ is a local orthonormal frame on (M,g). The Cotton tensor is a $(0,3)$-type tensor and it is defined by

$$C(X,Y)Z = (divR)(X,Y)Z - \frac{1}{2(d-1)}\{(Xr)g(Y,Z) - (Yr)g(X,Z)\}, \tag{1.4}$$

where div denotes the divergence. It is worthy to remember that the vanishing of Cotton tensor implies that M is conformally flat when the dimension of M is 3. In view of (1.2) and (1.4), the expression of Bach tensor (1.3) takes the form (see [6])

$$\mathcal{B}(X,Y) = \frac{1}{d-2}\left\{\sum_{i=1}^{d}(\nabla_{e_i}C)(e_i,X)Y + \sum_{i=1}^{d}S(e_i,e_j)W(X,e_i,e_j,Y)\right\}. \tag{1.5}$$

The Bach tensor is a tensor built up from pure geometry, and thereby captures necessary features of a space being conformally Einstein in an intrinsic way and it is conformally invariant in dimension 4. Before 1968, it was the only known conformally invariant tensor that is algebraically independent of the Weyl tensor. It is well known that the Weyl conformal curvature tensor W vanishes when dimension of M is 3, and hence the expression of Bach tensor transforms into

$$\mathcal{B}(X,Y) = \sum_{i=1}^{3}(\nabla_{e_i}C)(e_i,X)Y.$$

If (M,g) is locally conformally related to an Einstein space, $\mathcal{B}$ has to vanish, but there are Riemannian manifolds with $\mathcal{B}=0$ which are not conformally related to Einstein spaces [25]. By (1.3), it is easy to see that Bach flatness is a natural generalization of Einstein and conformal flatness. For more details about Bach tensor, we refer to reader [3,12,23,25] and references therein.

On the other hand, Kenmotsu manifolds known not only as a special case of almost contact metric manifolds (see [5]) but also an analogous of Hermitian manifolds, investigated by many authors. After Kenmotsu manifolds was first introduced by Kenmotsu in [18], later such manifolds were generalized to almost Kenmotsu manifolds by Janssens and Vanhecke [17] and then studied this notion rigorously by Dileo and Pastore [10,11], Kim and Pak [19] and many others.

Firstly, Ghosh and Sharma in [13] studied Bach tensor on contact metric manifolds. They particularly concentrate on Sasakian manifold and studied purely transversal Bach tensor. In this work, they proved that if a Sasakian manifold admits a purely transversal Bach tensor, then g has constant scalar curvature $\geq 2n(2n+1)$, with equality holds if and only if g is Einstein, and the Ricci tensor of g has a constant norm. Also, they studied (κ,μ)-contact manifolds with divergence free Cotton tensor and vanishing Bach tensor in [15]. Their work provided essential criteria under which such curvature properties are satisfied, enriching the theory of contact geometry and paving the way for further investigations into related structures. Building on this foundational work, several geometers have recently extended the classification of curvature tensors like the Cotton and Bach tensors to other classes of almost contact and Lorentzian manifolds. Notably, these curvature tensors are classified by the geometers Ghosh [14], Naik et al. [21], De and Majhi [8] and Li et al. [20] on Kenmotsu manifolds, CoKahler manifolds, $(\kappa,\mu)'$-almost Kenmotsu manifolds and a class of Lorentzian manifolds, respectively. By taking motivation of these pioneer work, we are interested to characterize the divergence free Cotton tensor and vanishing Bach tensor in the framework of a class of almost Kenmotsu manifolds. Some related developments can be found in [1,16,26,27,28].

The paper is organized as follows: Section 2 is concerned with the basic formulas and properties of almost Kenmotsu manifolds. In section 3, we classify $(\kappa,\mu)'$-almost Kenmotsu manifold whose metrics have (i) divergence free Cotton tensor and (ii) vanishing Bach tensor.

2. Preliminaries

In this section, we recall some basic notions and properties of almost Kenmotsu manifolds which will be useful for the establishment of our main results. An almost contact structure on a $(2n+1)$-dimensional smooth manifold M is a triplet (φ,ξ,η), where φ is a (1,1)-type tensor field, ξ a global vector field and η a 1-form, such that

$$\varphi^2 = -id + \eta\otimes\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity endomorphism, which imply that $\varphi(\xi)=0$, $\eta\circ\varphi=0$ and $rank(\varphi)=2n$. Generally, ξ is called the Reeb vector field or characteristic vector field. A Riemannian metric g on M is said to be compatible with the almost contact structure (φ,ξ,η) if

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields $X,Y\in TM$. An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure and it is denoted by (M,φ,ξ,η,g). The fundamental 2-form Φ of an almost contact metric manifold M is defined by $\Phi(X,Y)=g(X,\varphi Y)$ for any vector fields X and Y. We define an almost complex structure J on the product manifold $M\times\mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X denotes the vector field tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function defined on the product manifold $M\times\mathbb{R}$. It is well known that the normality of an almost contact structure is expressed by the vanishing of the tensor $N_\varphi=[\varphi,\varphi]+2d\eta\otimes\xi$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ (see [5]).

According to Janssens and Vanhecke [17], an almost contact metric manifold satisfying $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$ is called an almost Kenmotsu manifold. On an almost Kenmotsu manifold M, we set $\ell=R(\cdot,\xi)\xi$, $h=\frac{1}{2}L_\xi\varphi$ and $h'=h\circ\varphi$, where R denotes the curvature tensor of M and L is the Lie differentiation. The above mentioned tensor field plays an important role in the study of geometry of almost Kenmotsu manifolds. According to [10,11], the three (1,1)-type tensor fields ℓ, h and h' are all symmetric and satisfy the following equations

$$\nabla_X\xi = X - \eta(X)\xi + h'X, \tag{2.3}$$

$$h\xi = \ell\xi = 0, \quad trh = trh' = 0, \quad h\varphi + \varphi h = 0, \tag{2.4}$$

$$\varphi\ell\varphi - \ell = 2(h^2 - \varphi^2), \tag{2.5}$$

for any $X,Y\in TM$, where ∇ and tr denotes the Levi-Civita connection of g and the trace operator respectively. An almost contact metric manifold is said to be η-Einstein if the Ricci tensor S satisfies

$$S = ag + b\eta\otimes\eta, \tag{2.6}$$

where a,b are arbitrary functions on M.

The last term of Bach tensor in (1.5) can be expressed as

$$\sum_{i,j=1}^{2n+1} S(e_i,e_j)W(X,e_i,e_j,Y) = -\sum_{i=1}^{2n+1} g(QW(X,e_i)Y,e_i),$$

where we applied $d=2n+1$. Thus, the expression of Bach tensor (1.5) transforms into

$$\mathcal{B}(X,Y) = \frac{1}{2n-1}\sum_{i=1}^{2n+1}\left\{(\nabla_{e_i}C)(e_i,X)Y - g(QW(X,e_i)Y,e_i)\right\}. \tag{2.7}$$

3. Characterization of Cotton and Bach tensor on $(\kappa,\mu)'$-almost Kenmotsu manifolds

In [11], Dileo and Pastore introduced the notion of $(\kappa,\mu)'$-nullity distribution on an almost Kenmotsu manifold. If the Reeb vector field ξ of an almost Kenmotsu manifold M belongs to the $(\kappa,\mu)'$-nullity distribution, that is,

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y), \tag{3.1}$$

for some constants κ and μ, then M^{2n+1} is called $(\kappa,\mu)'$-almost Kenmotsu manifold.
In the aforementioned relation, substitute Y by ξ to obtain $\ell = -\kappa\varphi^2 + \mu h'$. Putting this relation into equation (2.5) and employing (2.4), we obtain

$$h'^2 = (\kappa+1)\varphi^2.$$

By (2.1), it follows from above relation that $\kappa \leq -1$. Obviously, we see that the tensor field $h' = 0$ identically if and only if $\kappa = -1$ and $h' \neq 0$ everywhere if and only if $\kappa < -1$. From [11] we remark that on an almost Kenmotsu manifold with ξ belonging to $(\kappa,\mu)'$-nullity distribution and $\kappa < -1$, we have $\mu = -2$. Following [11] and using $h'^2 = (\kappa+1)\varphi^2$, in case of $\kappa < -1$ we denote by $[\lambda]'$ and $[-\lambda]'$ the eigenspaces of h' corresponding to two eigenvalues $\lambda > 0$ and $\lambda < 0$ respectively. Obviously, we have $\lambda = \sqrt{-1-\kappa}$. Some interesting results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions were studied in the papers [10,24,29,32,22].

Throughout this section, we consider those $(\kappa,\mu)'$-almost Kenmotsu manifolds with $\kappa < -1$, i.e., $h' \neq 0$ everywhere. The following result is deduced directly from Dileo and Pastore [11].

Lemma 3.1 *[33] On a $(\kappa,\mu)'$-almost Kenmotsu manifold with $\kappa < -1$ the Ricci operator is given by*

$$QX = -2nX + 2n(\kappa+1)\eta(X)\xi - 2nh'X. \tag{3.2}$$

In this case, the scalar curvature r is $2n(\kappa - 2n)$.

First, we consider a non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold whose metrics have divergence free Cotton tensor and prove the following result.

Theorem 3.1 *Let M be a non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold of dimension $2n+1$. Then M admits a divergence free Cotton tensor if and only if M is locally isometric to the Riemannian product of and $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof: First we invoke Lemma 3.1 to obtain

$$\begin{aligned}(\nabla_Y Q)X =& 2n(\kappa+1)\eta(X)(Y + h'Y) - 2n(\nabla_Y h')X \\ &+ 2n(\kappa+1)\{g(X,Y) - 2\eta(X)\eta(Y) + g(h'X,Y)\}\xi.\end{aligned}$$

It follows from the above equation that

$$\begin{aligned}(\nabla_X Q)Y - (\nabla_Y Q)X &= 2n\{(\nabla_Y h')X - (\nabla_X h')Y\} \\ &+2n(\kappa+1)\{\eta(Y)(X + h'X) - \eta(X)(Y + h'Y)\}.\end{aligned} \tag{3.3}$$

Since the scalar curvature of non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold is constant, the Cotton tensor (1.4) transforms into

$$C(X,Y)Z = g((\nabla_X Q)Y - (\nabla_Y Q)X, Z). \tag{3.4}$$

Making use of (3.3) in the foregoing equation we obtain

$$\begin{aligned}C(X,Y)Z =& 2n\{g((\nabla_Y h')X - (\nabla_X h')Y, Z)\} \\ &+ 2n(\kappa+1)\{\eta(Y)g(X + h'X, Z) - \eta(X)g(Y + h'Y, Z)\}.\end{aligned} \tag{3.5}$$

With the help of $h'^2 = (\kappa+1)\varphi^2$, it has been proved by Dileo and Pastore in [11] that on M, the following equation

$$(\nabla_X h')Y = g((\kappa+1)X - h'X, Y)\xi + \eta(Y)((\kappa+1)X - h'X) - 2(\kappa+1)\eta(X)\eta(Y)\xi,$$

holds. Making use of the preceding equation in (3.5) yields

$$C(X,Y)Z = 2n(\kappa+2)\{g(h'X,Z)\eta(Y) - g(h'Y,Z)\eta(X)\}. \tag{3.6}$$

Differentiating this along U and using (2.3) we have

$$(\nabla_U C)(X,Y)Z = 2n(\kappa+2)\{g((\nabla_U h')X,Z)\eta(Y) + g(h'X,Z)(\nabla_U \eta)(Y) - g((\nabla_U h')Y,Z)\eta(X) - g(h'Y,Z)(\nabla_U \eta)(X)\}. \tag{3.7}$$

According to Corollary 4.1 of Delio and Pastore [11], and Lemma 3.4 of Wang and Liu [30] we have that

$$(i)\ tr(\nabla_X h') = 0 \quad and$$
$$(ii)\ (div h')X = 2n(\kappa+1)\eta(X).$$

Thus, setting $U = X = e_i$ in (3.7), summing over $i = 1, \ldots, 2n+1$ and using (i), (ii) and (2.3) we obtain

$$\sum_{i=1}^{2n+1} (\nabla_{e_i} C)(e_i, Y)Z = 2n(\kappa+2)\{(2n+1)(\kappa+1)\eta(Y)\eta(Z) - (\kappa+1)g(Y,Z) - (2n-1)g(h'Y,Z)\}. \tag{3.8}$$

Suppose we assume that the Cotton tensor is divergence free, then replacing Y by $h'Y$ in the above equation and using $h'^2 = (\kappa+1)\varphi^2$, we ultimately obtain

$$2n(\kappa+2)\{(2n-1)(\kappa+1)(g(Y,Z) - \eta(Y)\eta(Z)) - (\kappa+1)g(h'Y,Z)\} = 0.$$

Contracting the foregoing equation with Y and Z, and using (2.4) we get

$$4n(2n-1)(\kappa+2)(\kappa+1) = 0. \tag{3.9}$$

In view of $\kappa < -1$, it follows from (3.9) that $\kappa = -2$. As a result of Corollary 4.2 and Proposition 4.1 of Dileo and Pastore [11] we obtain that M is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.

Conversely, suppose M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, then from Proposition 4.2 of [4] or Remark 4.1 of [11] we observe that such a product is locally symmetric, that is, $\nabla R = 0$, this implies that the Ricci tensor is symmetric $(i.e., \nabla S = 0)$. This together with (3.4) shows that $C = 0$, which implies that $divC = 0$, that is, Cotton tensor is divergence free. This completes the proof. □

Remark 3.1 Our Theorem 3.1 is in analogy with Theorem 1.1 of Ghosh-Sharma [15] in almost Kenmotsu geometry.

Corollary 3.1 *A 3-dimensional non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifold is conformally flat if and only if it is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

It is well known that $divW = \frac{2n-2}{2n-1}C$. From this we may observe that W is harmonic and is equivalent to $C = 0$. Thus we state the following;

Corollary 3.2 *A non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifold M of dimension $2n+1$ has harmonic Weyl tensor if and only if M is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Since $divR = 0$ imples that $divW = 0$, thus we have the following;

Corollary 3.3 *Let M be a $(2n+1)$-dimensional non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold. Then the curvature tensor of M is harmonic if and only if M is locally isometric to Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

Remark 3.2 The above Corollary 3.2 and Corollary 3.3 have been proved by Wang and Liu [31]. Here we observe that Corollary 3.3 is just Theorem 1.1 of [31], which means that our Theorem 3.1 generalizes the corresponding results shown in [31].

Now, we end up this section by considering Bach flat tensor on non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold, and prove the following fruitful result.

Theorem 3.2 *A Bach flat non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold M is 3-dimensional and is locally isometric to the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$.*

Proof: Since Weyl conformal curvature tensor is trace-free, thus from (3.2) the last term of (2.7) can be written as

$$\begin{aligned}\sum_{i=1}^{2n+1} g(QW(X,e_i)Y,e_i) &= \sum_{i=1}^{2n+1}\{-2ng(W(X,e_i)Y,e_i)\\ &\quad + 2n(\kappa+1)g(W(X,e_i)Y,\xi)g(e_i,\xi) - 2ng(h'W(X,e_i)Y,e_i)\}\\ &= \sum_{i=1}^{2n+1}\{-2n(\kappa+1)g(W(X,\xi)\xi,Y) - 2ng(W(X,e_i)Y,h'e_i)\}.\end{aligned}\tag{3.10}$$

By virtue of $h'^2 = (\kappa+1)\varphi^2$, we find

$$\sum_{i=1}^{2n+1} g(h'^2 e_i,e_i) = -2n(\kappa+1).\tag{3.11}$$

On non-Kenmotsu $(\kappa,\mu)'$-almost Kenmotsu manifold, the expression of curvature tensor is of the form (see [7,9]);

$$\begin{aligned}R(X,Y)Z &= \kappa\eta(Z)\{\eta(Y)X-\eta(X)Y\} + \kappa\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi\\ &\quad + \{g(Y-h'Y,Z)\eta(X) - g(X-h'X,Z)\eta(Y)\}\xi\\ &\quad + \eta(Z)\{\eta(Y)(X-h'X)-\eta(X)(Y-h'Y)\}\\ &\quad - \{g(Y+h'Y,Z)(X+h'X) - g(X+h'X,Z)(Y+h'Y)\}.\end{aligned}\tag{3.12}$$

The application of (3.12) together with (2.4) and (3.11), one can easily calculate

$$\begin{aligned}\sum_{i=1}^{2n+1} g(R(X,e_i)Y,h'e_i) &= (\kappa-2n(\kappa+1))g(h'X,Y) + 2(1-n)(\kappa+1)g(X,Y)\\ &\quad - 2(n+1)(\kappa+1)\eta(X)\eta(Y).\end{aligned}\tag{3.13}$$

Utilization of (3.13) in (1.2) and remember the fact that h' is trace-free provides

$$\begin{aligned}\sum_{i=1}^{2n+1} g(W(X,e_i)Y,h'e_i) &= (\kappa-2n(\kappa+1))g(h'X,Y) + 2(1-n)(\kappa+1)g(X,Y)\\ &\quad - 2(n+1)(\kappa+1)\eta(X)\eta(Y) - \frac{1}{2n-1}\{g(Qh'X,Y)\\ &\quad + g(h'QX,Y) - tr(Qh')g(X,Y)\} + \frac{r}{2n-1}g(h'X,Y).\end{aligned}\tag{3.14}$$

On the other hand, as a result of (3.2) and $h'^2 = (\kappa+1)\varphi^2$, we see that

$$h'Q = Qh' = -2nh' - 2nh'^2 = -2nh' - 2n(\kappa+1)\varphi^2,$$

and hence $tr(Qh') = 4n^2(\kappa+1)$. Insert these facts and $r = 2n(\kappa - 2n)$ in (3.14) to obtain

$$\sum_{i=1}^{2n+1} g(W(X,e_i)Y, h'e_i) = \frac{4n(\kappa+1)(1-n)}{2n-1} g(h'X,Y) + \frac{(2n-2)(\kappa+1)}{2n-1} g(X,Y) + \frac{(2+2n-4n^2)(\kappa+1)}{2n-1}\eta(X)\eta(Y). \tag{3.15}$$

Further, making use of (3.1) and (3.2) along with $r = 2n(\kappa-2n)$ in (1.2) we obtain

$$g(W(X,\xi)\xi, Y) = \frac{2(n-1)}{2n-1} g(h'X,Y). \tag{3.16}$$

Employing (3.15) and (3.16) in (3.10), one can get

$$\sum_{i=1}^{2n+1} g(QW(X,e_i)Y, e_i) = \frac{4n(1-2n)(1-n)(\kappa+1)}{2n-1} g(h'X,Y) - \frac{2n(2n-2)(\kappa+1)}{2n-1} g(X,Y) - \frac{2n(2+2n-4n^2)(\kappa+1)}{2n-1}\eta(X)\eta(Y). \tag{3.17}$$

Since Bach tensor $\mathcal{B}$ vanishes, thus the making use of (3.8) and (3.17) in (2.7) provides

$$2n(\kappa+2)\{(2n+1)(\kappa+1)\eta(X)\eta(Y) - (\kappa+1)g(X,Y) - (2n-1)g(h'X,Y)\} - \frac{4n(1-2n)(1-n)(\kappa+1)}{2n-1} g(h'X,Y) + \frac{2n(2n-2)(\kappa+1)}{2n-1} g(X,Y) + \frac{2n(2+2n-4n^2)(\kappa+1)}{2n-1}\eta(X)\eta(Y) = 0. \tag{3.18}$$

Substituting ξ for X and Y in the foregoing equation we find

$$\kappa = -\frac{2n}{2n-1}. \tag{3.19}$$

Replacing X by $h'X$ in (3.18), we have

$$0 = 2n(\kappa+2)\{-(\kappa+1)g(h'X,Y) - (2n-1)g(h'^2X,Y)\} - \frac{4n(1-2n)(1-n)(\kappa+1)}{2n-1} g(h'^2X,Y) + \frac{2n(2n-2)(\kappa+1)}{2n-1} g(h'X,Y).$$

Contracting the foregoing equation along X and Y, and then using (3.11) we get

$$\kappa = -\frac{12n^2 - 14n + 4}{8n^2 - 10n + 3}. \tag{3.20}$$

Now, we compare (3.19) with (3.20) to achieve $2n^3 - 5n^2 + 4n - 1 = 0$, which is equivalent to

$$(n-1)^2(n - \frac{1}{2}) = 0.$$

It follows from the above equation that either $n = 1$ or $n = \frac{1}{2}$. By these solution sets of n, we conclude that the only possibility is $n = 1$, that is, M is 3-dimensional. By virtue of $n = 1$ in (3.20), we have $\kappa = -2$. From Proposition 4.1 and Corollary 4.2 of Dileo and Pastore [11] we deduce that M is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. This completes the proof. □

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