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Some notes on certain analytic functions

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ABSTRACT: In this paper, we obtain several results on certain analytic functions using Nunokawa's well-known lemma and generalize some recent findings by emphasizing key points. This note primarily aims to highlight that these results, along with other related outcomes, can be further improved and refined.

Key Words: Analytic functions, Nunokawa's lemma.

Contents

1. Introduction

Let $\mathcal{A}_0(n)$ be the class of functions $f(z)$ of the form

$$f(z) = f(0) + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (n \in \mathbb{N}),$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, where $a_n \neq 0$ and $f(0) > 0$. Also, let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (n \in \mathbb{N}),$$

which are analytic in $\mathbb{U}$, where $a_n \neq 0$. If $f(z) \in \mathcal{A}(n)$ satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \; z \in \mathbb{U},$$

for some real $\alpha\,(0 \leq \alpha < n)$, then $f(z)$ is said to be in the class $\mathcal{S}(n, \alpha)$ [3].

In 2019, Yavuz and Owa [3] obtained the following results on certain analytic functions:

Theorem 1.1 [3, Theorem 2.1] *For $f(z) \in \mathcal{A}_0(n)$, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\operatorname{Re} f(z) > 0 \quad (|z| < |z_0| < 1)$$

and

$$\operatorname{Re} f(z_0) = 0, \quad f(z_0) \neq 0.$$

Then we have

$$\frac{z_0 f'(z_0)}{f(z_0)} = ik,$$

where k is real and $|k| \geq n$.

Theorem 1.2 [3, Theorem 2.4] *If $f(z) \in \mathcal{A}_0(n)$ satisfies*

$$\left| \operatorname{Im} \left(\frac{zf'(z)}{f(z)} \right) \right| < n \quad (z \in \mathbb{U}),$$

then $\operatorname{Re} f(z) > 0\,(z \in \mathbb{U})$.

The main purpose of this note is to point out the obtained results in Theorems 1.1-1.2 can be improved.

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2. Main results

First, we recall the result of Theorem 1.1 from [3] holds assuming that

$$f(z) \neq -f(0), \forall z \in \mathbb{U}. \tag{2.1}$$

The above assumption is essential because the function w given by the equation (2.4) in [3] should be analytic in $\mathbb{U}$. Therefore, the assumption (2.1) should appear in the assumption of Theorem 1.1.

Next, the following lemma presented by Nunokawa will be used in our investigation:

Lemma 2.1 [1,2] *Let $p(z) = 1 + \sum_{m \geq n}^{\infty} c_m z^m$, $c_n \neq 0$, be an analytic function in $|z| < 1$ with $p(0) = 1$ and $p(z) \neq 0$ for $z \in \mathbb{U}$. If there exists a point z_0 with $|z_0| < 1$, such that*

$$|\arg p(z)| < \frac{\beta\pi}{2}, \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\beta\pi}{2},$$

for some $\beta > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{n(a + a^{-1})}{2} \geq n \quad \text{when} \quad \arg p(z_0) = \frac{\beta\pi}{2}$$

and

$$k \leq -\frac{n(a + a^{-1})}{2} \leq -n \quad \text{when} \quad \arg p(z_0) = -\frac{\beta\pi}{2},$$

where

$$[p(z_0)]^{1/\beta} = \pm ia, \quad \text{and} \quad a > 0.$$

Applying Lemma 2.1, we get a real generalization of Theorem 1.1 in the following result.

Theorem 2.1 *Let $f(z) = f(0) + \sum_{m \geq n}^{\infty} a_m z^m$, $a_n \neq 0$, be an analytic function in $|z| < 1$ with $f(z) \neq 0$ for $z \in \mathbb{U}$. We suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\left|\arg \frac{f(z)}{f(0)}\right| < \frac{\beta\pi}{2}, \quad \text{for} \quad |z| < |z_0|$$

and

$$\left|\arg \frac{f(z_0)}{f(0)}\right| = \frac{\beta\pi}{2},$$

for some $\beta > 0$, then we have

$$\frac{z_0 f'(z_0)}{f(z_0)} = ik\beta,$$

for some k is real and $|k| \geq \frac{n(a + a^{-1})}{2} \geq n$ where

$$\left[\frac{f(z_0)}{f(0)}\right]^{1/\beta} = \pm ia, \quad \text{and} \quad a > 0.$$

Proof: We define the function $p(z)$ by

$$p(z) := \frac{f(z)}{f(0)}.$$

Then, $p(z)$ is an analytic function in $|z| < 1$ with $p(0) = 1$ and $p(z) \neq 0$ for $z \in \mathbb{U}$. By the assumption of the theorem using Lemma 2.1 to $p(z)$, we observe that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{z_0 f'(z_0)}{f(z_0)} = ik\beta.$$

This completes the assertion of the theorem. □

By setting $\beta = 1$, Theorem 2.1 yields the following corollary.

Corollary 2.1 *Let $f(z) = f(0) + \sum_{m \geq n}^{\infty} a_m z^m$, $a_n \neq 0$, be an analytic function in $|z| < 1$ with $f(z) \neq 0$ for $z \in \mathbb{U}$. We suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\mathrm{Re}\left(\frac{f(z)}{f(0)}\right) > 0 \quad (|z| < |z_0| < 1)$$

and

$$\mathrm{Re}\left(\frac{f(z_0)}{f(0)}\right) = 0.$$

Then we have

$$\frac{z_0 f'(z_0)}{f(z_0)} = ik,$$

for some k is real and $|k| \geq \frac{n(a + a^{-1})}{2} \geq n$ where

$$\frac{f(z_0)}{f(0)} = \pm ia, \quad \text{and} \quad a > 0.$$

Remark 2.1 Theorem 2.1 a real generalization of Theorem 1.1, because:

(i) It is not necessary to assume that $f(0) > 0$, only to assume that $f(z) \neq 0$ for all $z \in \mathbb{U}$.

(ii) It is not necessary to assume that (2.1) holds.

(iii) We get a better delimitation for k; instead of $|k| \geq n$ we have $|k| \geq \frac{n(a + a^{-1})}{2} \geq n$, where

$$\left[\frac{f(z_0)}{f(0)}\right]^{1/\beta} = \pm ia, \quad \text{and} \quad a > 0.$$

In the next result we get a small generalization of Theorem 1.2.

Theorem 2.2 *Let $f(z) = f(0) + \sum_{m \geq n}^{\infty} a_m z^m$, $a_n \neq 0$, be an analytic function in $|z| < 1$ satisfies*

$$\left|\mathrm{Im}\left(\frac{z f'(z)}{f(z)}\right)\right| < n \quad (z \in \mathbb{U}), \tag{2.2}$$

then $\mathrm{Re}\left(\frac{f(z)}{f(0)}\right) > 0 \, (z \in \mathbb{U})$.

Proof: From the assumption (2.2), we easily have $f(z) \neq 0$ for $z \in \mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\mathrm{Re}\left(\frac{f(z_0)}{f(0)}\right) = 0.$$

Thus, from Corollary 2.1 we obtain

$$\frac{z_0 f'(z_0)}{f(z_0)} = ik,$$

for some k is real and $|k| \geq \dfrac{n(a+a^{-1})}{2} \geq n$ where

$$\frac{f(z_0)}{f(0)} = \pm ia, \quad \text{and} \quad a > 0.$$

that is,

$$\left|\mathrm{Im}\left(\frac{z_0 f'(z_0)}{f(z_0)}\right)\right| = |k| \geq n.$$

This contradicts (2.2). This completes our proof. □

3. Conclusion

In this paper, we have applied Nunokawa's lemma [1,2] to generalize and enhance the results of Theorems 1.1 and 1.2, obtaining a stronger and more refined outcome. Additionally, our findings extended existing results by emphasizing key points.

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