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Domatic Polynomials of $\Gamma(Z(R))$: The Zero-Divisor Graphs of Commutative Rings

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ABSTRACT: The domatic polynomial $DP(G,x)$ of a graph G is defined as $DP(G,x) = \sum_{j=1}^{d(G)} dp(G,j)x^j$, where $dp(G,j)$ represents the number of domatic partition of G with size j. In this paper, we find domatic number and domatic polynomial of $\Gamma(Z_n)$ where $n \in \{2s, s^2, st, s^2t, stu, s^\alpha\}$ for distinct prime numbers s, t and u with $\alpha > 2$ and their roots. Further, we discuss a characterization on $DP(\Gamma(R), x)$. Finally, we establish that their domatic polynomials possess the properties of log-concavity and unimodality

Key Words: Zero divisor graphs, commutative ring, domatic partition, domatic number, unimodal, log-concave.

Contents

1. Introduction

We consider a simple graph $G = (V, E)$ of order $|V| = n$. The set $N_G(q) = \{p|pq \in E(G)\}$ is called the open neighborhood of a vertex $p \in V$ and the set $N_G[q] = N_G(q) \cup \{q\}$ is called the closed neighbourhood of $q \in V$. In G, a subset $T \subseteq V$ is called a *dominating set* if $N_G[T] = V(G)$, or every vertex in $V \setminus T$ has at least one neighbor in T. The minimum cardinality of a dominating set in G is represented by the domination number $\gamma(G)$. For more details on domination, we refer [9,12,13]. The concept of the domination polynomial $D(G,x)$ of a graph G was defined by Alikhani and Peng [4] in 2009. A *domatic partition* of a graph G is a partition of the vertex set into disjoint dominating sets. The domatic number $d(G)$ is the maximum size of a domatic partition of a graph G. Cockayne and Hedetniemi [19] introduced the domatic number of a graph G. More details on the domatic number can be seen in [18,19,20,21].

Graph polynomials are a well-developed field that can be utilized to analyze graph properties. The study of the number of domatic partitions in a graph has recently attracted considerable attention from researchers. Firstly, we state the definition of the domatic polynomial of a graph G.

Definition 1.1 *[3] Let $\mathcal{DP}(G,j)$ be the family of domatic partitions of a graph G with cardinality j, and let $dp(G,j) = |\mathcal{DP}(G,j)|$. Then the domatic polynomial $DP(G,x)$ of G is defined as*

$$DP(G,x) = \sum_{j=1}^{d(G)} dp(G,j)x^j,$$

where $d(G)$ is the domatic number of G.

Beck introduced the zero-divisor graph for commutative rings in his work in [7]. Beck initially focused on the coloring of rings, defining a graph where the vertex set included all elements of the ring. Later, Anderson and Livingston [5] revised this definition, specifically formulating the zero-divisor graph for

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commutative rings. They formulated the zero-divisor graph of a commutative ring based on the nonzero zero-divisors of the ring.

In recent years, the exploration of zero-divisor graphs has progressed in various ways. Actually, it is the interplay between the ring theoretic properties of a ring R and the graph-theoretic properties of its zero divisor graph [2,5]. There are many papers which studied some parameters and topological indices of the zero-divisor graphs. For more details, see [1,6,14].

In section 2, we give some definitions and results about zero-divisor graphs. In section 3, we find the domatic number and domatic polynomial of $\Gamma(Z_n)$, where $n \in \{2s, s^2, st, s^2t, stu, s^\alpha\}$ for distinct prime numbers s, t and u with $\alpha > 2$. Further, we discuss a characterization on $DP(\Gamma(R), x)$. Also, we establish that their domatic polynomials possess the properties of log-concavity and unimodality. In section 4, we summarize our research work.

2. Preliminaries

This section includes definitions and results that will be used throughout the paper. First, we define the zero-divisor of a graph.

Definition 2.1 *[5] Let Z_n denote the ring of integers modulo n. The zero-divisor graph $\Gamma(Z_n)$ is the simple, undirected graph without loops, where the vertices correspond to the nonzero zero divisors of Z_n and two distinct vertices u and v in $\Gamma(Z_n)$ are adjacent if their product equals zero in Z_n.*

Example 2.1 *[10] For the graph $\Gamma(Z_{75})$, we have $|V(\Gamma(Z_{75})| = 34$ and $|E(\Gamma(Z_{75})| = 86$. This graph has shown in Figure 1.*

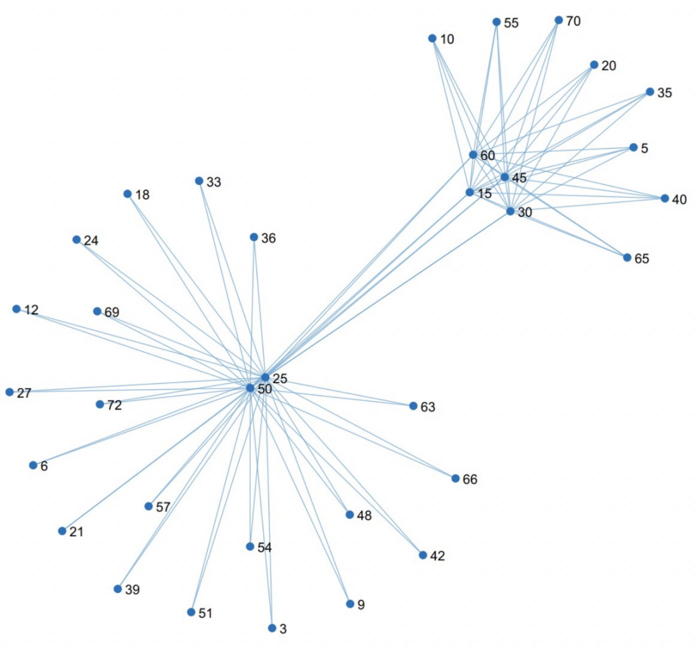

Figure 1: Zero-divisor graph $\Gamma(\mathbb{Z}_{75})$, see [10]

An integer c is called a *proper divisor* of n if $1 < c < n$ and $c|n$. Let $c_1, c_2, \ldots, c_k$ be the distinct proper divisors of n. For $1 \leq j \leq k$, consider the following sets:

$$V_{c_j} = \{y \in Z_n : gcd(y, n) = c_j\}.$$

The sets $V_{c_1}, \ldots, V_{c_k}$ are pairwise disjoint and we can partition the vertex set of $\Gamma(Z_n)$ as

$$V(\Gamma(Z_n)) = V_{c_1} \cup V_{c_2} \cup \ldots \cup V_{c_k}.$$

The following lemma determines the size of each vertex subset in $\Gamma(Z_n)$.

Lemma 2.1 *[17] Let n be a positive integer having distinct divisors $c_1, c_2, \ldots, c_q$. If $V_{c_j} = \{y \in Z_n : gcd(y,n) = c_j\}$ for $j = 1, 2, \ldots, q$, then $|V_{c_j}| = \phi(\frac{n}{c_j})$, where ϕ is the Euler's totient function.*

Lemma 2.2 *[8] For $j, k \in \{1, 2, \ldots, m\}$, a vertex of V_{c_j} is adjacent to a vertex of V_{c_k} in $\Gamma(Z_n)$ if and only if n divides $c_j c_k$.*

Corollary 2.1 *[8]*

i) For each $j \in \{1, 2, \ldots, m\}$, the subgraph of $\Gamma(Z_n)$ induced by the vertex set V_{c_j}, denoted as $\Gamma(V_{c_j})$, is either the complete graph $K_{\phi(n/c_j)}$ or its complement graph $\bar{K}_{\phi(n/c_j)}$. Specifically, $\Gamma(V_{c_j})$ forms $K_{\phi(n/c_j)}$ if and only if n divides c_j^2.

ii) For $j, k \in \{1, 2, \ldots, m\}$, with $j \neq k$, any vertex in V_{c_j} is either adjacent to all the vertices in V_{c_k} or to none in $\Gamma(Z_n)$.

Definition 2.2 *A finite sequence of real numbers $(b_0, b_1, b_2, \ldots, b_p)$ is called unimodal if there exist $j \in \{0, 1, \ldots, p\}$, called the mode of sequence, such that $b_0 \leq \ldots \leq b_{j-1} \leq b_k \geq b_{j+1} \geq \ldots \geq b_p$; the mode is unique if $b_{j-1} < b_j > b_{j+1}$. If the sequence of a polynomial's coefficients is unimodal, then the polynomial is called unimodal.*

A finite sequence of real numbers $(b_0, b_1, b_2, \ldots, b_p)$ is called log-concave, if for all $1 \leq j \leq p-1$,

$$b_j^2 \geq b_{j-1} b_{j+1}.$$

If b_i's are non-negative and all the zeros of $P(y)$ are real. The fundamental method for studying unimodal and log-concave sequences relies on Newton's inequalities, as stated in [11]:

$$b_j^2 \geq b_{j+1} b_{j-1} \left(1 + \frac{1}{j}\right)\left(1 + \frac{1}{p-j}\right), \quad \text{for} \quad j = 1, \ldots, p-1.$$

Any log-concave polynomial with positive coefficients (or a sequence of positive integers) is known to be unimodal.

We give the following results [15,16] which will be used in proving the Theorem 3.1 - 3.3 and Theorem 3.7 - 3.8.

Result 2.1 *Let G be a star graph. Then $d(G) = 2$, and $DP(G, x) = x^2 + x$.*

Result 2.2 *Let G be a complete graph. Then $d(G) = n$, and*

$$DP(G, x) = \sum_{j=1}^{n} dp \begin{Bmatrix} n \\ j \end{Bmatrix} x^j,$$

where $dp\left\{ {n \atop j} \right\}$ is the number of j-domatic partitions of G.

Result 2.3 *Let $G = K_{m,n}$ be a complete bipartite graph with $m \neq n$, where $m = 2$. Then $d(G) = 2$, and*

$$DP(G, x) = x + (m^n - 1)x^2.$$

3. Domatic number and domatic polynomial of graphs $\Gamma(Z_n)$

This section explores the domatic number and domatic polynomial of $\Gamma(Z_n)$, where $n \in \{2s, s^2, st, s^2t, stu, s^\alpha\}$ for distinct prime numbers s, t and u with $\alpha > 2$. Then we prove that these domatic polynomials have two real roots only. Also, we discuss that their characterization and unimodality as well.

Theorem 3.1 *For $\Gamma(Z_n)$, if $n = 2s$, where s is prime number. Then $d(\Gamma(Z_n)) = 2$, and*

$$DP(\Gamma(Z_{2s}), x) = x^2 + x.$$

Proof: The integers 2 and s are the proper divisors of $2s$, so the vertex set of $\Gamma(Z_{2s})$ can be divided into two distinct subsets as $V_2 = \{2y : y = 1, 2, \ldots, s-1\}$ and $V_s = \{s\}$. By Corollary 2.0A, $\Gamma(Z_{2s})$ is the star graph $S_{\phi(s)}$. Hence $d(\Gamma(Z_{2s})) = 2$. Since $DP(S_n, x) = x^2 + x$ and $\phi(s) = s - 1$, so we have the results. □

Theorem 3.2 *For $\Gamma(Z_n)$, if $n = s^2$, where s is prime number. Then $d(\Gamma(Z_n)) = s - 1$, and*

$$DP(\Gamma(Z_{s^2}), x) = \sum_{j=1}^{s-1} dp\begin{Bmatrix} s-1 \\ j \end{Bmatrix} x^j.$$

Proof: The integer s is only proper divisor of s^2. That is, $V_s = \{s, 2s, \ldots, (s-1)s\}$. So, $|\Gamma(Z_{s^2}| = s - 1$ and distinct vertices of $\Gamma(Z_{s^2})$ are adjacent. Therefore, by Corollary 2.0A, the graph $\Gamma(Z_{s^2})$ is isomorphic to complete graph K_{s-1}. Hence $d(\Gamma(Z_{s^2})) = s - 1$.

Since $DP(K_n, x) = \sum_{j=1}^{n} dp\{^n_j\} x^j$, and $\phi(s) = s - 1$, so we have the results. □

Theorem 3.3 *For $\Gamma(Z_n)$, if $n = st$, where s and t are distinct prime numbers. Then $d(\Gamma(Z_n)) = 2$, and*

$$DP(\Gamma(Z_{st}), x) = x + ((s-1)^{(t-1)} - 1)x^2.$$

Proof: The integers s and t are the proper divisor of $n = st$. In this case, we have two partite sets with the cardinality of $s - 1$ and $t - 1$ respectively. i.e.,

$$\begin{aligned} V_s &= \{sy : y = 1, 2, \ldots, t-1\}, \\ V_t &= \{ty : y = 1, 2, \ldots, s-1\}, \end{aligned}$$

and every pair of graph vertices in the two sets are adjacent. Therefore, by Corollary 2.0A, $\Gamma(Z_{st})$ isomorphic to complete bipartite graph $K_{s-1,t-1}$. Hence $d(\Gamma(Z_{st})) = 2$.
Since $DP(K_{mn}, x) = x + (m^n - 1)x^2$ where $m = 2$ and $n \geq 2$ with $\phi(s) = s - 1$, $\phi(t) = t - 1$, so we have the results. □

Theorem 3.4 *For $\Gamma(Z_n)$, if $n = s^2t$, where s and t are distinct prime numbers. Then $d(\Gamma(Z_n)) = min\{s, t\}$, and*

$$DP(\Gamma(Z_{s^2t}), x) = \sum_{j=1}^{min\{s,t\}} dp(\Gamma(Z_{s^2t}), j)x^j.$$

Proof: Let $n = s^2t$. In this case, s, s^2, t and st are proper divisors only. Then we have four partite sets with the cardinalities $st - 1$, $s^2 - 1$, $t - 1$ and $s - 1$, as given below.

$$\begin{aligned} V_s &= \{sy : y = 1, 2, \ldots, st-1, s \nmid y, t \nmid y\}, \\ V_t &= \{ty : y = 1, 2, \ldots, s^2-1, s \nmid y\}, \\ V_{s^2} &= \{s^2x : y = 1, 2, \ldots, t-1\}, \\ V_{st} &= \{sty : y = 1, 2, \ldots, s-1\}, \end{aligned}$$

Since $n = s^2t$, for every $x, y \in Z_n, xy = 0$ iff $x \in V_t, y \in V_{s^2}$ or $x \in V_{s^2}, y \in V_{st}$ or $x \in V_s, y \in V_{st}$ or $x, y \in V_{st}$. In zero-divisor graph $\Gamma(\mathbb{Z}_{s^2t})$, vertices in V_{st} can dominate V_{s^2}, V_s and itself as well. Similarly, vertices in V_{s^2} can dominate V_t and V_{st}. Also, vertices in V_t can dominate V_{s^2} and vertices in V_s can dominate V_{st}. This follows that the maximum size of a domatic partition of $\Gamma(Z_{s^2t})$ is $min\{s,t\}$. Hence $d(\Gamma(Z_n)) = min\{s,t\}$. Consequently, we have

$$DP(\Gamma(Z_{s^2t}), x) = \sum_{j=1}^{min\{s,t\}} dp(\Gamma(Z_{s^2t}), j)x^j.$$

□

Theorem 3.5 *For $\Gamma(Z_n)$, if $n = stu$, where s, t and u are distinct prime numbers then $d(\Gamma(Z_n)) = min\{s,t,u\}$, and*

$$DP(\Gamma(Z_{stu}), x) = \sum_{j=1}^{min\{s,t,u\}} dp(\Gamma(Z_{stu}), j)x^j.$$

Proof: Let $n = stu$. In this case, s, t, u, st, su and tu are proper divisors only. Then we have six partite sets with the cardinalities $tu-1$, $su-1$, $u-1$, $t-1$ and $s-1$, as given below.

$$\begin{aligned}
V_s &= \{sy : y = 1, 2, \ldots, tu-1, t \nmid y, u \nmid y\},\\
V_t &= \{ty : y = 1, 2, \ldots, su-1, s \nmid y, u \nmid y\},\\
V_u &= \{uy : y = 1, 2, \ldots, st-1, s \nmid y, t \nmid y\},\\
V_{st} &= \{sty : y = 1, 2, \ldots, u-1\},\\
V_{su} &= \{suy : y = 1, 2, \ldots, t-1\},\\
V_{tu} &= \{tuy : y = 1, 2, \ldots, s-1\},
\end{aligned}$$

Since $n = stu$, for every $x, y \in Z_n, xy = 0$ iff $x \in V_u, y \in V_{st}$ or $x \in V_{st}, y \in V_{su}$ or $x \in V_{st}, y \in V_{tu}$ or $x \in V_{tu}, y \in V_{su}$ or $x \in V_{su}, y \in V_t$ or $x \in V_{tu}, y \in V_s$. In zero-divisor graph $\Gamma(\mathbb{Z}_{stu})$, vertices in V_u can dominate V_{st}, vertices in V_{st} can dominate V_{su} and V_{tu}, vertices in V_{su} can dominate V_{st}, V_{tu} and V_t. Similarly, vertices in V_{tu} can dominate V_{st}, V_{su} and V_s. Lastly, vertices in V_s and V_t can dominate V_{tu} and V_{su} respectively. This follows that the maximum size of a domatic partition of $\Gamma(Z_{stu})$ is $min\{s,t,u\}$. Hence $d(\Gamma(Z_n)) = min\{s,t,u\}$. Consequently

$$DP(\Gamma(Z_{stu}), x) = \sum_{j=1}^{min\{s,t,u\}} dp(\Gamma(Z_{stu}), j)x^j.$$

□

Theorem 3.6 *For $\Gamma(Z_n)$, if $n = s^\alpha$, where s is prime number and $\alpha > 2$ then $d(\Gamma(Z_n)) = s$, and*

$$DP(\Gamma(Z_{s^\alpha}), x) = \sum_{j=1}^{s} dp(\Gamma(Z_{s^\alpha}), j)x^j.$$

Proof: Let $n = s^\alpha, \alpha > 2$. In this case, $s, s^2, \ldots, s^{\alpha-1}$ are proper divisors only. The vertex set of $\Gamma(Z_{s^\alpha})$ consists of the disjoint union of the following sets $V_s, V_{s^2}, \ldots, V_{s^{\alpha-1}}$.

$$\begin{aligned}
V_s &= \{sy : y = 1, 2, \ldots, s^{\alpha-1}-1, s \nmid y\},\\
V_s^2 &= \{s^2y : y = 1, 2, \ldots, s^{\alpha-2}-1, s \nmid y\},\\
&\vdots\\
V_{s^{\alpha-1}} &= \{s^{\alpha-1}y : y = 1, 2, \ldots, s-1\},
\end{aligned}$$

In zero-divisor graph $\Gamma(\mathbb{Z}_{s^\alpha})$, vertices in $V_{s^{\alpha-1}}$ can dominate all other vertices of $s\Gamma(Z_{s^\alpha})$. This implies that maximum size of a domatic partition of $\Gamma(Z_{s^\alpha})$ is s, since $V_{s^{\alpha-1}}$ has cardinality of $s-1$. Hence, $d(\Gamma(Z_{s^\alpha})) = s$. Consequently, we have

$$DP(\Gamma(Z_{s^\alpha}), x) = \sum_{j=1}^{s} dp(\Gamma(Z_{s^\alpha}), j)x^j.$$

□

Theorem 3.7 *Let $DP(\Gamma(R), x)$ be a domatic polynomial of $\Gamma(R)$. If $R \cong Z_{2s}$, Z_{st}, Z_{s^2t}, Z_{stu} for $\min\{s,t,u\} = 2$, Z_{s^α}, for $\alpha > 2 = s$, where s, t and u are prime numbers, then R has exactly two real roots, one of which is 0.*

Proof: If $R \cong Z_{2s}$ or Z_{st} where s, t are prime numbers, then their domatic polynomials are given by

$$DP(\Gamma(Z_{2s}), x) = x^2 + x, \text{ or } DP(\Gamma(Z_{st}), x) = x + ((s-1)^{(t-1)} - 1)x^2.$$

Since, both the domatic polynomials of $\Gamma(R)$ are quadratic without a constant term. It implies that R cannot have complex roots. Therefore, R has exactly two real roots where 0 is one of them. Similarly, it can be shown that if $R \cong Z_{s^2t}$, Z_{stu} for $\min\{s,t,u\} = 2$, or Z_{s^α}, for $\alpha > 2 = s$, then R also has precisely two real roots, with 0 as one of them. □

Theorem 3.8 *Let $DP(\Gamma(R), x)$ be a domatic polynomial of $\Gamma(R)$ and s is prime number. $DP(\Gamma(R), x) = x^2 + x$, iff $R \cong Z_{2s}$, or Z_{s^α} for $s = 2$, and $3 \leq \alpha \leq 4$.*

Proof: Let $DP(\Gamma(R), x) = x^2 + x$. That is, $\Gamma(R)$ always has 1 domatic partition of cardinality 1 and 2 respectively. It implies that $\Gamma(R) \cong S_{s-1}$. Also, $V(\Gamma(Z_{2s})) = V_2 \bigcup V_s = \{2, 4, 6, \ldots, 2.(s-1)\} \bigcup \{s\}$.

For all $a, b \in V(\Gamma(Z_{2s}))$, we have $ab = 0$. That is, $\Gamma(Z_{2s}) \cong S_{s-1}$. Therefore, we have $Z_{2s} \cong R$. Similarly, we can prove that $\Gamma(Z_{s^\alpha}) \cong R$ for $s = 2$ and $\alpha = 3$ or 4 since every vertex $u \in V_{s^{\alpha-1}}$ is connected to all vertices in $\Gamma(Z_{s^\alpha})$.

Conversely, let $R \cong Z_{2s}$. Since $V_2 = \{2y : y = 1, 2, \ldots, s-1\}$ and $V_s = \{s\}$ are proper distinct subsets of $\Gamma(Z_{2s})$. For all $a, b \in V_2$ or V_s, we have $ab = 0$. Therefore, $\Gamma(Z_{2s}) \cong S_{s-1}$. Hence $DP(\Gamma(Z_{2s}), x) = x^2 + x$. Similarly, we can easily prove that $DP(\Gamma(Z_{s^\alpha}), x) = x^2 + x$ for $\alpha = 3$ or 4 and $s = 2$. □

Theorem 3.9 *The domatic polynomial of $\Gamma(Z_n)$, where $n \in \{2s, s^2, st, s^2t, stu, s^\alpha\}$ is always log-concave and unimodal for distinct primes s, t and u and $\alpha > 2$.*

Proof: Let $n = s^2$, then domatic polynomial of $\Gamma(Z_n)$ is a unimodal because its coefficients initially increase up to a certain point and then decrease afterwards (see proof of Theorem 3.2).

Now, let $n = 2s$ then the domatic polynomial of $\Gamma(Z_{2s})$ is

$$DP(\Gamma(Z_{2s}), x) = x^2 + x.$$

We have $1 = b_1^2 \geq b_0 b_2 = 0$. For $1 \leq j \leq 2$, we get $0 = b_j^2 \geq b_{j-1}b_{j+1} = 0$. Therefore, the polynomial $DP(\Gamma(Z_{2s}), x)$ exhibits log-concavity, which consequently implies its unimodality. Similarly, we can prove that the domatic polynomial of $\Gamma(Z_n)$, where $n \in \{st, s^2t, stu, s^\alpha\}$ is unimodal as its proved in Theorem 3.3 - 3.6. □

4. Conclusion

In this article, we explored the domatic number and domatic polynomial of zero-divisor graphs associated with the rings Z_n where $n \in \{2s, s^2, st, s^2t, stu, s^\alpha\}$ for distinct prime numbers s, t and u with $\alpha > 2$ and their characterization too. Also, we prove that their domatic polynomials are unimodal. Our findings contribute to a deeper understanding of the interplay between the algebraic properties of rings and the combinatorial characteristics of their zero-divisor graphs.

Future research may extend these results to other classes of commutative rings or develop computational methods for efficiently determining domatic polynomials of larger graphs.

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