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Coefficient Estimates For Subclasses Of Bazilevič Type Functions Associated With Quasi-Subordination

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ABSTRACT: In the present paper, by making use of principle of quasi-subordination between two analytic functions the authors introduce two novel subclasses of analytic functions related to Bazilevič type functions. We investigate the coefficient estimates, the classical Fekete-Szegö problem and the bounds of inverse coefficients for the function belonging to these classes. Some important and useful special cases of the main results in the form of corollaries are pointed out.

Key Words: Analytic function, univalent function, quasi-subordination, Fekete-Szegö functional.

1. Introduction and Motivation

Let $\mathcal{A}$ signifies the family of all holomorphic functions $f(\xi)$ normalized by $f(0) = 0$ and $f'(0) = 1$ defined in the domain of open unit disk $\Delta := \{\xi \in \mathbb{C} : |\xi| < 1\}$. In view of the above normalization, the function $f(\xi)$ admits a Taylor-Maclaurin's series expansion of the form:

$$f(\xi) = \xi + \sum_{n=2}^{\infty} l_n \xi^n \quad (\xi \in \Delta). \tag{1.1}$$

By $\mathcal{S}$ we mean the subclass of $\mathcal{A}$ that is univalent in Δ (see [1]). For two functions $f,\ g \in \mathcal{A}$, we call the function f is subordinate to another function g or g is superordinate to f, represented as $f \prec g$ if there exists an analytic function $\omega(\xi)$ with $\omega(0) = 0$ and $|\omega(\xi)| < 1$ such that

$$f(\xi) = g(\omega(\xi)) \quad (\xi \in \Delta). \tag{1.2}$$

Based on the above definition of subordination, Ma and Minda [5] introduced the classes $S^*(\phi)$ and $C(\phi)$ to those functions $f \in \mathcal{S}$ that satisfies $\frac{\xi f'(\xi)}{f(\xi)} \prec \phi(\xi)$ and $1 + \frac{\xi f''(\xi)}{f'(\xi)} \prec \phi(\xi)$ respectively where the function ϕ is an analytic function with the positive real part in Δ, $\phi(\Delta)$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) \geq 0$. The classes $S^*(\phi)$ and $C(\phi)$ include several well-known subclasses of starlike and convex functions as a special cases.
Further, let $f, g \in \mathcal{A}$. We say f is quasi-subordination to another function g in the unit disk Δ if there exist the function $\omega(\xi)$(with constant coefficient zero) and $\varphi(\xi)$ which are analytic and bounded by one in the unit disk Δ such that

$$\frac{f(\xi)}{\varphi(\xi)} \prec g(\xi) \quad (\xi \in \Delta). \tag{1.3}$$

We denote this quasi-subordination by

$$f(\xi) \prec_q g(\xi) \quad (\xi \in \Delta). \tag{1.4}$$

Equivalently, we may write relation (1.4) as

$$f(\xi) = \varphi(\xi) g(\omega(\xi)) \quad (\xi \in \Delta). \tag{1.5}$$

It may be noted that when $\varphi(\xi) \equiv 1$, then $f(\xi) = g(\omega(\xi))$ which implies $f(\xi) \prec g(\xi)$ in Δ. Further, if $\omega(\xi) = \xi$, then $f(\xi) = \varphi(\xi) g(\xi)$ and it is said that $f(\xi)$ is majorized by $g(\xi)$ and written as $f(\xi) \ll g(\xi)$ in Δ. Thus, from the above discussion it is clear that quasi-subordination is a generalization of subordination as well as majorization. For recent expository works on quasi-subordination, see [6,9,10,12,14].
Now we recall the concept of Bazilevič function introduced by Singh [15](also see [4]).

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Definition 1.1 (see [4,15]) A function $f \in \mathcal{A}$ given by (1.1) is said to be in the function class $\mathcal{B}_1(\lambda)$, the class of Bazilevič function of order λ $(\lambda \geq 0)$ if and only if

$$\Re\left[\left(\frac{f(\xi)}{\xi}\right)^{\lambda-1} f'(\xi)\right] > 0 \quad (\xi \in \Delta).$$

Note that $\mathcal{B}_1(0) = \mathcal{S}^*$ and $\mathcal{B}_1(1) = \mathcal{R}$ where $\mathcal{S}^*$ and $\mathcal{R}$ are familiar classes of starlike and bounded turning functions respectively and each of the above classes are necessarily subclasses of $\mathcal{S}$. Motivated by above works of researchers and making use of the concept of quasi-subordination between two analytic functions, we here introduce two novel subclasses of $\mathcal{A}$ as follows:

Definition 1.2 A function $f \in \mathcal{A}$ represented as (1.1) belongs to the function class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$ if it satisfies the following condition:

$$\left[\left(\frac{\xi}{f(\xi)}\right)^{1-\lambda} f'(\xi)\right]^{\alpha} - 1 \prec_q \phi(\xi) - 1 \quad (\lambda \geq 0, \alpha > 0; \xi \in \Delta) \tag{1.6}$$

where the power are considered to be having only principal value.

For $\alpha = 1,\ \lambda = 0$ and $\alpha = \lambda = 1$, the classes $\mathcal{S}_q^{1,0}(\phi)$ and $\mathcal{S}_q^{1,1}(\phi)$ which respectively reduce to the classes $\mathcal{S}_q^*(\phi)$ and $\mathcal{R}_q(\phi)$ studied earlier by Mohd and Darus [6].
From Definition 1.2, it follows that a function f is in the class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$ if and only if there exists an analytic function $\psi(\xi)$ with $|\psi(\xi)| \leq 1$ $(\xi \in \Delta)$ such that

$$\frac{\left[\left(\frac{\xi}{f(\xi)}\right)^{1-\lambda} f'(\xi)\right]^{\alpha} - 1}{\psi(\xi)} \prec (\phi(\xi) - 1) \quad (\xi \in \Delta). \tag{1.7}$$

If we take $\psi(\xi) \equiv 1$ $(\xi \in \Delta)$ in (1.7), then class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$ reduces to $\mathcal{S}^{\alpha,\lambda}(\phi)$ satisfies the condition:

$$\left[\left(\frac{\xi}{f(\xi)}\right)^{1-\lambda} f'(\xi)\right]^{\alpha} \prec \phi(\xi) \quad (\xi \in \Delta). \tag{1.8}$$

Definition 1.3 A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{C}_q^{\lambda}(\phi)$ $(0 \leq \lambda \leq 1)$ if it satisfies the following quasi-subordination condition:

$$\left[\frac{\xi^{2-\lambda} f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right] \prec_q \phi(\xi) - 1. \tag{1.9}$$

For $\lambda = 0$ the class $\mathcal{C}_q^0(\phi)$ reduces to the class $\mathcal{C}_q^*(\phi)$ which is studied earlier by Mohd and Darus [6]. From Definition 1.3, it follows that a function f is in the class $\mathcal{C}_q^{\lambda}(\phi)$ if and only if there exists an analytic function $\psi(\xi)$ with $|\psi(\xi)| \leq 1$ $(\xi \in \Delta)$ such that

$$\frac{\left[\frac{\xi(f''(\xi))}{(f'(\xi))^{1-\lambda}}\right]}{\psi(\xi)} \prec (\phi(\xi) - 1) \quad (\xi \in \Delta). \tag{1.10}$$

If we take $\psi(\xi) \equiv 1$ $(\xi \in \Delta)$ in (1.10), then class $\mathcal{C}_q^{\lambda}(\phi)$ reduces to $\mathcal{C}^{\lambda}(\phi)$ satisfies the condition:

$$\left[\frac{\xi(f''(\xi))}{(f'(\xi))^{1-\lambda}}\right] + 1 \prec \phi(\xi) \quad (\xi \in \Delta). \tag{1.11}$$

The study of coefficient problems for various subclasses of class $\mathcal{A}$ is an exemplary problems in Geometric Function Theory. For function $f \in \mathcal{S}$ given by (1.1), there holds a sharp inequality for the functional $|a_3 - a_2^2|$. Fekete-Szegö [2] obtained sharp bound of the functional $|a_3 - \mu a_2^2|$ for $f \in \mathcal{S}$ with real μ $(0 \leq \mu \leq 1)$. Since then, the problem of determining the sharp upper bounds for such a nonlinear functional for any compact family of function $f \in \mathcal{A}$ is generally known as the classical Fekete-Szegö problem. Fekete-Szegö problems for various subclasses of $\mathcal{S}$ have been investigated by many authors [3,8,11,13,16].

The aim of the present article is to investigate the coefficient estimates, Fekete-Szegö inequality and inverse coefficients estimate of the function belonging to the classes $\mathcal{S}_q^{\alpha,\lambda}(\phi)$ and $\mathcal{C}_q^{\lambda}(\phi)$. Further, the results based on majorization as a particular cases are pointed out.

Let Ω be the family of analytic functions of the form:

$$\omega(\xi) = \omega_1\xi + \omega_2\xi^2 + \cdots \quad (\xi \in \Delta) \tag{1.12}$$

satisfying the condition $|\omega(\xi)| < 1$.

We need the following lemma in order to derive our main results.

Lemma 1.1 *(see [3], p. 10) If $\omega \in \Omega$, then for any complex number ν:*

$$|\omega_1| \leq 1, \qquad |\omega_2 - \nu\omega_1^2| \leq 1 + (|\nu| - 1)|\omega_1|^2 \leq max\{1, |\nu|\}. \tag{1.13}$$

The result is sharp for the functions $\omega(\xi) = \xi$ or $\omega(\xi) = \xi^2$.

2. Main Results

Unless otherwise stated, we assume throughout the paper that

$$f(\xi) = \xi + l_2\xi^2 + l_3\xi^3 + \cdots, \qquad \psi(\xi) = d_0 + d_1\xi + d_2\xi^2 + \cdots,$$
$$\phi(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots, \qquad \omega(\xi) = \omega_1\xi + \omega_2\xi^2 + \cdots.$$

We prove the following result for the class $S_q^{\alpha,\lambda}(\phi)$.

Theorem 2.1 *If $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$ $(\alpha > 0, \ \lambda \geq 0)$, then*

$$|l_2| \leq \frac{c_1}{\alpha(1+\lambda)}, \tag{2.1}$$

and for any complex parameter γ, we have

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} max\left\{1, \left|\frac{c_2}{c_1} - Kc_1\right|\right\}, \tag{2.2}$$

where

$$K = \frac{2\gamma(\lambda+2) - [(\lambda+3) - \alpha(1+\lambda)^2]}{2\alpha(1+\lambda)^2}. \tag{2.3}$$

The above estimates are sharp.

Proof: Let the function $f \in \mathcal{A}$ represented by (1.1) belongs to the class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$. Then by Definition 1.2 there exists a Schwarz function $\omega(\xi)$ and an analytic function $\psi(\xi)$ given as above such that

$$\left[\frac{\xi^{1-\lambda} f'(\xi)}{(f(\xi))^{1-\lambda}}\right]^\alpha - 1 = \psi(\xi)[\phi(\omega(\xi)) - 1] \quad (\xi \in \Delta). \tag{2.4}$$

Using the series expansion for $f(\xi)$ and $f'(\xi)$ from (1.1) we obtain

$$\left[\frac{\xi^{1-\lambda} f'(\xi)}{(f(\xi))^{1-\lambda}}\right]^\alpha - 1 = \alpha(1+\lambda)l_2\xi + \alpha\left[(\lambda+2)l_3 + \frac{1}{2}\left(\alpha(1+\lambda)^2 - (\lambda+3)\right)l_2^2\right]\xi^2 + \cdots \tag{2.5}$$

Also,

$$\phi(\omega(\xi)) - 1 = c_1\omega_1\xi + (c_1w_2 + c_2w_1^2)\xi^2 + \cdots,$$

and

$$\psi(\xi)[\phi(\omega(\xi))-1] = c_1 d_0 \omega_1 \xi + [c_1 d_1 \omega_1 + d_0(c_1\omega_2 + c_2\omega_1^2)]\xi^2 + \cdots . \tag{2.6}$$

Using (2.5) and (2.6) in the relation (2.4) and equating the coefficients of ξ and ξ^2 in the resulting equation, we get

$$l_2 = \frac{c_1 d_0 \omega_1}{\alpha(1+\lambda)} \quad (\lambda \geq 0, \quad \alpha > 0), \tag{2.7}$$

and

$$l_3 = \frac{c_1 d_1}{\alpha(2+\lambda)}\omega_1 + \frac{c_1 d_0}{\alpha(2+\lambda)}\omega_2 + \frac{2\alpha(1+\lambda)^2 d_0 c_2 - [\alpha(1+\lambda)^2 - (\lambda+3)]c_1^2 d_0^2}{2\alpha^2(1+\lambda)^2(2+\lambda)}\omega_1^2. \tag{2.8}$$

Thus, for any complex number γ, we have

$$|l_3 - \gamma l_2^2| = \frac{c_1}{\alpha(2+\lambda)}\left[d_1\omega_1 + \left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right)d_0 - Kc_1 d_0^2\omega_1^2\right], \tag{2.9}$$

where K is given by (2.3).
Since $\psi(\xi)$ is analytic and bounded in the open unit disk Δ, hence upon using ([7], p. 172), we have for some τ with $|\tau| \leq 1$:

$$|d_0| \leq 1 \qquad \textit{and} \qquad d_1 = (1-d_0^2)\tau. \tag{2.10}$$

Putting the value of d_1 from (2.10) in the relation (2.9), we obtain

$$l_3 - \gamma l_2^2 = \frac{c_1}{\alpha(2+\lambda)}\left[\tau\omega_1 + \left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right)d_0 - (Kc_1\omega_1^2 + \tau\omega_1))d_0^2\right]. \tag{2.11}$$

If $d_0 = 0$, then (2.11) gives

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)}. \tag{2.12}$$

Suppose $d_0 \neq 0$. In such case, consider the function

$$L(d_0) = \tau\omega_1 + \left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right)d_0 - (Kc_1\omega_1^2 + \tau\omega_1)d_0^2. \tag{2.13}$$

The expression (2.13) is a quadratic polynomial in d_0 and hence analytic in $|d_0| \leq 1$. The maximum value of $L(d_0)$ occurs at $d_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. Therefore,

$$\begin{aligned} Max|L(d_0)| &= Max_{0\leq\theta<2\pi}|L(e^{i\theta})| = L(1) \\ &= \left|\omega_2 - \left(Kc_1 - \frac{c_1}{c_1}\right)\omega_1^2\right|. \end{aligned} \tag{2.14}$$

Taking modulus on the both sides of (2.11) and making use of (2.14) in (2.11) we get

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)}\left|\omega_2 - \left(Kc_1 - \frac{c_2}{c_1}\right)\omega_1^2\right|. \tag{2.15}$$

An application of Lemma 1.1 we obtain

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} max\left\{1, \left|\frac{c_2}{c_1} - Kc_1\right|\right\}. \tag{2.16}$$

The assertion mentioned in (2.2) follows from (2.12) and (2.16). The result is sharp for the function $f(\xi)$ given by

$$\left[\frac{\xi^{1-\lambda}f'(\xi)}{(f(\xi))^{1-\lambda}}\right]^\alpha = \phi(\xi),$$

or

$$\left[\frac{\xi^{1-\lambda}f'(\xi)}{(f(\xi))^{1-\lambda}}\right]^\alpha = \phi(\xi^2).$$

The proof of Theorem 2.1 is complete. □

Taking $\lambda = 0$ and $\alpha = 1$ in the Theorem 2.1, we obtain the following sharp result for the class $\mathcal{S}_q^*(\phi)$.

Corollary 2.1 *(see [9], Corollary 2.2) Let the function $f \in \mathcal{A}$ given by (1.1) belongs to the function class $\mathcal{S}_q^*(\phi)$. Then*

$$|l_2| \leq c_1,$$

and for any complex number γ,

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{2} max\left\{1, \left|\frac{c_2}{c_1} + (1-2\gamma)c_1\right|\right\},$$

The result is sharp.

Remark 2.1 *The above Fekete-Szegö type inequalities for the class $\mathcal{S}_q^*(\phi)$ improve similar result obtained by earlier researchers [6].*

The next theorem is devoted to the result based on majorization.

Theorem 2.2 *If a function $f \in \mathcal{A}$ of the form (1.1) satisfies*

$$\left[\left(\frac{\xi}{f(\xi)}\right)^{1-\lambda} f'(\xi)\right]^{\alpha} - 1 \ll [\phi(\xi) - 1] \quad (\xi \in \Delta), \tag{2.17}$$

then

$$|l_2| \leq \frac{c_1}{\alpha(1+\lambda)}, \tag{2.18}$$

and for any complex number γ, we have

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} max\left\{1, \left|\frac{c_2}{c_1} - Kc_1\right|\right\} \tag{2.19}$$

where K is given by (2.3). The results is sharp.

Proof: Assume that (2.17) holds true. From the definition of majorization, there exists an analytic function $\psi(\xi)$ such that

$$\left[\left(\frac{\xi}{f(\xi)}\right)^{1-\lambda} f'(\xi)\right]^{\alpha} - 1 = \psi(\xi)[\phi(\xi) - 1] \quad (\xi \in \Delta). \tag{2.20}$$

Proceeding similarly as in Theorem 2.1 and setting $\omega(\xi) = \xi$ so that $\omega_1 = 1$ and $\omega_k = 0$ for $k \geq 2$, we get

$$l_2 = \frac{c_1 d_0}{\alpha(1+\lambda)},$$

which implies

$$|l_2| \leq \frac{c_1}{\alpha(1+\lambda)} \quad (|d_0| \leq 1). \tag{2.21}$$

Also,

$$l_3 - \gamma l_2^2 = \frac{c_1}{\alpha(2+\lambda)}[\tau + \frac{c_2}{c_1} d_0 - (Kc_1 + \tau)d_0^2]. \tag{2.22}$$

If $d_0 = 0$ then the relation (2.22) yields

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} \quad (|\tau| \leq 1). \tag{2.23}$$

If $d_0 \neq 0$ then consider the function

$$R(d_0) = \tau + \frac{c_2}{c_1} d_0 - (Kc_1 + \tau)d_0^2. \tag{2.24}$$

□

The relation given by (2.24) is a quadratic polynomial in d_0 and hence analytic in $|d_0| \leq 1$. The maximum value of $|R(d_0)|$ occurs at $d_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. Thus, we have

$$max_{0\leq\theta<2\pi} R(e^{i\theta}) = |R(1)| = \frac{c_2}{c_1} - Kc_1.$$

From relation (2.22) we obtain

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} \left| \frac{c_2}{c_1} - Kc_1 \right|. \tag{2.25}$$

The estimate (2.19) stated in the theorem follows from (2.23) and (2.25) . The result is sharp for the function given by

$$\left[\frac{\xi^{1-\lambda} f'(\xi)}{(f(\xi))^{1-\lambda}}\right]^{\alpha} = \phi(\xi) \quad (\xi \in \Delta),$$

which completes the proof of Theorem 2.2.
The following theorem gives a result related to the class $\mathcal{S}^{\alpha,\lambda}(\phi)$.

Theorem 2.3 *Suppose $f \in \mathcal{A}$ given by (1.1) is in the function class $\mathcal{S}^{\alpha,\lambda}(\phi)$ $(\alpha > 0,\ \lambda \geq 0)$. Then*

$$|l_2| \leq \frac{c_1}{\alpha(\lambda+1)}, \tag{2.26}$$

and for any complex number γ, we have

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} max\left\{1, \left|\frac{c_2}{c_1} - Kc_1\right|\right\}, \tag{2.27}$$

where K is given by (2.3) and the result is sharp.

Proof: The proof of the above theorem is similar to that of Theorem 2.1. Let $f \in \mathcal{S}^{\alpha,\lambda}(\phi)$. If $\psi(\xi) \equiv 1$ then $d_0 = 1,\ \ d_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (2.7) and (2.8) and by an application of Lemma 1.1 we obtain the desired result. The result is sharp for the function $f(\xi)$ given by

$$\left[\frac{\xi^{1-\lambda} f'(\xi)}{(f(\xi))^{1-\lambda}}\right]^{\alpha} = \phi(\xi)$$

or

$$\left[\frac{\xi^{1-\lambda} f'(\xi)}{(f(\xi))^{1-\lambda}}\right]' = \phi(\xi^2).$$

This completes the proof of Theorem 2.3 . □

Now we derive the bound of Fekete-Szegö functional $|l_3 - \gamma l_2^2|$ for the class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$ when γ is a real a parameter.

Theorem 2.4 *Suppose the function $f \in \mathcal{A}$ given by (1.1) is in the function class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$. Then for any real number γ, we have*

$$|l_3 - \gamma l_2^2| \leq \begin{cases} \frac{c_1}{\alpha(2+\lambda)}\left[\frac{c_2}{c_1} + \frac{(\lambda+3)-\alpha(\lambda+1)^2-2\gamma(\lambda+2)}{2\alpha(1+\lambda)^2} c_1\right] & \gamma \leq \delta, \\ \frac{c_1}{\alpha(2+\lambda)} & \delta \leq \gamma \leq \delta + 2\epsilon, \\ -\frac{c_1}{\alpha(2+\lambda)}\left[\frac{c_2}{c_1} + \frac{(\lambda+3)-\alpha(\lambda+1)^2-2\gamma(\lambda+2)}{2\alpha(1+\lambda)^2} c_1\right] & \gamma \geq \delta + 2\epsilon, \end{cases} \tag{2.28}$$

where

$$\delta = \frac{(\lambda+3) - \alpha(\lambda+1)^2}{2(\lambda+2)} - \frac{\alpha(\lambda+1)^2}{\lambda+2}\left(\frac{1}{c_1} - \frac{c_2}{c_1^2}\right), \tag{2.29}$$

and

$$\epsilon = \frac{\alpha(1+\lambda)^2}{(\lambda+2)c_1}. \tag{2.30}$$

Each of the estimate mentioned in (2.28) is sharp.

Proof: For real values of γ, the above bounds can be obtained from (2.2) respectively under the following cases:

$$Kc_1 - \frac{c_2}{c_1} \leq -1, \qquad -1 \leq Kc_1 - \frac{c_2}{c_1} \leq 1 \qquad Kc_1 - \frac{c_2}{c_1} \geq 1,$$

where K is given by (2.3). Note that:
(i) When $\gamma < \delta$ or $\gamma > \delta + 2\epsilon$, then the equality holds if and only if $\omega(\xi) = \xi$ or one of its rotations.
(ii) When $\delta < \gamma < \delta + 2\epsilon$, then the inequality holds if and only if $\omega(\xi) = \xi^2$ or one of its rotations.
(iii) Equality holds for $\gamma = \delta$ if and only if $\omega(\xi) = \frac{\xi(\xi+\beta)}{1+\beta\xi}$ $(0 \leq \beta \leq 1)$ or one of its rotations, while for $\gamma = \delta + 2\epsilon$, the equality holds if and only if $\omega(\xi) = -\frac{\xi(\xi+\beta)}{1+\beta\xi}$ $(0 \leq \beta \leq 1)$ or one of its rotations. □

The second part of the assertion mentioned in the relation (2.28) can be improved further as follows:

Theorem 2.5 *Let $f \in \mathcal{S}_q^{\alpha,\lambda}(\phi)$. Then for real number γ, we have*

$$|l_3 - \gamma l_2^2| + (\gamma - \delta)|l_2|^2 \leq \frac{c_1}{\alpha(2+\lambda)} \quad \delta \leq \gamma \leq \delta + \epsilon, \tag{2.31}$$

and

$$|l_3 - \gamma l_2^2| + (\delta + 2\epsilon - \gamma)|l_2|^2 \leq \frac{c_1}{\alpha(2+\lambda)} \quad \delta + \epsilon \leq \gamma \leq \delta + 2\epsilon, \tag{2.32}$$

where δ_1 and ϵ are given by (2.29) and (2.30) respectively.

Proof: Let $f \in \mathcal{S}_q^{\alpha,\lambda}(\phi)$. For real parameter γ satisfying $\delta \leq \gamma \leq \delta + \epsilon$ and using (2.7) and (2.15) we get

$$|l_3 - \gamma l_2^2| + (\gamma - \delta)|l_2|^2 \leq \frac{c_1}{\alpha(2+\lambda)} \left[|\omega_2| - (\gamma - \delta - \epsilon)\frac{\lambda+2}{\alpha(1+\lambda)^2} c_1|\omega_1|^2 + \frac{c_1(\lambda+2)}{\alpha(1+\lambda)^2}(\gamma - \delta)|\omega_1|^2 \right].$$

By virtue of Lemma 1.1 we get

$$|l_3 - \gamma l_2^2| + (\gamma - \delta)|l_2|^2 \leq \frac{c_1}{\alpha(2+\lambda)}[1 - |\omega_1|^2 + |\omega_1|^2],$$

which gives the assertion (2.31).
Further, if $\delta + \epsilon \leq \gamma \leq \delta + 2\epsilon$, then again from (2.7) and (2.15) and application of Lemma 1.1, one can obtain

$$\begin{aligned} |l_3 - \gamma l_2^2| + (\delta + 2\epsilon - \gamma)|l_2|^2 \leq & \frac{c_1}{\alpha(2+\lambda)} \left[|\omega_2| + \frac{(\lambda+2)c_1}{\alpha(1+\lambda)^2}(\gamma - \delta - \epsilon)|\omega_1|^2 \right. \\ & \left. + \frac{(\lambda+2)c_1}{\alpha(1+\lambda)^2}(\delta + 2\epsilon - \gamma)|\omega_1|^2 \right] \\ \leq & \frac{c_1}{\alpha(2+\lambda)}[1 - |\omega_1|^2 + |\omega_1|^2], \end{aligned}$$

which estimates (2.32). □

Next we prove the following result for the class $\mathcal{C}_q^{\lambda}(\phi)$.

Theorem 2.6 *If $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{C}_q^{\lambda}(\phi)$ $(\lambda \geq 0)$, then*

$$|l_2| \leq \frac{c_1}{2}, \tag{2.33}$$

and for any complex parameter γ, we have

$$|l_3-\gamma l_2^2|\leq \frac{c_1}{6}max\left\{1,\left|\frac{c_2}{c_1}-tc_1\right|\right\}, \tag{2.34}$$

where

$$t=\frac{2(\lambda-1)+3\gamma}{2}. \tag{2.35}$$

The above estimates are sharp.

Proof: Let the function $f\in\mathcal{A}$ represented by (1.1) belongs to the class $\mathcal{C}_q^{\lambda}(\phi)$. Then by Definition 1.3 there exists a Schwarz function $\omega(\xi)$ and an analytic function $\psi(\xi)$ given as above such that

$$\left[\frac{\xi^{2-\lambda}f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right]=\psi(\xi)\left[\phi(\omega(\xi))-1\right]. \tag{2.36}$$

Using the series expansion for $f(\xi)$, $f'(\xi)$ and $f''(\xi)$ from (1.1) we obtain

$$\left[\frac{\xi^{2-\lambda}f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right]=2l_2\xi+\left[4(\lambda-1)l_2^2+6l_3\right]\xi^2+\cdots \tag{2.37}$$

Using (2.6) and (2.37) in the relation (2.36) and equating the coefficients of ξ and ξ^2 in the resulting equation, we get

$$l_2=\frac{c_1d_0\omega_1}{2}\quad(\lambda\geq 0) \tag{2.38}$$

and

$$l_3=\frac{1}{6}\left[c_1d_1\omega_1+c_1d_0\omega_2+d_0(c_2-(\lambda-1)c_1^2d_0)\omega_1^2\right]. \tag{2.39}$$

Thus, for any complex number γ, we have

$$l_3-\gamma l_2^2=\frac{c_1}{6}\left[d_1\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)d_0-tc_1d_0^2\omega_1^2\right], \tag{2.40}$$

where t is given by (2.35).
Since $\psi(\xi)$ is analytic and bounded in the open unit disk Δ, hence upon using ([7], p. 172), we have for some τ with $|\tau|\leq 1$:

$$|d_0|\leq 1\qquad \textit{and}\qquad d_1=(1-d_0^2)\tau. \tag{2.41}$$

Putting the value of d_1 from (2.41) in the relation (2.40), we obtain

$$l_3-\gamma l_2^2=\frac{c_1}{6}\left[\tau\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)d_0-(tc_1\omega_1^2+\tau\omega_1))d_0^2\right]. \tag{2.42}$$

If $d_0=0$, then (2.42) gives

$$|l_3-\gamma l_2^2|\leq\frac{c_1}{6}. \tag{2.43}$$

Suppose $d_0\neq 0$. In such case, consider the function

$$T(d_0)=\tau\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)d_0-(tc_1\omega_1^2+\tau\omega_1)d_0^2. \tag{2.44}$$

The expression (2.44) is a quadratic polynomial in d_0 and hence analytic in $|d_0|\leq 1$. The maximum value of $T(d_0)$ occurs at $d_0=e^{i\theta}$ $(0\leq\theta<2\pi)$. Therefore,

$$\begin{aligned}Max|T(d_0)|&=Max_{0\leq\theta<2\pi}|T(e^{i\theta})|=T(1)\\&=\left|\omega_2-\left(tc_1-\frac{c_2}{c_1}\right)\omega_1^2\right|.\end{aligned} \tag{2.45}$$

Taking modulus on the both sides of (2.42) and making use of (2.45) in (2.42) we get

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{6}\left|\omega_2 - \left(tc_1 - \frac{c_2}{c_1}\right)\omega_1^2\right|. \tag{2.46}$$

An application of Lemma 1.1 we obtain

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{6} max\left\{1, \left|\frac{c_2}{c_1} - tc_1\right|\right\}. \tag{2.47}$$

The assertion mentioned in (2.34) follows from (2.43) and (2.47). The result is sharp for the function $f(\xi)$ given by

$$\left[\frac{\xi^{2-\lambda} f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right] = \phi(\xi),$$

or

$$\left[\frac{\xi^{2-\lambda} f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right] = \phi(\xi^2).$$

The proof of Theorem 2.6 is complete. □

Taking $\lambda = 0$ in the Theorem 2.6, we obtain the following sharp result for the class $\mathcal{C}_q^*(\phi)$.

Corollary 2.2 *(see [9], Corollary 2.3) Let the function $f \in \mathcal{A}$ given by (1.1) belongs to the function class $\mathcal{C}_q^*(\phi)$. Then*

$$|l_2| \leq \frac{c_1}{2},$$

and for any $\gamma \in \mathbb{C}$,

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{6} max\left\{1, \left|\frac{c_2}{c_1} + \left(1 - \frac{3\gamma}{2}\right)c_1\right|\right\},$$

The result is sharp.

Remark 2.2 *The above Fekete-Szegö type inequalities for the class $\mathcal{C}_q^*(\phi)$ improve similar result obtained by earlier researchers [6].*

The next result is based on majorization.

Theorem 2.7 *If a function $f \in \mathcal{A}$ of the form (1.1) satisfies*

$$\left[\frac{\xi^{2-\lambda} f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right] \ll [\phi(\xi) - 1] \quad (\xi \in \Delta), \tag{2.48}$$

then

$$|l_2| \leq \frac{c_1}{2}, \tag{2.49}$$

and for any complex number γ, we have

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{6} max\left\{1, \left|\frac{c_2}{c_1} - tc_1\right|\right\} \tag{2.50}$$

where t is given by (2.35). The results is sharp.

Proof: Assume that (2.48) holds true. From the definition of majorization, there exists an analytic function $\psi(\xi)$ such that

$$\left[\frac{\xi^{2-\lambda} f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right] = \psi(\xi)[\phi(\xi) - 1] \quad (\xi \in \Delta). \tag{2.51}$$

Proceeding similarly as in Theorem 2.6 and setting $\omega(\xi) = \xi$ so that $\omega_1 = 1$ and $\omega_k = 0$ for $k \geq 2$, we get

$$l_2 = \frac{c_1 d_0}{2},$$

which implies

$$|l_2| \leq \frac{c_1}{2} \quad (|d_0| \leq 1). \tag{2.52}$$

Also,

$$l_3 - \gamma l_2^2 = \frac{c_1}{6}[\tau + \frac{c_2}{c_1} d_0 - (tc_1 + \tau)d_0^2]. \tag{2.53}$$

If $d_0 = 0$ then the relation (2.53) yields

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{6} \quad (|\tau| \leq 1). \tag{2.54}$$

If $d_0 \neq 0$ then consider the function

$$M(d_0) = \tau + \frac{c_2}{c_1} d_0 - (tc_1 + \tau)d_0^2. \tag{2.55}$$

The relation given by (2.55) is a quadratic polynomial in d_0 and hence analytic in $|d_0| \leq 1$. The maximum value of $|M(d_0)|$ occurs at $d_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. Thus, we have

$$max_{0 \leq \theta < 2\pi} M(e^{i\theta}) = |M(1)| = \frac{c_2}{c_1} - Kc_1.$$

From relation (2.53) we obtain

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{6} \left| \frac{c_2}{c_1} - tc_1 \right|. \tag{2.56}$$

The estimate (2.50) stated in the theorem follows from (2.54) and (2.56) . The result is sharp for the function given by

$$\left[\frac{\xi^{2-\lambda} f''(\xi)}{(\xi f'(\xi))^{1-\lambda}} \right] = \phi(\xi) \quad (\xi \in \Delta),$$

which completes the proof of Theorem 2.7.
The following theorem gives a result related to the class $\mathcal{C}^\lambda(\phi)$.

□

Theorem 2.8 *Suppose $f \in \mathcal{A}$ given by (1.1) is in the function class $\mathcal{C}^\lambda(\phi)$ $(\lambda \geq 0)$. Then*

$$|l_2| \leq \frac{c_1}{2}, \tag{2.57}$$

and for any complex number γ, we have

$$|l_3 - \gamma l_2^2| \leq \frac{c_1}{6} max \left\{ 1, \left| \frac{c_2}{c_1} - tc_1 \right| \right\}, \tag{2.58}$$

where t is given by (2.35) and the result is sharp.

Proof: The proof of the above theorem is similar to that of Theorem 2.6. Let $f \in \mathcal{C}^\lambda(\phi)$. If $\psi(\xi) \equiv 1$ then $d_0 = 1, \quad d_n = 0 \quad (n \in \mathbb{N})$. Therefore, in view of (2.38) and (2.39) and by an application of Lemma 1.1 we obtain the desired result. The result is sharp for the function $f(\xi)$ given by

$$\left[\frac{\xi^{2-\lambda} f''(\xi)}{(\xi f'(\xi))^{1-\lambda}} \right] = \phi(\xi)$$

or

$$\left[\frac{\xi^{2-\lambda}f''(\xi)}{(\xi f'(\xi))^{1-\lambda}}\right] = \phi(\xi^2).$$

This completes the proof of Theorem 2.8 . □

Now we derive the bound of Fekete-Szegö functional $|l_3 - \gamma l_2^2|$ for the class $\mathcal{C}_q^{\lambda}(\phi)$ when γ is a real a parameter.

Theorem 2.9 *Suppose the function $f \in \mathcal{A}$ given by (1.1) is in the function class $\mathcal{C}_q^{\lambda}(\phi)$. Then for any real number γ, we have*

$$|l_3 - \gamma l_2^2| \leq \begin{cases} \frac{c_1}{6}\left[\frac{c_2}{c_1} + \frac{2(1-\lambda)-3\gamma}{2}c_1\right] & \gamma \leq \rho, \\ \frac{c_1}{6} & \rho \leq \gamma \leq \rho + 2\varsigma, \\ -\frac{c_1}{6}\left[\frac{c_2}{c_1} + \frac{2(1-\lambda)-3\gamma}{2}c_1\right] & \gamma \geq \rho + 2\varsigma, \end{cases} \tag{2.59}$$

where

$$\beta = \frac{2(c_2 - c_1)}{3c_1^2} - \frac{2(\lambda - 1)}{3}, \tag{2.60}$$

and

$$\varsigma = \frac{4}{3c_1}. \tag{2.61}$$

Each of the estimate mentioned in (2.59) is sharp.

Proof: The proof of Theorem 2.9 follows a similar approach to that of Theorem 2.4 and hence we omit the details proof.

□

The next part of the assertion mentioned in the relation (2.59) can be improved further as follows:

Theorem 2.10 *Let $f \in \mathcal{C}_q^{\lambda}(\phi)$. Then for real number γ, we have*

$$|l_3 - \gamma l_2^2| + (\gamma - \rho)|l_2|^2 \leq \frac{c_1}{6} \quad \rho \leq \gamma \leq \rho + \varsigma, \tag{2.62}$$

and

$$|l_3 - \gamma l_2^2| + (\rho + 2\varsigma - \gamma)|l_2|^2 \leq \frac{c_1}{6} \quad \rho + \varsigma \leq \gamma \leq \rho + 2\varsigma, \tag{2.63}$$

where ρ and ς are given by (2.60) and (2.61) respectively.

Proof: The proof of Theorem 2.10 follows the same approach to that of Theorem 2.5 and hence we omit the details proof. □

3. Inverse Coefficient Bounds For The Classes $\mathcal{S}_q^{\alpha,\lambda}(\phi)$ and $\mathcal{C}_q^{\lambda}(\phi)$.

For every univalent functions f defined on the domain of open unit disk $\mathbb{D}$, the famous Koebe one-quarter theorem asserts that its inverse f^{-1} exists at least on a disk of radius 1/4 with the Taylor-Maclaurin series expansion of the form:

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} e_n w^n \quad (|w| < \frac{1}{4}), \tag{3.1}$$

where

$$e_2 = -l_2, \tag{3.2}$$

and

$$e_3 = 2l_2^2 - l_3. \tag{3.3}$$

Theorem 3.1 *Suppose that the function $f^{-1} \in \mathcal{A}$ given by (1.1) belongs to the class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$. Then*

$$|e_2| \leq \frac{c_1}{\alpha(1+\lambda)}, \tag{3.4}$$

and for any complex parameter γ, we have

$$|e_3 - \gamma e_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} max\left\{1, \left|\frac{c_2}{c_1} - Pc_1\right|\right\}, \tag{3.5}$$

where

$$P = \frac{(3\lambda+5) + \alpha(1+\lambda)^2 - 2\gamma(2+\lambda)}{2\alpha(1+\lambda)^2}. \tag{3.6}$$

The above estimates are sharp.

Proof: Let the function $f^{-1} \in \mathcal{A}$ be in the class $\mathcal{S}_q^{\alpha,\lambda}(\phi)$. Then using the relation (2.7) and (2.8) in the relation (3.2) and (3.3), we get

$$e_2 = l_2 = \frac{-c_1 d_0 \omega_1}{\alpha(1+\lambda)}, \quad (\lambda \geq 0, \quad \alpha > 0) \tag{3.7}$$

$$e_3 = 2l_2^2 - l_3 = \frac{\left[(3\lambda+5) + \alpha(1+\lambda)^2\right] c_1^2 d_0^2 - 2\alpha(1+\lambda)^2 d_0 c_2}{2\alpha^2(1+\lambda)^2(2+\lambda)} \omega_1^2 - \frac{c_1 d_1}{\alpha(2+\lambda)} \omega_1 - \frac{c_1 d_0}{\alpha(2+\lambda)} \omega_2. \tag{3.8}$$

Thus, for any complex number γ, we have

$$e_3 - \gamma e_2^2 = \frac{c_1}{\alpha(2+\lambda)} \left[d_1\omega_1 + \left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right) d_0 - Pc_1 d_0^2 \omega_1^2\right], \tag{3.9}$$

where P is given by (3.6).
Since $\psi(\xi)$ is analytic and bounded in the open unit disk Δ, hence upon using ([7], p. 172), we have for some τ with $|\tau| \leq 1$:

$$|d_0| \leq 1 \qquad \text{and} \qquad d_1 = (1 - d_0^2)\tau. \tag{3.10}$$

Putting the value of d_1 from (3.10) in the relation (3.9), we obtain

$$e_3 - \gamma e_2^2 = \frac{-c_1}{\alpha(2+\lambda)} \left[\tau\omega_1 + d_0\left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right) - (Pc_1\omega_1^2 + \tau\omega_1))d_0^2\right]. \tag{3.11}$$

If $d_0 = 0$, then (3.11) gives

$$|e_3 - \gamma e_2^2| \leq \frac{c_1}{\alpha(2+\lambda)}. \tag{3.12}$$

Suppose $d_0 \neq 0$. In such case, consider the function

$$U(d_0) = \tau\omega_1 + d_0\left(\omega_2 + \frac{c_2}{c_1}\omega_1^2\right) - (Pc_1\omega_1^2 + \tau\omega_1)d_0^2. \tag{3.13}$$

The expression (3.13) is a quadratic polynomial in d_0 and hence analytic in $|d_0| \leq 1$. The maximum value of $U(d_0)$ occurs at $d_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. Therefore,

$$\begin{aligned} Max|U(d_0)| &= Max_{0\leq\theta<2\pi}|U(e^{i\theta})| = U(1) \\ &= \left|\omega_2 - \left(Pc_1 - \frac{c_1}{c_1}\right)\omega_1^2\right|. \end{aligned} \tag{3.14}$$

Taking modulus on the both sides of (3.11) and making use of (3.14) in (3.11) we get

$$|e_3 - \gamma e_2^2| \leq \frac{c_1}{\alpha(2+\lambda)} \left|\omega_2 - \left(Pc_1 - \frac{c_2}{c_1}\right)\omega_1^2\right|. \tag{3.15}$$

An application of Lemma 1.1 we obtain

$$|e_3-\gamma e_2^2|\leq \frac{c_1}{\alpha(2+\lambda)}max\left\{1,\left|\frac{c_2}{c_1}-Pc_1\right|\right\}. \tag{3.16}$$

The assertion mentioned in (3.5) follows from (3.12) and (3.16).
The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Suppose that the function $f^{-1}\in\mathcal{A}$ given by (1.1) belongs to the class $\mathcal{C}_q^{\lambda}(\phi)\quad(\lambda\geq 0)$. Then*

$$|e_2|\leq \frac{c_1}{2}, \tag{3.17}$$

and for any complex parameter γ, we have

$$|e_3-\gamma e_2^2|\leq \frac{c_1}{6}max\left\{1,\left|\frac{c_2}{c_1}-Qc_1\right|\right\}, \tag{3.18}$$

where

$$Q=\frac{2(\lambda+2)-3\gamma}{2}. \tag{3.19}$$

The above estimates are sharp.

Proof: Let the function $f^{-1}\in\mathcal{A}$ be in the class $\mathcal{C}_q^{\lambda}(\phi)$. Then using the relation (2.38) and (2.39) in the relation (3.2) and (3.3), we get

$$e_2=-l_2=\frac{-c_1d_0\omega_1}{2},\quad(\lambda\geq 0) \tag{3.20}$$

$$e_3=\frac{-c_1}{6}\left[d_1\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)d_0-(\lambda+2)c_1d_0^2\omega_1^2\right]. \tag{3.21}$$

Thus, for any complex number γ, we have

$$e_3-\gamma e_2^2=\frac{-c_1}{6}\left[d_1\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)d_0-Qc_1d_0^2\omega_1^2\right], \tag{3.22}$$

where P is given by (3.19).
Since $\psi(\xi)$ is analytic and bounded in the open unit disk Δ, hence upon using ([7], p. 172), we have for some τ with $|\tau|\leq 1$:

$$|d_0|\leq 1\qquad and\qquad d_1=(1-d_0^2)\tau. \tag{3.23}$$

Putting the value of d_1 from (3.23) in the relation (3.22), we obtain

$$e_3-\gamma e_2^2=\frac{-c_1}{6}\left[\tau_1\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)d_0-(Qc_1\omega_1^2+\tau\omega_1)d_0^2\right]. \tag{3.24}$$

If $d_0=0$, then (3.11) gives

$$|e_3-\gamma e_2^2|\leq\frac{c_1}{6}. \tag{3.25}$$

Suppose $d_0\neq 0$. In such case, consider the function

$$V(d_0)=\tau\omega_1+\left(\omega_2+\frac{c_2}{c_1}\omega_1^2\right)d_0-(Qc_1\omega_1^2+\tau\omega_1)d_0^2. \tag{3.26}$$

The expression (3.26) is a quadratic polynomial in d_0 and hence analytic in $|d_0|\leq 1$. The maximum value of $V(d_0)$ occurs at $d_0=e^{i\theta}\quad(0\leq\theta<2\pi)$. Therefore,

$$\begin{aligned}Max|V(d_0)|&=Max_{0\leq\theta<2\pi}|V(e^{i\theta})|=V(1)\\&=\left|\omega_2-\left(Qc_1-\frac{c_2}{c_1}\right)\omega_1^2\right|.\end{aligned} \tag{3.27}$$

Taking modulus on the both sides of (3.24) and making use of (3.27) in (3.24) we get

$$|e_3 - \gamma e_2^2| \leq \frac{c_1}{6}\left|\omega_2 - \left(Qc_1 - \frac{c_2}{c_1}\right)\omega_1^2\right|. \tag{3.28}$$

An application of Lemma 1.1 we obtain

$$|e_3 - \gamma e_2^2| \leq \frac{c_1}{6} max\left\{1, \left|\frac{c_2}{c_1} - Qc_1\right|\right\}. \tag{3.29}$$

The assertion mentioned in (3.18) follows from (3.25) and (3.29).
The proof of Theorem 3.2 is complete. □

Concluding Remarks: In the present paper, the authors introduced two novel subclasses of Bazilevič type functions by means of quasi-subordination. Some initial coefficient bounds, Fekete-Szegö functional and the bounds of inverse coefficients are determined for the said classes. The researchers can make use of (p, q)-calculus to define more general classes and result of this paper generalized accordingly.

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References

1. P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

2. M. Fekete and G. Szegö, *Eine Bemerkung, Über ungrade schlichet funktionen*, *J. Math. Soc.*, **8**(1933), 85-89.

3. F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, *Proc. Amer. Math. Soc.*, **20**(1969), 8-12.

4. O. S. Kwon, D. K. Thomas and Y. J. Sim, *The fifth coefficient of Bazilevič* functions, *Symmetry*,2020, 12, 1228. Doi: 10.3390/sym 12081228.

5. W. C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, in Proceeding of the International Conference on Complex Analysis, Tianjin, 1992.

6. M. H. Mohd and M. Darus, *Fekete-Szegö problems for quasi-subordination classes*, *Abstr. Appl. Anal.*(2012). Art. ID: 192956. DOI: 10.1155/2012/192956.

7. Z. Nehari, *Conformal Mapping*, *Dover, New York*(1975)(repriniting of the 1952 edition).

8. T. Panigrahi and G. Murugusundarmoorthy, *The Fekete-Szegö inequality for subclass of analytic functions of complex order*, *Adv. Studies Contemp. Math.*, **24**(1)(2014), 67-75.

9. T. Panigrahi and R. K. Raina, *Fekete-Szegö problems for generalized Sakaguchi type functions associated with quasi-subordination*, *Stud. Univ. Babeş-bolyai Math.*, **63**(3)(2018), 329-340; DOI:10.24193/submath.2018.3.05.

10. T. Panigrahi and R. K. Raina, *Fekete-Szegö coefficient functional for quasi-subordination class*, *Afr. Math.*, DOI: 10.1007/s13370-016-0477-1.

11. V. Ravichandran, M. Darus, M. H. Khan and K. G. Subramanian, *Fekete-Szegö inequality for certain class of analytic functions*, *Austral. J. Math. Anal. Appl.*, **4**(1)(2004), 1-7.

12. S. G. A. Shah, S. Al-Sa'di, S. Hussain, A. Tasleem, A. Rasheed, I. Z. Cheema amd M. Darus, *Fekete-Szegö functional for a class of non-Bazilevic functions related to quasi-subordination*, *Demonstratio Math.*, **56**(2023), Art. ID: 20220232.

13. T. N. Shanmugham and S. Sivasubramanian, *On the Fekete-Szegö problem for some subclasses of analytic functions*, *J. Inequal. Pure Appl. Math.*, **6**(3)(2005), Art. ID: 71, 6 pages.

14. P. Sharma and R. K. Raina, *On a Sakaguchi type class of analytic function associated with quasi-subordination*, *Comment. Math, Univ. St. Paul.*, **64**(1)(2015), 59-70.

15. R. Singh, *On Bazilevic functions*, *Proc. Amer. Math. Soc.*, **38**(1973), 261-271.

16. N. Tuneski and M. Darus. *Fekete-Szegö functional for non-Bazilevic functions*, *Acta. Math. Acad. Paedagog. Nyhazi. (N. S.)*, **18**(2)(2002), 63-65.

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