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Unit group of group algebras of non abelian group of order up to 30

Diksha Upadhyay and Harish Chandra*

ABSTRACT: In this paper, we characterize the structure of unit group of semisimple group algebra F_qG, where G is non abelian group of order up to 30 and F_q is a field of order $q(= p^k)$, p is a prime number. In particular, we have characterized the structure of unit group of group algebra of 7 non abelian groups of order 16, $C_7 \rtimes C_3$ of order 21, $C_9 \rtimes C_3$ of order 27, $C_7 \rtimes C_4$ of order 28 and $C_5 \times S_3$ of order 30. Unit groups of semisimple group algebras for non abelian groups up to order 30 have now been thoroughly studied.

Key Words: Group rings, unit group, conjugacy classes, dihedral groups.

Contents

1. Introduction

Let G be a finite group, F_q be a finite field with $q = p^k$ elements for an odd prime p, and F_qG is the group algebra of group G over the finite field F_q. Determination of unit group of group algebra has never been easy. Many efforts have been made to determine the algebraic structure of unit group $U(F_qG)$ of group algebra F_qG. Let D_n be the dihedral group of order $2n$, C_n be the cyclic group of order n, and S_n be the symmetric group of degree n. For a finite field F, the structure of $U(FS_3)$ has been discussed by Sahai and Ansari in [19]. Sahai and Ansari [10], have discussed the structure of $U(FD_4)$, $U(FD_8)$, $U(FD_{10})$, $U(FD_{16})$, and $U(FD_{20})$ for a finite field F of characteristic $p > 0$. Unit group structure of generalized Quaternion group has been discussed in [14]. M. Khan [6], has been obtained the structure of unit group of FD_5 over a finite field F. In [4], there is a complete characterization of structure of unit group of group algebra of group of order 12. The structure of $U(FD_7)$, $U(FD_{12})$, $U(FD_{14})$ and $U(FD_{24})$ has been discussed in [11]. In [5], Gildea has characterized the structure of the unit group of the group algebra of the non abelian group of order 16 with exponent 4 for a field of characteristic $p = 2$. Complete characterization of unit groups of group algebras of groups of order 18, 20 and 24 has been discussed in [13,1,17]. Sahai and Ansari [12], have obtained the structure of the unit groups of the semisimple group algebras $U(FD_{11})$, $U(FD_{13})$, $U(FD_{17})$, $U(FD_{19})$ and $U(FD_{23})$. Further, a note on the structure of $U(F(C_3^2 \rtimes C_3))$, $U(F(C_3 \times D_5))$ and $U(FD_{15})$ is given in [18,16,8]. In this paper, we give the complete characterization of structure of unit group of semisimple group algebra of remaining non abelian group of order up to 30.

* Corresponding author.

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2. Preliminaries

Let F be any arbitrary finite field, ζ be a primitive eth root of unity, and e represent the exponent of G. Then T be the multiplicative group consisting of those element t, taken modulo e, for which $\zeta \mapsto \zeta^t$ defines an automorphism of $F(\zeta)$ over F i.e.
$T = \{t : \zeta \mapsto \zeta^t$ is an automorphism of $F(\zeta)$ over $F\}$.

For any p-regular element $g \in G$, we can denote γ_g as the sum of all of its conjugates and let cyclotomic F-classes of γ_g is denoted by

$$S(\gamma_g) = \{\gamma_{g^t} : t \in T\}.$$

Let us now recall the two subsequent findings related to the cyclotomic F-classes.

Theorem 2.1 *[3] The number of simple components of $FG/J(FG)$ and the number of cyclotomic F-classes in G are equal.*

Theorem 2.2 *[3] Let j be the number of cyclotomic F-classes in G. If K_i, $1 \leq i \leq j$, are the simple components of center of $FG/J(FG)$ and S_i, $1 \leq i \leq j$, are the cyclotomic F-classes in G, then $|S_i| = [K_i : F]$ for each i after suitable ordering of the indices.*

Theorem 2.3 *[7] Let F be a finite field with prime power order q. If e is such that $gcd(e, q) = 1$, ζ is the primitive eth root of unity and z is the order of q modulo e, then we have $T = \{1, q, q^2, ..., q^{z-1}\}$ mod e.*

Theorem 2.4 *[9] If RG is a semisimple group algebra, then*

$$RG = R(G/G') \oplus \Delta(G, G'),$$

where G' is the commutator subgroup of G, $R(G/G')$ is the sum of all commutative simple components of RG, and $\Delta(G, G')$ is the sum of all others.

Theorem 2.5 *[2] If $R = \bigoplus_{t=1}^{j} M_{n_t}(F_{q_t})$ is a summand of a semisimple group ring $F_qG(q = p^k)$, then p does not divide any of the n_t.*

3. Unit Group of Group Algebras of Non Abelian Groups of Order 16

Characterizing the structure of unit groups of F_qG, where G is a non-abelian group of order 16, is the primary goal of this section. Up to isomorphism, there are 9 non-abelian groups of order 16, namely $G_1 = D_8$, $G_2 = Q_{16}$, $G_3 = SD_{16}$, $G_4 = M_4(2)$, $G_5 = C_4 \circ D_4$, $G_6 = C_2^2 \rtimes C_4$, $G_7 = C_4 \rtimes C_4$, $G_8 = C_2 \times D_4$ and $G_9 = C_2 \times Q_8$. Here D_n is dihedral group of order $2n$, SD_{16} is the semi dihedral group of order 16, $M_4(2)$ is modular maximal-cyclic group and $C_4 \circ D_4$ is the central product of C_4 and D_4. The structure of unit group of group algebra of G_1 and G_2 has already been discussed. Now we discuss the unit groups of group algebra of remaining groups.

The group $G_3 = SD_{16}$ has the following presentation:
$G_3 = < x, y, z, w | x^2w^{-1}, y^{-1}x^{-1}yxz^{-1}, z^{-1}x^{-1}zxw^{-1}, w^{-1}x^{-1}wx, y^2, z^{-1}y^{-1}zyw^{-1}, w^{-1}y^{-1}wy, z^2w^{-1}, w^{-1}z^{-1}wz, w^2 >$.
There are 7 conjugacy classes of G_3 which are shown in the following Table:
Clearly from Table 1, it can be observed that the exponent of G_3 is 8. Also $G_3^{'} = C_4$. Next we give the Wedderburn decomposition for $p > 2$.

Theorem 3.1 *The Wedderburn decomposition of F_qG_3 for $p > 2$, where $q = p^k$ is given by*

$$F_qSD_{16} \cong \begin{cases} F_q^4 \oplus M_2(F_q)^3 & \text{for} \quad q \equiv \{1, 3\} \mod 8 \\ F_q^4 \oplus M_2(F_q) \oplus M_2(F_{q^2}) & \text{for} \quad q \equiv \{5, 7\} \mod 8. \end{cases}$$

Table 1:

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xz, xw, xzw\}$	4
y	$\{y, yz, yw, yzw\}$	2
z	$\{z, zw\}$	4
w	$\{w\}$	2
xy	$\{xy, xyz\}$	8
xyw	$\{xyw, xyzw\}$	8

Proof: Since for $p > 2$, F_qG_3 is semisimple. So it's Wedderburn Decomposition is provided by

$$F_qG_3 \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t), \tag{3.1}$$

and F_t is a finite extension of F_q. Now, first we discuss the case when k is even. If k is even, then $p^k \equiv 1$ mod 8. Here $|S(\gamma_g)| = 1$ for all $g \in G_3$. Theorems 2.1, 2.2 imply that

$$F_qG_3 \cong F_q \bigoplus_{t=1}^{6} M_{n_t}(F_t). \tag{3.2}$$

Now by using dimension formula, we have

$$15 = \sum_{t=1}^{6} n_t^2, n_t \geq 1, \forall\, t. \tag{3.3}$$

Here only one possibility of n_t's for the above equation is (1,1,1,2,2,2). Therefore the Equation (3.2) becomes

$$F_qG_3 \cong F_q^4 \oplus M_2(F_q)^3. \tag{3.4}$$

We now consider the case when k is odd. For this $p^k \equiv \{1, 3, 5, 7\}$ mod 8. For $p^k \equiv \{1, 3\}$ mod 8, we have $|S(\gamma_g)| = 1$ for all $g \in G_3$. The Wedderburn decomposition in this case is given by Equation (3.4). Next, for $p^k \equiv \{5, 7\}$ mod 8, we have $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyw}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for all the remaining representative of the conjugacy classes. As it is clearly known that $G_3/G_3' = C_2^2$. Now using Theorem 2.5 and the Wedderburn structure of FC_2^2 from [15], Equation (3.2) becomes

$$F_qG_3 \cong F_q^4 \oplus M_2(F_q) \oplus M_2(F_{q^2}). \tag{3.5}$$

Hence, the unit group structure is given by:

$$U(F_qSD_{16}) \cong \begin{cases} C_{p^k-1}^4 \oplus GL_2(F_q)^3 & \text{for} \quad q \equiv \{1, 3\} \mod 8 \\ C_{p^k-1}^4 \oplus GL_2(F_q) \oplus GL_2(F_{q^2}) & \text{for} \quad q \equiv \{5, 7\} \mod 8. \end{cases}$$

□

The group $G_4 = M_4(2)$ has the following presentation:

$G_4 =< x, y, z, w | x^2z^{-1}, y^{-1}x^{-1}yxw^{-1}, z^{-1}x^{-1}zx, w^{-1}x^{-1}wx, y^2, z^{-1}y^{-1}zy,$
$w^{-1}y^{-1}wy, z^2w^{-1}, w^{-1}z^{-1}wz, w^2 >.$

There are 10 conjugacy classes of G_4 which is shown in Table 2.
Now from Table 2, it can be observed that the exponent of G_4 is 8. Also $G_4' = C_2$. Next we give the Wedderburn decomposition for $p > 2$.

Table 2:

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw\}$	8
y	$\{y, yw\}$	2
z	$\{z\}$	4
w	$\{w\}$	2
xy	$\{xy, xyw\}$	8
xz	$\{xz, xzw\}$	8
yz	$\{yz, yzw\}$	4
zw	$\{zw\}$	4
xyz	$\{xyz, xyzw\}$	8

Theorem 3.2 *The Wedderburn decomposition of F_qG_4 for $p > 2$, where $q = p^k$ is given by*
$$F_qM_4(2) \cong \begin{cases} F_q^8 \oplus M_2(F_q)^2 & \textit{for} \quad q \equiv \{1,5\} \mod 8 \\ F_q^4 \oplus F_{q^2}^2 \oplus M_2(F_{q^2}) & \textit{for} \quad q \equiv \{3,7\} \mod 8. \end{cases}$$

Proof: We can see that for $p > 2$, F_qG_4 is semisimple. So it's Wedderburn decomposition is given by

$$F_qG_4 \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{3.6}$$

Let k is even, then $p^k \equiv 1 \mod 8$ and $|S(\gamma_g)| = 1$ for all $g \in G_4$. Using Theorem 2.1 and Theorem 2.2, we get

$$F_qG_4 \cong F_q \bigoplus_{t=1}^{9} M_{n_t}(F_t) \tag{3.7}$$

and

$$15 = \sum_{t=1}^{9} n_t^2, n_t \geq 1, \forall\, t.$$

So possible values of n_t's for the above equation is (1,1,1,1,1,1,1,2,2) and hence Equation (3.7) becomes

$$F_qG_4 \cong F_q^8 \oplus M_2(F_q)^2. \tag{3.8}$$

If k is odd, then $p^k \equiv \{1,3,5,7\} \mod 8$. So if $p^k \equiv \{1,5\} \mod 8$, then $T = \{1,5\}$ and $|S(\gamma_g)| = 1$ for all $g \in G_4$. Therefore, Wedderburn decomposition in this case is given by Equation (3.8). If $p^k \equiv \{3,7\} \mod 8$, then $T = \{1,3,7\}$ and $S(\gamma_x) = \{\gamma_x, \gamma_{xz}\}$, $S(\gamma_z) = \{\gamma_z, \gamma_{zw}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyz}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for all the remaining representative of the conjugacy classes. After using Theorem 2.5 the Wedderburn decomposition is given by

$$F_qG_4 \cong F_q \bigoplus_{t=1}^{3} M_{n_t}(F_t) \bigoplus_{t=4}^{6} M_{n_t}(F_{t^2}) \tag{3.9}$$

and

$$15 = \sum_{t=1}^{3} n_t^2 \oplus 2\sum_{t=4}^{6} n_t^2, n_t \geq 1, \forall\, t,$$

which further implies that the possible choice of n_t's is (1,1,1,1,1,1,1,2,2). Since $G_4/G_4^{'} = C_2 \times C_4$, hence using the Wedderburn structure of $F(C_2 \times C_4)$ from [15], Equation (3.9) becomes

$$F_qG_4 \cong F_q^4 \oplus F_{q^2}^2 \oplus M_2(F_{q^2}). \tag{3.10}$$

Hence, the unit group structure is given by:
$$U(F_qM_4(2)) \cong \begin{cases} C^8_{p^k-1} \oplus GL_2(F_q)^2 & \text{for} \quad q \equiv \{1,5\} \mod 8 \\ C^4_{p^k-1} \oplus C^2_{p^{2k}-1} \oplus GL_2(F_{q^2}) & \text{for} \quad q \equiv \{3,7\} \mod 8. \end{cases}$$
□

Presentation for the group $G_5 = C_4 \circ D_4$ is as follows:

$G_5 =< x, y, z, w | x^2, y^{-1}x^{-1}yxw^{-1}, z^{-1}x^{-1}zx, w^{-1}x^{-1}wx, y^2, z^{-1}y^{-1}zy,$
$w^{-1}y^{-1}wy, z^2w^{-1}, w^{-1}z^{-1}wz, w^2 >$.

Number of conjugacy classes of G_5 are 10, which are shown in Table 3:

Table 3:

Representative	**Elements in the class**	**Order of element**
1	$\{1\}$	1
x	$\{x, xw\}$	2
y	$\{y, yw\}$	2
z	$\{z\}$	4
w	$\{w\}$	2
xy	$\{xy, xyw\}$	4
xz	$\{xz, xzw\}$	4
yz	$\{yz, yzw\}$	4
zw	$\{zw\}$	4
xyz	$\{xyz, xyzw\}$	2

It is evident from Table 3 that the exponent of G_5 is 4, $G_5^{'} = C_2$ and $G_5/G_5^{'} = C_2^3$. Next we give the Wedderburn decomposition for $p > 2$.

Theorem 3.3 *The Wedderburn decomposition of F_qG_5 for $p > 2$, where $q = p^k$ is given by*
$$F_q(C_4 \circ D_4) \cong \begin{cases} F_q^8 \oplus M_2(F_q)^2 & \text{for} \quad q \equiv 1 \mod 4 \\ F_q^8 \oplus M_2(F_{q^2}) & \text{for} \quad q \equiv 3 \mod 4. \end{cases}$$

Proof: Since for $p > 2$, F_qG_5 is semisimple. Therefore, Wedderburn decomposition is given by

$$F_qG_5 \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{3.11}$$

When k is even, $p^k \equiv 1 \mod 4$ and $|S(\gamma_g)| = 1$ for all $g \in G_5$. Theorems 2.1, 2.2 imply that

$$F_qG_5 \cong F_q \bigoplus_{t=1}^{9} M_{n_t}(F_t), \tag{3.12}$$

and

$$15 = \sum_{t=1}^{9} n_t^2, n_t \geq 1, \forall\, t.$$

Possibility of n_t's for the above equation is (1,1,1,1,1,1,1,2,2). Therefore, the above equation implies that

$$F_qG_5 \cong F_q^8 \oplus M_2(F_q)^2. \tag{3.13}$$

If k is odd, then $p^k \equiv \{1,3\} \mod 4$. Now for $p^k \equiv 1 \mod 4$, we have $|S(\gamma_g)| = 1$ for all $g \in G_5$. Therefore Wedderburn decomposition is given by Equation (3.13). Next, for $p^k \equiv 3 \mod 4$, we have

$T = \{1,3\}$ and $S(\gamma_z) = \{\gamma_z, \gamma_{zw}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for all the remaining representative of the conjugacy classes. Now using Theorem 2.5, we get that

$$F_qG_4 \cong F_q \bigoplus_{t=1}^{7} M_{n_t}(F_t) \oplus M_{n_t}(F_{t^2}), \tag{3.14}$$

and

$$15 = \sum_{t=1}^{7} n_t^2 \oplus 2n_t^2, n_t \geq 1, \forall\, t,$$

which shows that the possible choice of n_t's is (1,1,1,1,1,1,1,2,2). As it is already known that $G_5/G_5^{'} = C_2^3$, therefore using the Wedderburn structure of FC_2^3 from [15], Equation (3.14) becomes

$$F_qG_5 \cong F_q^8 \oplus M_2(F_{q^2}). \tag{3.15}$$

Now, the structure of unit group is given by:

$$U(F_q(C_4 \circ D_4)) \cong \begin{cases} C_{p^k-1}^8 \oplus GL_2(F_q)^2 & \text{for} \quad q \equiv 1 \mod 4 \\ C_{p^k-1}^8 \oplus GL_2(F_{q^2}) & \text{for} \quad q \equiv 3 \mod 4. \end{cases}$$ □

The group $G_6 = C_2^2 \rtimes C_4$ has the following presentation:
$G_6 =< x, y, z, w | x^2w^{-1}, y^{-1}x^{-1}yxz^{-1}, z^{-1}x^{-1}zx, w^{-1}x^{-1}wx, y^2, z^{-1}y^{-1}zy,$
$w^{-1}y^{-1}wy, z^2, w^{-1}z^{-1}wz, w^2 >$.
Here list of conjugacy classes of G_6 are shown in the Table 4. From Table 4, it can be observed that the

Table 4:

Representative	**Elements in the class**	**Order of element**
1	$\{1\}$	1
x	$\{x, xz\}$	4
y	$\{y, yz\}$	2
z	$\{z\}$	2
w	$\{w\}$	2
xy	$\{xy, xyz\}$	4
xw	$\{xw, xzw\}$	4
yw	$\{yw, yzw\}$	2
zw	$\{zw\}$	2
xyw	$\{xyw, xyzw\}$	4

exponent of G_6 is 4. Also $G_6^{'} = C_2$ and $G_6/G_6^{'} = C_2 \times C_4$. Next, we give the Wedderburn decomposition for $p > 2$.

Theorem 3.4 *The Wedderburn decomposition of F_qG_6 for $p > 2$, where $q = p^k$ is given by*
$$F_q(C_2^2 \rtimes C_4) \cong \begin{cases} F_q^8 \oplus M_2(F_q)^2 & \text{for} \quad q \equiv 1 \mod 4 \\ F_q^4 \oplus F_{q^2}^2 \oplus M_2(F_q)^2 & \text{for} \quad q \equiv 3 \mod 4. \end{cases}$$

Proof: As for $p > 2$, F_qG_6 is semisimple. Therefore, it's Wedderburn Decomposition is provided by

$$F_qG_6 \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{3.16}$$

Now if k is even, then $p^k \equiv 1 \mod 4$. Here $|S(\gamma_g)| = 1$ for all $g \in G_6$. Theorems 2.1, 2.2 imply that

$$F_qG_6 \cong F_q \bigoplus_{t=1}^{9} M_{n_t}(F_t) \tag{3.17}$$

and

$$15=\sum_{t=1}^{9}n_t^2, n_t\geq 1, \forall\, t.$$

Here possible value of n_t's is only (1,1,1,1,1,1,1,2,2). Therefore, the above equation implies that

$$F_qG_6\cong F_q^8\oplus M_2(F_q)^2. \tag{3.18}$$

For k is odd, $p^k\equiv\{1,3\} \mod 4$. Now for $p^k\equiv\{1\} \mod 4$, we have $T=\{1,3\}$ and $|S(\gamma_g)|=1$ for all the representative g of conjugacy classes of G_6. Therefore Wedderburn decomposition in this case is given by Equation (3.18). Next, for $p^k\equiv 3 \mod 4$, we have $S(\gamma_x)=\{\gamma_x,\gamma_{xw}\}$, $S(\gamma_{xy})=\{\gamma_{xy},\gamma_{xyw}\}$ and $S(\gamma_g)=\{\gamma_g\}$ for all the remaining representative of the conjugacy classes. Now using Theorem 2.5 the Wedderburn decomposition is given by

$$F_qG_6\cong F_q\bigoplus_{t=1}^{5}M_{n_t}(F_t)\bigoplus_{t=6}^{7}M_{n_t}(F_{t^2}) \tag{3.19}$$

and

$$15=\sum_{t=1}^{6}n_t^2\oplus 2\sum_{t=7}^{8}n_t^2, n_t\geq 1, \forall\, t,$$

which further implies that the possible choices of n_t's is (1,1,1,1,1,1,1,2,2). Now using the Wedderburn structure of $F(C_2\times C_4)$ [15], Equation (3.19) becomes

$$F_qG_5\cong F_q^4\oplus F_{q^2}^2\oplus M_2(F_q)^2. \tag{3.20}$$

Hence, the unit group structure is given by:

$$U(F_q(C_2^2\rtimes C_4))\cong\begin{cases} C_{p^k-1}^8\oplus GL_2(F_q)^2 & \text{for} \quad q\equiv 1 \mod 4\\ C_{p^k-1}^4\oplus C_{p^{2k}-1}^2\oplus GL_2(F_q)^2 & \text{for} \quad q\equiv 3 \mod 4. \end{cases}$$ □

The group $G_7=C_4\rtimes C_4$ has the following presentation:

$G_7=< x,y,z,w|x^2w^{-1},y^{-1}x^{-1}yxz^{-1},z^{-1}x^{-1}zx,w^{-1}x^{-1}wx,y^2z^{-1},z^{-1}y^{-1}zy,$
$w^{-1}y^{-1}wy,z^2,w^{-1}z^{-1}wz,w^2>$.

There are 10 conjugacy classes of G_7 which is shown in the following Table:

Table 5:

Representative	**Elements in the class**	**Order of element**
1	$\{1\}$	1
x	$\{x,xz\}$	4
y	$\{y,yz\}$	4
z	$\{z\}$	2
w	$\{w\}$	2
xy	$\{xy,xyz\}$	4
xw	$\{xw,xzw\}$	4
yw	$\{yw,yzw\}$	4
zw	$\{zw\}$	2
xyw	$\{xyw,xyzw\}$	4

Here we can see that the exponent of G_7 is 4, $G_7^{'}=C_2$ and $G_7/G_7^{'}=C_2\times C_4$. Next we give the Wedderburn decomposition for $p>2$.

Theorem 3.5 *The Wedderburn decomposition of F_qG_7 for $p>2$, where $q=p^k$ is given by*
$F_q(C_4 \rtimes C_4) \cong \begin{cases} F_q^8 \oplus M_2(F_q)^2 & \text{for } q \equiv 1 \mod 4 \\ F_q^4 \oplus F_{q^2}^2 \oplus M_2(F_q)^2 & \text{for } q \equiv 3 \mod 4. \end{cases}$

Proof: F_qG_7 is semisimple when $p>2$. So it's Wedderburn Decomposition is given by

$$F_qG_7 \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{3.21}$$

Now, if k is even, then $p^k \equiv 1 \mod 4$ and $|S(\gamma_g)| = 1$ for all $g \in G_7$. After using Theorem 2.1, 2.2, we have

$$F_qG_7 \cong F_q \bigoplus_{t=1}^{9} M_{n_t}(F_t) \tag{3.22}$$

and

$$15 = \sum_{t=1}^{9} n_t^2, n_t \geq 1, \forall\, t.$$

Here we get the unique possibility of n_t's which is (1,1,1,1,1,1,1,2,2). Therefore, the above equation implies that

$$F_qG_7 \cong F_q^8 \oplus M_2(F_q)^2. \tag{3.23}$$

If k is odd, then $p^k \equiv \{1,3\} \mod 4$. Now for $p^k \equiv \{1\} \mod 4$, we have $|S(\gamma_g)| = 1$ for all $g \in G_7$. Therefore Wedderburn decomposition in this case is given by Equation (3.23). Next, for $p^k \equiv 3 \mod 4$, we have $S(\gamma_x) = \{\gamma_x, \gamma_{xw}\}$, $S(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{xyw}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for all the remaining representative of the conjugacy classes. Now using Theorem 2.5, the Wedderburn decomposition is given by

$$F_qG_7 \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_t) \bigoplus_{t=6}^{7} M_{n_t}(F_{t^2}) \tag{3.24}$$

and

$$15 = \sum_{t=1}^{6} n_t^2 \oplus 2\sum_{t=7}^{8} n_t^2, n_t \geq 1, \forall\, t,$$

which further implies that the possible choices of n_t's is (1,1,1,1,1,1,1,2,2). Hence after using the Wedderburn structure of $F(C_2 \times C_4)$ [15], Equation (3.24) becomes

$$F_qG_7 \cong F_q^4 \oplus F_{q^2}^2 \oplus M_2(F_q)^2. \tag{3.25}$$

The unit group structure is give by:

$$U(F_q(C_4 \rtimes C_4)) \cong \begin{cases} C_{p^k-1}^8 \oplus GL_2(F_q)^2 & \text{for } q \equiv 1 \mod 4 \\ C_{p^k-1}^4 \oplus C_{p^{2k}-1}^2 \oplus GL_2(F_q)^2 & \text{for } q \equiv 3 \mod 4. \end{cases} \qquad \square$$

The group $G_8 = C_2 \times D_4$ has the following presentation:
$G_8 =< x, y, z, w | x^2, y^{-1}x^{-1}yxw^{-1}, z^{-1}x^{-1}zx, w^{-1}x^{-1}wx, y^2, z^{-1}y^{-1}zy, w^{-1}y^{-1}wy, z^2, w^{-1}z^{-1}wz, w^2 >$.

There are 10 conjugacy classes of G_8, which are shown in Table 6.
Clearly from Table 6, it is obvious that the exponent of G_8 is 4. Also $G_8^{'} = C_2$ and $G_8/G_8^{'} = C_2^3$. Next we give the Wedderburn decomposition for $p>2$.

Theorem 3.6 *The Wedderburn decomposition of F_qG_8 for $p>2$, where $q=p^k$ is given by*
$F_q(C_2 \times D_4) \cong \{\, F_q^8 \oplus M_2(F_q)^2 \quad \text{for} \quad q \equiv 1,3 \mod 4.$

Table 6:

Representative	**Elements in the class**	**Order of element**
1	$\{1\}$	1
x	$\{x, xw\}$	2
y	$\{y, yw\}$	2
z	$\{z\}$	2
w	$\{w\}$	2
xy	$\{xy, xyw\}$	4
xz	$\{xz, xzw\}$	2
yz	$\{yz, yzw\}$	2
zw	$\{zw\}$	2
xyz	$\{xyz, xyzw\}$	4

Proof: Since for $p > 2$, F_qG_8 is semisimple. So it's Wedderburn Decomposition is provided by

$$F_qG_8 \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{3.26}$$

Now, if k is even, then $p^k \equiv 1 \mod 4$. Here $|S(\gamma_g)| = 1$ for all $g \in G_8$. Theorems 2.1, 2.2 imply that

$$F_qG_8 \cong F_q \bigoplus_{t=1}^{9} M_{n_t}(F_t) \tag{3.27}$$

and

$$15 = \sum_{t=1}^{9} n_t^2, n_t \geq 1, \forall\, t.$$

Above equation has only one possibility (1,1,1,1,1,1,1,2,2). Therefore, Equation (3.27) becomes

$$F_qG_8 \cong F_q^8 \oplus M_2(F_q)^2. \tag{3.28}$$

Now if k is odd, then $p^k \equiv \{1, 3\} \mod 4$ and for both cases, we have $|S(\gamma_g)| = 1$ for all the representative g of conjugacy classes of G_7. Therefore, Wedderburn decomposition in this case is given by Equation (3.28).

The unit group structure is given by:

$$U(F_q(C_2 \times D_4)) \cong \{\ C_{p^k-1}^8 \oplus GL_2(F_q)^2 \quad \text{for} \quad q \equiv 1, 3 \mod 4\ .$$

□

The group $G_9 = C_2 \times Q_8$ has the following presentation:

$G_9 =< x, y, z, w | x^2w^{-1}, y^{-1}x^{-1}yxw^{-1}, z^{-1}x^{-1}zx, w^{-1}x^{-1}wx, y^2w^{-1}, z^{-1}y^{-1}zy,$
$w^{-1}y^{-1}wy, z^2, w^{-1}z^{-1}wz, w^2 >.$

In Table 7 we have shown the conjugacy classes of G_9.
Now from Table 7, we can see that the exponent of G_9 is 4. Also $G_9' = C_2$ and $G_9/G_9' = C_2^3$. Next we give the Wedderburn decomposition for $p > 2$.

Theorem 3.7 *The Wedderburn decomposition of F_qG_9 for $p > 2$, where $q = p^k$ is given by* $F_q(C_2 \times Q_8) \cong \{\ F_q^8 \oplus M_2(F_q)^2$ *for* $q \equiv 1, 3 \mod 4$.

Table 7:

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xw\}$	4
y	$\{y, yw\}$	4
z	$\{z\}$	2
w	$\{w\}$	2
xy	$\{xy, xyw\}$	4
xz	$\{xz, xzw\}$	4
yz	$\{yz, yzw\}$	4
zw	$\{zw\}$	2
xyz	$\{xyz, xyzw\}$	4

Proof: Since for $p > 2$, F_qG_9 is semisimple. So it's Wedderburn Decomposition is provided by

$$F_qG_9 \cong F_q\bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{3.29}$$

If k is even, then $p^k \equiv 1 \mod 4$ and $|S(\gamma_g)| = 1$ for all $g \in G_8$. Theorems 2.1, 2.2 imply that

$$F_qG_9 \cong F_q\bigoplus_{t=1}^{9} M_{n_t}(F_t) \tag{3.30}$$

and

$$15 = \sum_{t=1}^{9} n_t^2, n_t \geq 1, \forall\, t,$$

from above we have only one possibility (1,1,1,1,1,1,1,2,2). Therefore, the above equation implies that

$$F_qG_9 \cong F_q^8 \oplus M_2(F_q)^2. \tag{3.31}$$

If k is odd, then $p^k \equiv \{1, 3\} \mod 4$. Now for $p^k \equiv \{1\} \mod 4$, we have $|S(\gamma_g)| = 1$ for all $g \in G_9$. Therefore, Wedderburn decomposition in this case is given by Equation (3.31). Next, for $p^k \equiv 3 \mod 4$, we have $S(\gamma_g) = \{\gamma_g\}$ for all the representative of the conjugacy classes. Now using Theorem 2.5 the Wedderburn decomposition is given by Equation (3.31).
Now, the unit group structure is given by:

$$U(F(C_2 \times Q_8)) \cong \{\ C_{p^k-1}^8 \oplus GL_2(F_q)^2 \quad \text{for} \quad q \equiv 1, 3 \mod 4.$$

□

4. Unit Group of Group Algebras of Non Abelian Groups of Order 21

The only non abelian group of order 21 is $C_7 \rtimes C_3$. In this section we give the structure of unit group of $F(C_7 \rtimes C_3)$.

The group $C_7 \rtimes C_3$ has the following presentation:

$$C_7 \rtimes C_3 =< x, y | x^3, y^{-1}x^{-1}yxy^{-1}, y^7 >.$$

There are 5 conjugacy classes of $C_7 \rtimes C_3$ which is shown in Table 8.
From Table 8, one can observe that the exponent of $C_7 \rtimes C_3$ is 21. Also $(C_7 \rtimes C_3)' = C_7$ and $(C_7 \rtimes C_3)/(C_7 \rtimes C_3)' = C_3$. Next we give the Wedderburn decomposition for $p \neq 3, 7$.

Table 8:

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xy^6, xy^5, xy^4, xy^3, xy^2, xy\}$	3
y	$\{y, y^2, y^4\}$	7
x^2	$\{x^2, x^2y^4, x^2y, x^2y^5, x^2y^2, x^2y^6, x^2y^3\}$	3
y^3	$\{y^3, y^5, y^6\}$	7

Theorem 4.1 *The Wedderburn decomposition of $F_q(C_7 \rtimes C_3)$ for $p \neq 3, 7$, where $q = p^k$ is given by*

$$F_q(C_7 \rtimes C_3) \cong \begin{cases} F_q^3 \oplus M_3(F_q)^2 & \text{for} \quad q \equiv 1, 4, 16 \mod 21 \\ F_q^3 \oplus M_3(F_{q^2})^2 & \text{for} \quad q \equiv 10, 19, 13 \mod 21 \\ F_q \oplus F_{q^2} \oplus M_3(F_q)^2 & \text{for} \quad q \equiv 2, 8, 11 \mod 21 \\ F_q \oplus F_{q^2} \oplus M_3(F_{q^2}) & \text{for} \quad q \equiv 17, 5, 20 \mod 21. \end{cases}$$

Proof: Since $F_q(C_7 \rtimes C_3)$ for $p \neq 3, 7$ is semisimple, we have

$$F_q(C_7 \rtimes C_3) \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{4.1}$$

If k is even, then $p^k \equiv 1, 4 \mod 21$. Here $|S(\gamma_g)| = 1$ for all representative $g \in F_q(C_7 \rtimes C_3)$. Further, Theorems 2.1, 2.2 imply that

$$F_q(C_7 \rtimes C_3) \cong F_q \bigoplus_{t=1}^{4} M_{n_t}(F_t) \tag{4.2}$$

and

$$20 = \sum_{t=1}^{4} n_t^2, n_t \geq 1, \forall\, t.$$

Here we get only one possibility of n_t for the above equation (1,1,3,3). Therefore, the above equation implies that

$$F_q(C_7 \rtimes C_3) \cong F_q^3 \oplus M_3(F_q)^2. \tag{4.3}$$

Now for the case when k is odd, we have $p^k \equiv \{2, 5, 8, 10, 11, 13, 16, 17, 19, 20\} \mod 21$. If $p^k \equiv 16 \mod 21$, then $|S(\gamma_g)| = 1$ for all the representative g of conjugacy classes of $(C_9 \rtimes C_3)$. Therefore, Wedderburn decomposition in this case is given by Equation (4.3). If, $p^k \equiv \{10, 19, 13\} \mod 21$, we have $T = \{1, 10, 16, 13, 4, 19\}$ and $S(\gamma_{g_y}) = \{\gamma_{g_y}, \gamma_{g_{y^2}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Hence after using the Wedderburn structure of FC_3 [15], the Wedderburn decomposition is given by

$$F_q(C_7 \rtimes C_3) \cong F_q^3 \oplus M_3(F_{q^2}). \tag{4.4}$$

Now for the case, $p^k \equiv \{2, 8, 11\} \mod 21$, we have $T = \{1, 2, 4, 8, 11, 16\}$, $S(\gamma_{g_x}) = \{\gamma_{g_x}, \gamma_{g_{x^2}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Now using the Wedderburn structure of FC_3 [15] and Theorems 2.1, 2.2, the Wedderburn decomposition is given by

$$F_q(C_7 \rtimes C_3) \cong F_q \oplus F_{q^2} \oplus M_3(F_q)^2. \tag{4.5}$$

Next, for $p^k \equiv \{17, 5, 20\} \mod 21$, we have $T = \{1, 4, 5, 16, 17, 20\}$ and $S(\gamma_{g_x}) = \{\gamma_{g_x}, \gamma_{g_{x^2}}\}$, $S(\gamma_{g_y}) = \{\gamma_{g_y}, \gamma_{g_{y^2}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Now using the Wedderburn structure of FC_3 [15] and Theorems 2.1, 2.2, the Wedderburn decomposition is given by

$$F_q(C_7 \rtimes C_3) \cong F_q \oplus F_{q^2} \oplus M_3(F_{q^2}). \tag{4.6}$$

The unit group structure is given by:

$$U(F_q(C_7 \rtimes C_3)) \cong \begin{cases} C^3_{p^{k-1}} \oplus GL_3(F_q)^2 & \text{for} \quad q \equiv 1,4,16 \mod 21 \\ C^3_{p^{k-1}} \oplus GL_3(F_{q^2})^2 & \text{for} \quad q \equiv 10,19,13 \mod 21 \\ C_{p^{k-1}} \oplus C_{p^{2k-1}} \oplus Gl_3(F_q)^2 & \text{for} \quad q \equiv 2,8,11 \mod 21 \\ C_{p^{k-1}} \oplus C_{p^{2k-1}} \oplus GL_3(F_{q^2}) & \text{for} \quad q \equiv 17,5,20 \mod 21. \end{cases}$$

□

5. Unit Group of Group Algebras of Non Abelian Groups of Order 27

Up to isomorphism, there are only two non abelian group of order 27, $C_3^2 \rtimes C_3$ and $C_9 \rtimes C_3$. The structure of unit group of $F(C_3^2 \rtimes C_3)$ has already been discussed (see [18]). In this section we give the structure of unit group of $F(C_9 \rtimes C_3)$.

The group $C_9 \rtimes C_3$ has the following presentation:

$$C_9 \rtimes C_3 =< x, y, z | x^3 z^{-1}, y^{-1}x^{-1}yxz^{-1}, z^{-1}x^{-1}zx, z^{-1}y^{-1}zy, z^3 >.$$

There are 11 conjugacy classes of $C_9 \rtimes C_3$ which is shown in Table 9.

Table 9:

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xz, xz^2\}$	9
y	$\{y, yz, yz^2\}$	3
z	$\{z\}$	3
x^2	$\{x^2, x^2z, x^2z^2\}$	9
xy	$\{xy, xyz, xyz^2\}$	9
y^2	$\{y^2, y^2z, y^2z^2\}$	3
z^2	$\{z^2\}$	3
x^2y	$\{x^2y, x^2yz, x^2yz^2\}$	9
xy^2	$\{xy^2, xy^2z, xy^2z^2\}$	9
x^2y^2	$\{x^2y^2, x^2y^2z, x^2y^2z^2\}$	9

Clearly from Table 9, it can be observed that the exponent of $C_9 \rtimes C_3$ is 9. Also $(C_9 \rtimes C_3)' = C_3$ and $C_9 \rtimes C_3/(C_9 \rtimes C_3)' = C_3^2$. Next we give the Wedderburn decomposition for $p \neq 3$.

Theorem 5.1 *The Wedderburn decomposition of $F_q(C_9 \rtimes C_3)$ for $p \neq 3$, where $q = p^k$ is given by*

$$F_q(C_9 \rtimes C_3) \cong \begin{cases} F_q^9 \oplus M_3(F_q) & \text{for} \quad q \equiv 1,4,7 \mod 9 \\ F_q \oplus F_{q^2}^4 \oplus M_3(F_{q^2}) & \text{for} \quad q \equiv 2,5,8 \mod 9. \end{cases}$$

Proof: Since $F_q(C_9 \rtimes C_3)$ for $p \neq 3$ is semisimple, we have

$$F_q(C_9 \rtimes C_3) \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{5.1}$$

If k is even, then $p^k \equiv 1,4,7 \mod 9$. Here $|S(\gamma_g)| = 1$ for all representative $g \in F_q(C_9 \rtimes C_3)$. Theorems 2.1, 2.2 imply that

$$F_q(C_9 \rtimes C_3) \cong F_q \bigoplus_{t=1}^{10} M_{n_t}(F_t) \tag{5.2}$$

and

$$26 = \sum_{t=1}^{10} n_t^2, n_t \geq 1, \forall\, t.$$

Here, n_t's has only one possibility for the above equation (1,1,1,1,1,1,1,1,1,3,3). Therefore, the above equation implies that

$$F_q(C_9 \rtimes C_3) \cong F_q^9 \oplus M_3(F_q). \tag{5.3}$$

If k is odd, then $p^k \equiv \{1,2,4,5,7,8\} \mod 9$. Now for $p^k \equiv \{1,4,7\} \mod 9$, we have $T = \{1,4,7\}$ and $|S(\gamma_g)| = 1$ for all $g \in C_9 \rtimes C_3$. Therefore Wedderburn decomposition in this case is given by Equation (5.3). Next, for $p^k \equiv \{2,5,8\} \mod 9$, we have $T = \{1,5,7,8,4,2\}$ and $S(\gamma_{g_x}) = \{\gamma_{g_x}, \gamma_{g_{x^2}}\}$, $S(\gamma_{g_y}) = \{\gamma_{g_y}, \gamma_{g_{y^2}}\}$, $S(\gamma_{g_z}) = \{\gamma_{g_z}, \gamma_{g_{z^2}}\}$, $S(\gamma_{g_{xy}}) = \{\gamma_{g_{xy}}, \gamma_{g_{x^2y^2}}\}$ and $S(\gamma_{g_{x^2y}}) = \{\gamma_{g_{x^2y}}, \gamma_{g_{xy^2}}\}$, and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Now using Theorem 2.5 the Wedderburn decomposition is given by

$$F_q(C_9 \rtimes C_3) \cong F_q \bigoplus_{t=1}^{5} M_{n_t}(F_{t^2}) \tag{5.4}$$

and

$$26 = 2\sum_{t=1}^{5} n_t^2, n_t \geq 1, \forall\, t,$$

which further implies that the possible choices of n_t's is (1,1,1,1,1,1,1,3,3). Hence after using structure of FC_3^2 [15], the Wedderburn decomposition is given by

$$F_q(C_9 \rtimes C_3) \cong F_q \oplus F_{q^2}^4 \oplus M_3(F_{q^2}). \tag{5.5}$$

Hence, the unit group structure is given by:

$$U(F_q(C_9 \rtimes C_3)) \cong \begin{cases} C_{p^k-1}^9 \oplus GL_3(F_q) & \text{for} \quad q \equiv 1,4,7 \mod 9 \\ C_{p^k-1} \oplus C_{p^{2k}-1}^4 \oplus GL_3(F_{q^2}) & \text{for} \quad q \equiv 2,5,8 \mod 9. \end{cases}$$ □

6. Unit Group of Group Algebras of Non Abelian Groups of Order 28

Up to isomorphism, there are only two non abelian group of order 28, D_{14} and $C_7 \rtimes C_4$. In this section we give the structure of unit group of $C_7 \rtimes C_4$ for semisimple case. The unit group structure of FD_{14} already has been discussed in [11].

The group $C_7 \rtimes C_4$ has the following presentation:
$C_7 \rtimes C_4 =< x, y, z | x^2y^{-1}, y^{-1}x^{-1}yx, z^{-1}x^{-1}zxz^{-5}, y^2, z^{-1}y^{-1}zy, z^7 >$.

Conjugacy classes of $C_9 \rtimes C_3$ are shown in the Table 10. From Table 10, it can be observed that

Table 10:

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xz^2, xz^4, xz^6, xz, xz^3, xz^5\}$	4
y	$\{y\}$	2
z	$\{z, z^6\}$	7
xy	$\{xy, xyz^2, xyz^4, xyz^6, xyz, xyz^3, xyz^5\}$	4
yz	$\{yz, yz^6\}$	14
z^2	$\{z^2, z^5\}$	7
yz^2	$\{yz^2, yz^5\}$	14
z^5y	$\{z^3, z^4\}$	7
yz^3	$\{yz^3, yz^4\}$	14

the exponent of $C_7 \rtimes C_4$ is 28. Also $(C_7 \rtimes C_4)' = C_7$ and $C_7 \rtimes C_4/(C_7 \rtimes C_4)' = C_4$. Next we give the Wedderburn decomposition for $p \neq 2, 7$.

Theorem 6.1 *The Wedderburn decomposition of $F_q(C_7 \rtimes C_4)$ for $p \neq 2, 7$, where $q = p^k$ is given by*

$$F_q(C_7 \rtimes C_4) \cong \begin{cases} F_q^4 \oplus M_2(F_q)^6 & \textit{for} \quad q \equiv 1, 13 \mod 28 \\ F_q^4 \oplus M_2(F_{q^3})^2 & \textit{for} \quad q \equiv 5, 9, 17, 25 \mod 28 \\ F_q^2 \oplus F_{q^2} \oplus M_2(F_q)^6 & \textit{for} \quad q \equiv 3, 11, 19, 23 \mod 28 \\ F_q^2 \oplus F_{q^2} \oplus M_2(F_{q^3})^2 & \textit{for} \quad q \equiv 3, 11, 19, 23 \mod 28. \end{cases}$$

Proof: Since $F_q(C_7 \rtimes C_4)$ for $p \neq 2, 7$ is semisimple, we have

$$F_q(C_7 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{6.1}$$

Now, for any k we have, $p^k \equiv 1, 3 \mod 4$ and $p^k \equiv 1, 2, 3, 4, 5, 6 \mod 7$. Which further implies $p^k \equiv \{1, 9, 17, 25, 5, 13, 15, 23, 3, 11, 19, 27\} \mod 28$ (Using Chinese Remainder Theorem). First we discuss the case when $p^k \equiv 1 \mod 28$, here $|S(\gamma_g)| = 1$ for all representative $g \in F_q(C_7 \rtimes C_4)$. Theorems 2.1, 2.2 imply that

$$F_q(C_7 \rtimes C_4) \cong F_q \bigoplus_{t=1}^{9} M_{n_t}(F_t) \tag{6.2}$$

and

$$27 = \sum_{t=1}^{9} n_t^2, n_t \geq 1, \forall\, t.$$

There are two valid combinations of n_t's for the above equation (1,1,1,1,1,1,1,2,4) and (1,1,1,2,2,2,2,2,2). Now using the structure of FC_4 [15] the possible choice for n_t's is (1,1,1,2,2,2,2,2,2). Therefore, the above equation implies that

$$F_q(C_7 \rtimes C_4) \cong F_q^4 \oplus M_2(F_q)^6. \tag{6.3}$$

Next, when $p^k \equiv 9 \mod 28$ we have $T = \{1, 9, 25\}$ and $S(\gamma_{g_z}) = \{\gamma_{g_z}, \gamma_{g_{z^2}}, \gamma_{g_{z^3}}\}$, $S(\gamma_{g_{yz}}) = \{\gamma_{g_{yz}}, \gamma_{g_{yz^2}}, \gamma_{g_{yz^3}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Now using the structure of FC_4 [15] and Theorems 2.1, 2.2, the Wedderburn decomposition is given by

$$F_q(C_7 \rtimes C_4) \cong F_q^4 \oplus M_2(F_{q^3})^2. \tag{6.4}$$

Now, when $p^k \equiv 15 \mod 28$, we have $T = \{1, 15\}$ and $S(\gamma_{g_x}) = \{\gamma_{g_x}, \gamma_{g_{xy}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Again using the structure of FC_4 Theorems 2.1, 2.2, the Wedderburn decomposition is given by

$$F_q(C_7 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \oplus M_2(F_q)^6. \tag{6.5}$$

For $p^k \equiv 23 \mod 28$, we have $T = \{1, 23, 25, 15, 9, 11\}$ and $S(\gamma_{g_x}) = \{\gamma_{g_x}, \gamma_{g_{xy}}\}$, $S(\gamma_{g_z}) = \{\gamma_{g_z}, \gamma_{g_{z^2}}, \gamma_{g_{z^3}}\}$, $S(\gamma_{g_{yz}}) = \{\gamma_{g_{yz}}, \gamma_{g_{yz^2}}, \gamma_{g_{yz^3}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Now using the structure of FC_4 [15] and Theorems 2.1, 2.2, the Wedderburn decomposition is given by

$$F_q(C_7 \rtimes C_4) \cong F_q^2 \oplus F_{q^2} \oplus M_2(F_{q^3})^2. \tag{6.6}$$

Hence, the unit group structure is given by:

$$U(F_q(C_7 \rtimes C_4)) \cong \begin{cases} C_{p^k-1}^4 \oplus GL_2(F_q)^6 & \text{for} \quad q \equiv 1, 13 \mod 28 \\ C_{p^k-1}^4 \oplus GL_2(F_{q^3})^2 & \text{for} \quad q \equiv 5, 9, 17, 25 \mod 28 \\ C_{p^k-1}^2 \oplus C_{p^{2k}-1} \oplus Gl_2(F_q)^6 & \text{for} \quad q \equiv 3, 11, 19, 23 \mod 28 \\ C_{p^k-1}^2 \oplus C_{p^{2k}-1} \oplus GL_2(F_{q^3})^2 & \text{for} \quad q \equiv 3, 11, 19, 23 \mod 28. \end{cases}$$

□

7. Unit Group of Group Algebras of Non Abelian Groups of Order 30

Up to isomorphism, there are three non abelian groups of order 30 namely D_{15}, $C_5 \times S_3$ and $C_3 \times D_5$. The unit group structure of D_{15} and $C_3 \times D_5$ already has been discussed (see [8], [16]). In this section we give the structure of unit group of $C_5 \times S_3$ for semisimple case. The group $C_5 \times S_3$ has the following presentation:

$C_5 \times S_3 =< x, y, z | x^2, y^{-1}x^{-1}yx, z^{-1}x^{-1}zxz^{-1}, y^5, z^{-1}y^{-1}zy, z^3 >.$

There are 15 conjugacy classes of $C_5 \times S_3$ which are shown in Table 11.

Table 11:

Representative	Elements in the class	Order of element
1	$\{1\}$	1
x	$\{x, xz, xz^2\}$	2
y	$\{y\}$	5
z	$\{z, z^2\}$	3
xy	$\{xy, xyz, xyz^2\}$	10
y^2	$\{y^2\}$	5
yz	$\{yz, yz^2\}$	15
xy^2	$\{xy^2, xy^2z, xy^2z^2\}$	10
y^3	$\{y^3\}$	5
y^2z	$\{y^2z, y^2z^2\}$	15
xy^3	$\{xy^3, xy^3z, xy^3z^2\}$	10
y^4	$\{y^4\}$	5
y^3z	$\{y^3z, y^3z^2\}$	15
xy^4y	$\{xy^4, xy^4z, xy^4z^2\}$	10
y^4z	$\{y^4z, y^4z^2\}$	15

Clearly from Table 11, it can be observed that the exponent of $C_5 \times S_3$ is 30. Also $(C_5 \times S_3)' = C_3$ and $C_5 \times S_3/(C_5 \times S_3)' = C_{10}$. Next we give the Wedderburn decomposition for $p \neq 2, 5, 7$.

Theorem 7.1 *The Wedderburn decomposition of $F_q(C_5 \times S_3)$ for $p \neq 2, 3, 5$, where $q = p^k$ is given by*

$$F_q(C_5 \times S_3) \cong \begin{cases} F_q^{10} \oplus M_2(F_q)^5 & \text{for} \quad q \equiv 1, 11 \mod 30 \\ F_q^2 \oplus F_{q^4}^2 \oplus M_2(F_q) \oplus M_2(F_{q^4}) & \text{for} \quad q \equiv 7, 13, 17, 23 \mod 30 \\ F_q^2 \oplus F_{q^2}^4 \oplus M_2(F_q) \oplus M_2(F_{q^2})^2 & \text{for} \quad q \equiv 19, 29 \mod 30. \end{cases}$$

Proof: Since $F_q(C_5 \times S_3)$ for $p \neq 2, 3, 5$ is semisimple, we have,

$$F_q(C_5 \times S_3) \cong F_q \bigoplus_{t=1}^{i-1} M_{n_t}(F_t). \tag{7.1}$$

If k is even, then $p^k \equiv 1 \mod 30$ and $p^k \equiv 19 \mod 30$. Now for $p^k \equiv 1 \mod 30$, we have $|S(\gamma_g)| = 1$ for all representative $g \in F_q(C_5 \times S_3)$. Theorems 2.1, 2.2 imply that

$$F_q(C_5 \rtimes S_3) \cong F_q \bigoplus_{t=1}^{14} M_{n_t}(F_t) \tag{7.2}$$

and

$$29 = \sum_{t=1}^{14} n_t^2, n_t \geq 1, \forall\, t.$$

There are two valid combinations of n_t's for the above equation (1,1,1,1,1,1,1,1,1,1,1,1,1,1,4) and (1,1,1,1,1,1,1,1,1,1,2,2,2,2,2). Now using the Wedderburn structure of FC_{10} [15] the possible choice for n_t's is (1,1,1,1,1,1,1,1,1,1,2,2,2,2,2). Therefore, the above equation implies that

$$F_q(C_5 \times S_3) \cong F_q^{10} \oplus M_2(F_q)^5. \tag{7.3}$$

Next, consider the case when $p^k \equiv 19 \mod 30$. For this case we have $T = \{1, 19\}$ and $S(\gamma_{g_y}) = \{\gamma_{g_y}, \gamma_{g_{y^4}}\}$, $S(\gamma_{g_{xy}}) = \{\gamma_{g_{xy}}, \gamma_{g_{xy^4}}\}$, $S(\gamma_{g_{y^2}}) = \{\gamma_{g_{y^2}}, \gamma_{g_{y^3}}\}$, $S(\gamma_{g_{yz}}) = \{\gamma_{g_{yz}}, \gamma_{g_{y^4z}}\}$, $S(\gamma_{g_{y^2z}}) = \{\gamma_{g_{y^2z}}, \gamma_{g_{y^3z}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Now using the structure of FC_{10} [15] and Theorems 2.1, 2.2, the Wedderburn decomposition is given by

$$F_q(C_5 \times S_3) \cong F_q^2 \oplus F_{q^2}^4 \oplus M_2(F_q) \oplus M_2(F_{q^2})^2. \tag{7.4}$$

We now consider the case when k is odd. For this $p^k \equiv \{1, 7, 13, 19, 11, 17, 23, 29\} \mod 30$. Now for $p^k \equiv 7 \mod 30$, we have $T = \{1, 7, 19, 13\}$ and $S(\gamma_{g_y}) = \{\gamma_{g_y}, \gamma_{g_{y^2}}, \gamma_{g_{y^3}}, \gamma_{g_{y^4}}\}$, $S(\gamma_{g_{xy}}) = \{\gamma_{g_{xy}}, \gamma_{g_{xy^2}}, \gamma_{g_{xy^3}}, \gamma_{g_{xy^4}}\}$,$S(\gamma_{g_{yz}}) = \{\gamma_{g_{yz}}, \gamma_{g_{y^2z}}, \gamma_{g_{y^3z}}, \gamma_{g_{y^4z}}\}$ and $S(\gamma_g) = \{\gamma_g\}$ for the remaining representative of the conjugacy classes. Now using the structure of FC_{10} [15] and Theorems 2.1, 2.2, the Wedderburn decomposition is given by

$$F_q(C_5 \times S_3) \cong F_q^2 \oplus F_{q^4}^2 \oplus M_2(F_q) \oplus M_2(F_{q^4}). \tag{7.5}$$

Hence the unit group structure is given by:

$$U(F_q(C_5 \times S_3)) \cong \begin{cases} C_{p^k-1}^{10} \oplus GL_2(C_{p^k-1})^5 & \text{for} \quad q \equiv 1, 11 \mod 30 \\ C_{p^k-1}^2 \oplus C_{p^{4k}-1}^2 \oplus GL_2(F_q) \oplus GL_2(F_{q^4}) & \text{for} \quad q \equiv 7, 13, 17, 23 \mod 30 \\ C_{p^k-1}^2 \oplus C_{p^{2k}-1}^4 \oplus GL_2(F_q) \oplus GL_2(F_{q^2})^2 & \text{for} \quad q \equiv 19, 29 \mod 30. \end{cases} \quad \square$$

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Diksha Upadhyay,
Department of Mathematics and Scientific Computing,
Madan Mohan Malaviya University of Technology,
India.
E-mail address: dikshaupadhyay111@gmail.com

and

Harish Chandra,
Department of Mathematics and Scientific Computing,
Madan Mohan Malaviya University of Technology,
India.
E-mail address: hcmsc@mmmut.ac.in