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Existence of Solutions for Nonlinear Boundary Value Problems for Second-Order Impulsive Differential Equations with a Deviating Argument

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ABSTRACT: In this paper, we study the existence of solutions for a second-order impulsive differential equation with a deviating argument in the following form:

$$-y'' = f(t, y(t), y(t-\tau(t)), y'(t)) + e(t), t \in J := [0,1], \quad t \neq t_k, k = 1, \dots, m,$$
$$y\left(t_k^+\right) - y\left(t_k^-\right) = I_k\left(y\left(t_k^-\right)\right), \quad k = 1, \dots, m,$$
$$y'\left(t_k^+\right) - y'\left(t_k^-\right) = \bar{I}_k\left(y'\left(t_k^-\right)\right), \quad k = 1, \dots, m,$$
$$y(0) = y(1) = 0,$$

where $0 = t_0 < t_1 < \cdots < t_m < 1$ be given, $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$ is a given function, $e, I_k, \bar{I}_k \in C(\mathbb{R}, \mathbb{R})$, . Let $J_0 = [0, t_1], (J_k = (t_k, t_{k+1}], k = 1, \dots, m), J' = J \backslash \{t_1, t_2, \dots, t_m\}$. By using the nonlinear alternative of Leray-Schauder.

Key Words: Impulsive differential equations, Boundary Value Problems, existence of solution, nonlinear alternative of Leray–Schauder.

Contents

1. Introduction

Impulsive differential equations represent a most important field in mathematics for describing various real-world phenomena such as in physics, engineering, medicine, and biology, etc.
Before the resolution of the impulsive differential equations it is important to study the existence of solutions. Thus, a diverse array of methods and analytical techniques has been employed to investigate the existence of solutions in impulsive differential equations.
Agarwal and O'Regan studied in [1] the existence of both unique and multiple solutions for a second-order impulsive differential equation with fixed impulse moments. The problem is formulated as follows:

$$\begin{aligned}
&y''(t) + \phi(t)\, f(t, y(t)) = 0, \quad t \in (0,1) \setminus \{t_1, \dots, t_m\},\\
&\Delta y(t_k) = I_k\big(y(t_k^-)\big), \quad k = 1, \dots, m,\\
&\Delta y'(t_k) = J_k\big(y(t_k^-)\big), \quad k = 1, \dots, m,\\
&y(0) = y(1) = 0,
\end{aligned}$$

where

$$f \in C\big([0,1] \times \mathbb{R}, \mathbb{R}\big), \quad \phi(t) \in C(0,1), \quad \text{and} \quad I_k,\, J_k \in C\big(\mathbb{R} \times \mathbb{R}\big).$$

Their methodology relies on the nonlinear alternative of Leray–Schauder type together with Krasnoselskii's fixed point theorem in a cone.

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In [5] Guo used the fixed point theory to analyze the existence of solutions for a specific class of second-order impulsive differential equations, expressed as follows:

$$\begin{aligned} & -x''=f\left(t,x,x'\right), \\ & \Delta x|_{t=t_k}=I_k\left(x\left(t_k\right)\right), \\ & \Delta x'|_{t=t_k}=\bar{I}_k\left(x\left(t_k\right),x'\left(t_k\right)\right), \quad k=1,2,\ldots,m, \\ & ax(0)-bx'(0)=x_0, \quad cx(1)+dx'(1)=x_0^*. \end{aligned}$$

This study was conducted within a Banach space E, where the function $f\in C(J\times E\times E,E)$, with $J=[0,1]$ and the impulse instants satisfying $0<t_1<\cdots<t_k<\cdots<t_m<1$. Additionally, the functions $I_k:E\to E$ and $\bar{I}_k:E\times E\to E$ are continuous mappings, $x_0,x_0^*\in E$, and $p=ac+ad+bc\neq 0$. Inspired by the above-mentioned works, in this paper we study the existence of solutions for a second-order impulsive differential equation,

$$\begin{gathered} -y''=f(t,y(t),y(t-\tau(t)),y'(t))+e(t), t\in J:=[0,1], \quad t\neq t_k, k=1,\ldots,m, \\ \Delta y(t_k)=y\left(t_k^+\right)-y\left(t_k^-\right)=I_k\left(y\left(t_k^-\right)\right), \quad k=1,\ldots,m, \\ \Delta y'(t_k)=y'\left(t_k^+\right)-y'\left(t_k^-\right)=\bar{I}_k\left(y'\left(t_k^-\right)\right), \quad k=1,\ldots,m, \\ y(0)=y(1)=0, \end{gathered} \tag{1.1}$$

where $0=t_0<t_1<\cdots<t_m<1$ be a partition of the interval $[0,1]$. $f\in C\big([0,1]\times\mathbb{R}^3,\mathbb{R}\big)$ is a given function and $\tau,\ e,\ I_k,\ \bar{I}_k\in C(\mathbb{R},\mathbb{R})$ are continuous functions.
Define the subintervals as follows: $J_0=[0,t_1]$ and $J_k=(t_k,t_{k+1}]$, $k=1,\ldots,m$. Let $J'=J\setminus\{t_1,t_2,\ldots,t_m\}$, and denote by y_k the restriction of any function y to the interval J_k.

To define solutions for (1.1), consider the space $PC(J,\mathbb{R})=\Big\{y:[0,1]\to\mathbb{R}\ \Big|\ y_k\in C(J_k,\mathbb{R}) \text{ for } k=0,\ldots,m, \text{ and } y(t_k^-),\ y(t_k^+) \text{ exist with } y(t_k^-)=y(t_k) \text{ for } k=1,\ldots,m\Big\}$. Equipped with the norm

$$\|y\|_{PC}=\max\left\{\|y_k\|_\infty, k=0,\ldots,m\right\}, \quad \|y_k\|_\infty=\sup_{t\in J_k}|y(t)|,$$

the space $PC(J,\mathbb{R})$ is a Banach space.

Similarly, define
$PC^1(J,\mathbb{R})=\Big\{y\in PC(J,\mathbb{R})\ \Big|\ y_k'\in C(J_k,\mathbb{R}) \text{ for } k=0,\ldots,m, \text{ and } y'(t_k^-),\ y'(t_k^+) \text{ exist with } y'(t_k^-)=y'(t_k) \text{ for } k=1,\ldots,m\Big\}$.
This space becomes a Banach space when it is endowed with the norm

$$\|y\|_{PC^1}=\max\left\{\|y\|_{PC},\ \|y'\|_{PC}\right\}.$$

2. Preliminaries

In this section, we present some results which will be needed in Section 3.

Lemma 2.1 *[4] Let X be a Banach space with $C\subset X$ closed and convex. Assume Ω is a relatively open subset of C with $0\in\Omega$ and $N:\bar{\Omega}\to C$ is a compact map. Then either,*
(i) N has a fixed point in $\bar{\Omega}$; or
(ii) there is a point $y\in\partial\Omega$ and $\lambda\in(0,1)$ with $y=\lambda N(y)$.

Lemma 2.2 *[5] If $y\in PC^1[J,E]\cap C^2\left[J',E\right]$ satisfies*

$$-y''(t)=f\left(t,y(t),y'(t)\right), \quad t\neq t_k(k=1,2,\ldots,m)$$

then

$$y'(t)=y'(0)-\int_0^t f\left(s,y(s),y'(s)\right)ds+\sum_{0<i_k<t}\left[y'\left(t_k^+\right)-y'\left(t_k\right)\right], \quad t\in J,$$

and

$$\begin{aligned}y(t) =& y(0) + x'(0)t - \int_0^t (t-s) f\left(s, y(s), y'(s)\right) ds \\ &+ \sum_{0<t_k<t} \left[y\left(t_k^+\right) - y\left(t_k\right)\right] + \sum_{0<t_k<t} \left[y'\left(t_k^+\right) - y'\left(t_k\right)\right]\left(t - t_k\right), \quad t \in J.\end{aligned}$$

Lemma 2.3 *$y \in PC^1[J,E] \cap C^2[J',E]$ is a solution of system (1.1) if and only if $y \in PC^1[J,\mathbb{R}]$ is a solution of equation*

$$\begin{aligned}y(t) =& \int_0^1 G(t,s)(f(s,y(s),y(s-\tau(s)),y'(s)) + e(s))ds \\ &+ \sum_{0<t_k<t} \left[I_k\left(y\left(t_k\right)\right) + \left(t - t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right] \\ &- t\sum_{k=1}^m \left[I_k\left(y\left(t_k\right)\right) + \left(1 - t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right],\end{aligned} \tag{2.1}$$

where

$$G(t,s) = \begin{cases}(1-t)s, & 0 \leqslant s \leqslant t, \\ (1-s)t, & t \leqslant s \leqslant 1,\end{cases}$$

Proof: *First suppose that $y \in PC^1[J,E] \cap C^2[J',E]$ is a solution to the boundary value problem (1.1).*

It follows from Lemma 2.2 , that

$$\begin{aligned}y(t) =& y(0) + y'(0)t - \int_0^t (t-s)(f(s,y(s),y(s-\tau(s)),y'(s)) + e(s))ds \\ &+ \sum_{0<t_k<t} \left[I_k\left(y\left(t_k\right)\right) + \left(t - t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right], \quad t \in J.\end{aligned} \tag{2.2}$$

For $t = 1$, we have

$$\begin{aligned}y(1) =& y(0) + y'(0) - \int_0^1 (1-s) f\left(s, y(s), y'(s)\right) ds \\ &+ \sum_{k=1}^m \left[I_k\left(y\left(t_k\right)\right) + \left(1 - t_k\right)\tilde{I}_k\left(y\left(t_k\right), y'\left(t_k\right)\right)\right]\end{aligned}$$

then

$$\begin{aligned}y'(0) =& \int_0^1 (1-s) f(s,y(s),y(s-\tau(s)),y'(s))ds \\ &- \sum_{k=1}^m \left[I_k\left(y\left(t_k\right)\right) + \left(1 - t_k\right)\tilde{I}_k\left(y\left(t_k\right), y'\left(t_k\right)\right)\right]\end{aligned} \tag{2.3}$$

Finally, substituting (2.3) into (2.2), we obtain

$$
\begin{aligned}
y(t) =& t\int_0^1 (1-s)f(s,y(s),y(s-\tau(s)),y'(s))ds - t\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\tilde{I}_k\left(y\left(t_k\right),y'\left(t_k\right)\right)\right] \\
&-\int_0^t (t-s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds + \sum_{0<t_k<t}\left[I_k\left(y\left(t_k\right)\right)+\left(t-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right] \\
=& t\int_0^t (1-s)f(s,y(s),y(s-\tau(s)),y'(s))ds + t\int_t^1 (1-s)f\left(s,y(s),y'(s)\right)ds \\
&- t\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\tilde{I}_k\left(y\left(t_k\right),y'\left(t_k\right)\right)\right] - \int_0^t (t-s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds \\
&+ \sum_{0<t_k<t}\left[I_k\left(y\left(t_k\right)\right)+\left(t-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right] \\
=& \int_0^1 G(t,s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds + \sum_{0<t_k<t}\left[I_k\left(y\left(t_k\right)\right)+\left(t-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right] \\
&- t\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y\left(t_k\right),y'\left(t_k\right)\right)\right]
\end{aligned}
$$

That is, $y(t)$ satisfies Equation (2.1).

Conversely, suppose $y \in PC^1[J,E]$ is a solution of (2.1).
Clearly,

$$
\left.\Delta y\right|_{t=t_k} = I_k\left(y\left(t_k\right)\right) \quad (k=1,2,\ldots,m)
$$

Direct differentiation implies, for $t \neq t_k$,

$$
\begin{aligned}
y'(t) =& -\int_0^t s(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds \\
&+\int_t^1 (1-s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds \\
&+\sum_{0<t_k<t}\bar{I}_k\left(y'\left(t_k\right)\right) \\
&-\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]
\end{aligned}
$$

and

$$
y''(t) = -f(t,y(t),y(t-\tau(t)),y'(t)) - e(t)
$$

so $y \in C^2\left[J',E\right]$ and

$$
\left.\Delta y'\right|_{t=t_k} = \bar{I}_k\left(y'\left(t_k\right)\right) \quad (k=1,2,\ldots,m).
$$

□

3. Main result

In this section, we examine the existence of solutions to problem (1.1). To proceed, we introduce the following conditions:

(H1) $f:[0,1]\times\mathbb{R}^3 \to \mathbb{R}$ is continuous, and $\tau, e, I_k, \bar{I}_k \in C(\mathbb{R},\mathbb{R})$.
(H2) The function f has the decomposition

$$
f(t,x,y,z) = h(t,x) + g(t,y) + p(t,z)
$$

such that

$$\lim_{|x|\to+\infty}\sup_{t\in[0,T]}\frac{|h(t,x(t))|}{|x(t)|}=r_0, \tag{3.1}$$

$$\lim_{|x|\to+\infty}\sup_{t\in[0,T]}\frac{|g(t,x(t-\tau(t)))|}{|x(t-\tau(t))|}=r_1, \quad \text{and} \tag{3.2}$$

$$\lim_{|x|\to+\infty}\sup_{t\in[0,T]}\frac{|p(t,x(t))|}{|x(t)|}=r_2, \tag{3.3}$$

where $r_i \geqslant 0, i=0,1,2$ are all constants, $g,h,$ and p are continuous on $\mathbb{R}\times\mathbb{R}$.
(H3) There are constants c_k and $\bar{c}_k$ such that for each $y\in PC^1(J,\mathbb{R})$, the inequalities $|I_k(y)|\le c_k$ and $|I_k(y')|\le \bar{c}_k$ hold for $k=1,\ldots,m$.
$(H4)$ $r_0+r_1+r_2<1$.

Theorem 3.1 *Suppose that the hypotheses* (H1)–(H4) *are satisfied. Then, the problem* (1.1) *has at least one solution* y.

Proof: *In order to reformulate the problem as a fixed point problem, define the mapping* $N\in PC^1(J,\mathbb{R})$ *as follows:*

$$\begin{aligned} Ny(t)=&\int_0^1 G(t,s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds\\ &+\sum_{0<t_k<t}\left[I_k\left(y\left(t_k\right)\right)+\left(t-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]\\ &-t\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]. \end{aligned} \tag{3.4}$$

We will show that N *is completely continuous. The proof will be given in several steps.*
Step 1. N *is continuous.*
Let y_n *be a sequence in* $PC^1(J,\mathbb{R})$ *such that* $\|y_n-y\|_{PC^1}\to 0(n\to\infty)$. *We will prove that* $\|N(y_n)-N(y)\|_{PC^1}\to 0(n\to\infty)$, *we have*

$$\begin{aligned} (Ny)'(t)=&\int_0^1 G'(t,s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds\\ &+\sum_{0<t_k<t}\bar{I}_k\left(y'\left(t_k\right)\right)-\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]. \end{aligned} \tag{3.5}$$

Then

$$\begin{aligned} |\left(Ny_n\right)(t)-(Ny)(t)|\le&\int_0^1|G(t,s)(f(s,y_n(s),y_n(s-\tau(s)),y_n'(s)))-G(t,s)(f(s,y(s),y(s-\tau(s)),y'(s)))|ds\\ &+\sum_{0<t_k<t}|\left[I_k\left(y_n\left(t_k\right)\right)+\left(t-t_k\right)\bar{I}_k\left(y_n'\left(t_k\right)\right)\right]-\left[I_k\left(y\left(t_k\right)\right)+\left(t-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]|\\ &+t\sum_{k=1}^{m}|\left[I_k\left(y_n\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y_n'\left(t_k\right)\right)\right]-t\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]|. \end{aligned} \tag{3.6}$$

And

$$\begin{aligned} |\left(Ny_n\right)'(t)-(Ny)'(t)|\le&\int_0^1|G'(t,s)(f(s,y_n(s),y_n(s-\tau(s)),y_n'(s)))-G'(t,s)(f(s,y(s),y(s-\tau(s)),y'(s)))|ds\\ &+\sum_{0<t_k<t}|\bar{I}_k\left(y_n'\left(t_k\right)\right)-\bar{I}_k\left(y'\left(t_k\right)\right)|\\ &+\sum_{k=1}^{m}|\left[I_k\left(y_n\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y_n'\left(t_k\right)\right)\right]-\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]|. \end{aligned} \tag{3.7}$$

Since $f, I, \bar{I}$ are continuous operators, then the right-hand sides in (3.6) and (3.7) tend to zero when $y_n \to y$. Thus N is continuous.

Step 2: *N Maps bounded sets into bounded sets in $PC^1(J,\mathbb{R})$. Let*

$$y \in D = \left\{ y \in PC^1(J,\mathbb{R}) : \|y\|_{PC^1} \leq q \right\}.$$

For each $t \in [0,1]$, we have

$$\begin{aligned} |Ny(t)| &\leq \int_0^1 G(t,s)| \left(f\left(s, y(s), y(s-\tau(s)), y'(s)\right) + e(s)\right)|ds \\ &+ \sum_{0<t_k<t} |I_k(y\left(t_k\right))| + (t-t_k)\,|\bar{I}_k(y'\left(t_k\right))| \\ &+ t\sum_{k=1}^{m} |I_k(y\left(t_k\right))| + (1-t_k)|\bar{I}_k\left(y'\left(y_k\right)\right)| \\ &\leqslant \int_0^1 G(t,s)|h(s,y(s))|ds + \int_0^1 G(t,s)|g(s,y(s-\tau(s))|ds \\ &+ \int_0^1 G(t,s)|p\left(s,y'(s)\right)|ds + \int_0^1 G(t,s)|e(s)|ds \\ &+ \sum_{k=1}^{m} \left[c_k + (t-t_k)\,\bar{c}_k\right] + \sum_{k=1}^{m} \left[tc_k + t(1-t_k)\bar{c}_k\right]. \end{aligned} \tag{3.8}$$

Let

$$\varepsilon = \frac{1 - r_0 - r_1 - r_2}{6}.$$

By using the hypothese (H4), we see $\varepsilon > 0$. One can find from assumptions (3.1)-(3.3) that there is a constant $D > 0$ such that

$$\begin{aligned} &\frac{|h(t,y)|}{|x|} < (r_0 + \varepsilon)\,, && \textit{for } t \in [0,T], |y| > D, \\ &\frac{|g(t,y)|}{|y|} < (r_1 + \varepsilon)\,, && \textit{for } t \in [0,T], |y| > D, \quad \textit{and} \\ &\frac{|p(t,y)|}{|y|} < (r_2 + \varepsilon)\,, && \textit{for } t \in [0,T], |y| > D, \end{aligned}$$

then

$$\begin{aligned} &|h(t,y)| < (r_0 + \varepsilon)\,|y|, && \textit{for } t \in [0,T], |y| > D, \\ &|g(t,y)| < (r_1 + \varepsilon)\,|y|, && \textit{for } t \in [0,T], |y| > D, \quad \textit{and} \\ &|p(t,y)| < (r_2 + \varepsilon)\,|y|, && \textit{for } t \in [0,T], |y| > D. \end{aligned} \tag{3.9}$$

Let

$$\begin{aligned} \Delta_1 &= \{t : t \in [0,T], |y(t)| \leqslant D\}, \\ \Delta_2 &= \{t : t \in [0,T], |y(t)| > D\}, \\ \Delta_3 &= \{t : t \in [0,T], |y(t-\tau(t))| \leqslant D\}, \\ \Delta_4 &= \{t : t \in [0,T], |y(t-\tau(t))| > D\}, \\ \Delta_5 &= \{t : t \in [0,T], |y'(t)| \leqslant D\}\,, \quad \textit{and} \\ \Delta_6 &= \{t : t \in [0,T], |y'(t)| > D\}\,. \end{aligned}$$

Then we have from (3.8) that

$$\begin{aligned}|Ny(t)| \leq & \int_{\Delta_1} G(t,s)|h(s,y(s))|ds + \int_{\Delta_2} G(t,s)|h(s,y(s))|ds + \int_{\Delta_3} G(t,s)|g(s,y(s-\tau(s))|ds \\ & + \int_{\Delta_4} G(t,s)|g(s,y(s-\tau(s))|ds + \int_{\Delta_5} G(t,s)|p\left(s,y'(s)\right)|ds + \int_{\Delta_6} G(t,s)|p\left(s,y'(s)\right)|ds \\ & + \int_0^1 G(t,s)|e(s)|ds + \sum_{k=1}^m \left[c_k + (t-t_k)\,\bar{c}_k\right] + \sum_{k=1}^m \left[tc_k + t(1-t_k)\bar{c}_k\right].\end{aligned}$$

Then we have from (3.5) that

$$\begin{aligned}|Ny(t)| & \leqslant (r_0 + r_1 + 2\varepsilon) \sup_{t\in[0,1]} \int_0^1 G(t,s)ds\|y\|_{PC} + (r_2+\varepsilon) \sup_{t\in[0,1]} \int_0^1 G(t,s)d\|y'\|_{PC} \\ & + \left[g_D + h_D + p_D + \|e\|_\infty\right] \int_0^1 G(t,s)ds + \sum_{k=1}^m \left[c_k + (t-t_k)\,\bar{c}_k\right] + \sum_{k=1}^m \left[tc_k + t(1-t_k)\bar{c}_k\right] \\ & \leqslant (r_0 + r_1 + r_2 + 3\varepsilon)\,\|y\|_{PC^1} \sup_{t\in[0,1]} \int_0^1 G(t,s)ds + \left[g_D + h_D + p_D + \|e\|_\infty\right] \int_0^1 |G(t,s)|ds \\ & + 2\sum_{k=1}^m \left[c_k + (t_k+1)\bar{c}_k\right] \\ & \leqslant q\,(r_0 + r_1 + r_2 + 3\varepsilon) \sup_{t\in[0,1]} \int_0^1 G(t,s)ds + \left[g_D + h_D + p_D + \|e\|_\infty\right] \int_0^1 G(t,s)ds \\ & + 2\sum_{k=1}^m \left[c_k + (t_k+1)\bar{c}_k\right]\end{aligned}$$

where

$$\begin{aligned} g_D &= \max_{t\in[0,T],|x|\leqslant D} |g(t,x)|, \quad h_D = \max_{t\in[0,T],|x|\leqslant D} |h(t,x)|, \quad \textit{and} \\ p_D &= \max_{t\in[0,T],|x|\leqslant D} |p(t,x)|.\end{aligned}$$

Thus

$$\|N(y)\|_{PC} \leq M_1, \tag{3.10}$$

where $M_1 := \left[q\,(r_0+r_1+r_2+3\varepsilon) + g_D + h_D + p_D + \|e\|_\infty\right] \sup_{t\in[0,1]} \int_0^1 G(t,s)ds + 2\sum_{k=1}^m \left[c_k + (t_k+1)\bar{c}_k\right].$
Similarly, from (3.5) we can get

$$\|(Ny)'\|_{PC} \leq M_2, \tag{3.11}$$

where $M_2 := \left[q\,(r_0+r_1+r_2+3\varepsilon) + g_D + h_D + p_D + \|e\|_\infty\right] \sup_{t\in[0,1]} \int_0^1 |G'(t,s)|ds + \sum_{k=1}^m \left[c_k + (t_k+2)\bar{c}_k\right].$
Then from (3.10) and (3.11) we have

$$\|(Ny)\|_{PC^1} \leq M, \tag{3.12}$$

where $M = \max\{M_1, M_2\}$.
Step 3: *N maps bounded sets into equicontinuous sets. Let* $t, \bar{t} \in J$, $\bar{t} < t$ *and D be a bounded set of* $PC^1(J,\mathbb{R})$ *as in step 2. Let* $y \in D$. *Then*

$$
\begin{aligned}
|(Ny)(t)-(Ny)(\bar{t})| &\le \int_0^1 |G(t,s)-G(\bar{t},s)||(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))|ds \\
&+\sum_{0<t_k<t}|\left[I_k(y_n(t_k))+(t-t_k)\bar{I}_k\left(y_n'(t_k)\right)\right]-\sum_{0<t_k<\bar{t}}\left[I_k(y(t_k))+(\bar{t}-t_k)\bar{I}_k\left(y'(t_k)\right)\right]| \\
&+t\sum_{k=1}^{m}|\left[I_k(y_n(t_k))+(1-t_k)\bar{I}_k\left(y_n'(t_k)\right)\right]-\bar{t}\sum_{k=1}^{m}\left[I_k(y(t_k))+(1-t_k)\bar{I}_k\left(y'(t_k)\right)\right]| \\
&\le \int_0^1 |G(t,s)-G(\bar{t},s)|\left[q\left(r_0+r_1+r_2+3\varepsilon\right)+g_D+h_D+p_D+\|e\|_\infty\right] \\
&+\sum_{0<t_k<\bar{t}}|(t-\bar{t})\bar{I}_k\left(y_n'(t_k)\right)-\sum_{\bar{t}<t_k<t}\left[I_k(y(t_k))+(\bar{t}-t_k)\bar{I}_k\left(y'(t_k)\right)\right]| \\
&+(t-\bar{t})\sum_{k=1}^{m}|\left[I_k(y_n(t_k))+(1-t_k)\bar{I}_k\left(y_n'(t_k)\right)\right]| \\
&\le \int_0^1 |G(t,s)-G(\bar{t},s)|\left[q\left(r_0+r_1+r_2+3\varepsilon\right)+g_D+h_D+p_D+\|e\|_\infty\right] \\
&+\sum_{0<t_k<\bar{t}}|(t-\bar{t})\bar{I}_k\left(y_n'(t_k)\right)-\sum_{\bar{t}<t_k<t}\left[I_k(y(t_k))+(\bar{t}-t_k)\bar{I}_k\left(y'(t_k)\right)\right]| \\
&+(t-\bar{t})\sum_{k=1}^{m}|\left[I_k(y_n(t_k))+(1-t_k)\bar{I}_k\left(y_n'(t_k)\right)\right]| \\
&\le \int_0^1 |G(t,s)-G(\bar{t},s)|\left[q\left(r_0+r_1+r_2+3\varepsilon\right)+g_D+h_D+p_D+\|e\|_\infty\right] \\
&+(t-\bar{t})\sum_{0<t_k<\bar{t}}\bar{c}_k+\sum_{\bar{t}<t_k<t}\left[c_k+(\bar{t}-t_k)\bar{c}_k\right]| \\
&+(t-\bar{t})\sum_{k=1}^{m}\left[c_k+(1-t_k)\bar{c}_k\right].
\end{aligned}
\tag{3.13}
$$

And similarly

$$
\begin{aligned}
|(Ny)'(t)-(Ny)'(\bar{t})| &\le \int_0^1 |G'(t,s)(f(s,y(s),y_n(s-\tau(s)),y'(s)))-G'(\bar{t},s)(f(s,y(s),y(s-\tau(s)),y'(s)))|ds \\
&+\sum_{0<t_k<t}|\bar{I}_k\left(y'(t_k)\right)-\sum_{0<t_k<\bar{t}}\bar{I}_k\left(y'(t_k)\right)| \\
&\le \int_0^1 |G'(t,s)(f(s,y(s),y_n(s-\tau(s)),y'(s)))-G'(\bar{t},s)(f(s,y(s),y(s-\tau(s)),y'(s)))|ds \\
&+\sum_{\bar{t}<t_k<t}|\bar{I}_k\left(y'(t_k)\right)| \\
&\le \int_0^1 |G'(t,s)-G'(\bar{t},s)|\left[q\left(r_0+r_1+r_2+3\varepsilon\right)+g_D+h_D+p_D+\|e\|_\infty\right]+\sum_{0<t_k<\bar{t}}\bar{c}_k.
\end{aligned}
\tag{3.14}
$$

As $t-\bar{t}\to 0$. The right-hand sides of the above inequalities tend to zero.
As a consequence of Steps 1, 2 and 3 together with the Ascoli-Arzela theorem, we can conclude that $N: PC^1(J,\mathbb{R})\to PC^1(J,\mathbb{R})$ is completely continuous.
A priori estimate. *Now we show that there exists a constant M' such that $\|y\|_{PC^1}\le M'$, where y is*

a solution to the problem (1.1). Let y represent a solution to problem (1.1), then by lemma 2.3 we have

$$\begin{aligned} y(t)=&\int_0^1 G(t,s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))ds\\ &+\sum_{0<t_k<t}\left[I_k\left(y\left(t_k\right)\right)+\left(t-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]\\ &-t\sum_{k=1}^{m}\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right], \end{aligned} \tag{3.15}$$

Then, by the same way, in step 2, we can prove that:

$$\begin{aligned} |y(t)| \leq& \left(r_0+r_1+r_2+3\varepsilon\right)\|y\|_{PC^1}\sup_{t\in[0,1]}\int_0^1|G(t,s)|ds+\left[g_D+h_D+p_D+\|e\|_\infty\right]\sup_{t\in[0,1]}\int_0^1|G(t,s)|ds\\ &+2\sum_{k=1}^{m}[c_k+(t_k+1)\bar{c}_k]\\ &:=A_1\|y\|_{PC^1}+B_1, \end{aligned}$$

And

$$\begin{aligned} |y'(t)| \leq& \int_0^1|G'(t,s)(f(s,y(s),y(s-\tau(s)),y'(s))+e(s))|ds\\ &+\sum_{0<t_k<t}|\bar{I}_k\left(y'\left(t_k\right)\right)|+\sum_{k=1}^{m}|\left[I_k\left(y\left(t_k\right)\right)+\left(1-t_k\right)\bar{I}_k\left(y'\left(t_k\right)\right)\right]|\\ &\leq \left(r_0+r_1+r_2+3\varepsilon\right)\|y\|_{PC^1}\sup_{t\in[0,1]}\int_0^1|G'(t,s)|ds+\left[g_D+h_D+p_D+\|e\|_\infty\right]\sup_{t\in[0,1]}\int_0^1|G'(t,s)|ds\\ &+\sum_{k=1}^{m}[c_k+(t_k+2)\bar{c}_k]\\ &:=A_2\|y\|_{PC^1}+B_2. \end{aligned}$$

Thus,

$$\|y\|_{PC^1}=\max\left(\|y\|_{PC},\|y'\|_{PC}\right)\leq A\|y\|_{PC^1}+B.$$

Where $A=\max\{A_1,A_2\}$ and $B=\max\{B_1,B_2\}$.
From hypothesis H4, we deduce that

$$\|y\|_{PC^1}\leq\frac{B}{1-A}:=M',$$

Set

$$\Omega=\left\{y\in C\left([0,t_1],\mathbb{R}\right):\|y\|<M'+1\right\}.$$

Due to the specific selection of Ω, there exists no point y on the boundary $\partial\Omega$ that satisfies the equation $y=\lambda N(y)$ for any λ in the interval $(0,1)$. As a result of Lemma 2.1, we can conclude that the operator N possesses a fixed point y within Ω, and this fixed point is a solution to problem (1.1). □

4. Example

In this section, we present an example to demonstrate the application of Theorem 3.1. Let us consider the impulsive differential equations as follows:

$$
\begin{aligned}
-y''(t) = \frac{1}{4}y'(t) + \frac{1}{5}y(t) - \frac{1}{10}y\left(t - \frac{1}{25}\sin t\right) + \cos t, \quad t \in [0,1], \quad t \neq t_k, \quad k = 1, \dots, 4,\\
y\left(t_k^+\right) - y\left(t_k^-\right) = \frac{1}{5}\sin(y(t_k)), \quad k = k = 1, \dots, 4,\\
y'\left(t_k^+\right) - y'\left(t_k^-\right) = \frac{1}{5}\sin(y'(t_k)), \quad k = k = 1, \dots, 4,\\
y(0) = y(1) = 0.
\end{aligned} \tag{4.1}
$$

Corresponding to problem (1.1), we have $m = 4$, $h(x) = \frac{1}{4}y$, $g(x) = \frac{1}{5}y$, $p(x) = \frac{1}{10}y$, $\tau(t) = \frac{1}{25}\sin t$, and $e(t) = \cos t, I_k(y(t)) = \bar{I}_k(y(t)) = \frac{1}{5}\cos(y(t))$. Then we have

$$
r_0 + r_1 + r_2 = \frac{1}{4} + \frac{1}{5} + \frac{1}{10} < 1
$$

We can easily show that all conditions $H1$–$H4$ of Theorem 3.1 are satisfied. Thus, by applying Theorem 3.1, it follows that problem (4.1) has at least one solution.

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