Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.76186

Existence and uniqueness of a solutions of nonlinear degenerated equations with measure data

E. Rami*, A. Barbara and E. Azroul

ABSTRACT: This paper is devoted to the study of the following of nonlinear degenerate equations of the type

$$(P) \quad \begin{cases} Au &= \mu \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where A is a Leray Lions operator acted from weighted Sobolev space $W_0^{1,p}(\Omega,\omega)$ into its dual $W^{-1,p'}(\Omega,\omega^*)$, and μ is a Radon measure does not charge the sets of null (p,ω)-capacity. We prove a decomposition theorem for measure does not charge the sets of null (p,ω)-capacity. We apply this result to prove existence and uniqueness of an entropy solution of problem (P).

Key Words: Weighted Sobolev spaces, radon measure, entropy solution, (p,ω)-capacity.

Contents

1. Introduction.

Let Ω be a bounded open set of $I\!R^N$, p be a real number $1 < p < \infty$ and $w = \{w_i(x) : 0 \le i \le N\}$ be a vector of weight functions (i.e., every component $w_i(x)$ is a measurable function strictly positive almost everywhere in Ω), satisfying some integrability conditions (see Section 2). Let $Au = -div(a(x,u,Du))$ be a Leray-Lions operator defined from the weighted Sobolev space $W_0^{1,p}(\Omega,\omega)$ into its dual $W^{-1,p'}(\Omega,\omega^*)$. Let μ be a Radon measure does not charge the sets of null (p,ω)-capacity.

In this paper we investigate the problem of existence and uniqueness of an entropy solutions for a class of nonlinear degenerate elliptic equations of the type

$$\begin{cases} -div(a(x,u,Du)) &= \mu \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Existence of solutions for (1.1) if A is a linear elliptic operator and μ is a Radon measure on Ω, have been obtained, using duality techniques, by G. Stampacchia in [6], the solutions verify a stronger formulation which ensures the uniqueness but be applied only to a linear problem. In the case of a datum μ in $L^1(\Omega)$, the first results were given by H. Brezis and W. Strauss [8], where more general equations (with maximal monotone graphs) were studied in [7]. In the case where μ is Radon measure and a is a monotone operator, a strict monotone, a p quasi monotone or a derived from a potential, the existence results proved by E. Rami, A. Barbara and E. Aroul see [4] and [5] where $\nu = f - divF$.

* Corresponding author.
2020 *Mathematics Subject Classification:* Primary 47A15; Secondary 46A32, 47D20.
Submitted March 19, 2025. Published September 01, 2025

In the case where μ is Radon measure and A is a monotone operator, existence results of (1.1), were proved by L. Boccardo and T.Gallouët [10,11] (see also [1] for $\mu \in L^1(\Omega)$), but the formulation does not ensure the uniqueness. To overcome this difficulty we use in this paper the framework of entropy solutions developed by Bénilan and *al.* [13] for the study of nonlinear elliptic problems with L^1 data.

For the measures not charging sets of null capacity was first observed by L. Boccardo and *al.* [12], the authors prove the existence and uniqueness of an entropy solutions for non degenerate problem $Au = -div(a(x, Du))$, and where the capacity defined from the Sobolev space $W_0^{1,p}(\Omega)$ (here $\omega = 1$).

Our main goal here is to prove the result of existence and uniqueness of an entropy solution for problem (1.1), where the operator $-div(a(x, u, Du))$ is defined on the weighted Sobolev spaces $W_0^{1,p}(\Omega, \omega)$ and where μ is a Radon measure does not charge the sets of null (p, ω)-capacity.

The plan of the paper is as follows. In Section 2 we give some preliminaries of weighted Sobolev spaces. In Section 3 we make precise all the assumptions on a, w and μ. In section 4 we give some technical results of measures and $(p, w)-$variational capacity. In Section 5 we give the definition of an entropy solution of (1.1) and we establish the existence of such a solution (Theorem 5.1). Section 6 is devoted to prove the uniqueness of an entropy solution (Theorem 6.1).

2. Preliminaries

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open subset on $I\!R^N$, $N \geq 1$. Let us suppose that $1 < p < \infty$ is a real number, and $\omega(x) = \{\omega_i(x)\}_{\{0 \leq i \leq N\}}$ is a vector of weight function. Further we suppose that every component $\omega_i(x)$ is a measurable function which is strictly positive, and satisfies

$$\omega_i \in L^1_{loc}(\Omega) \text{ and } {\omega_i}^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega). \tag{2.1}$$

We define the weighted Lebesgue space $L^p(\Omega, \omega_0)$ with weight ω_0, as the space of all real-valued measurable functions u for which

$$\|u\|_{p,\omega_0} = \Big(\int_\Omega |u(x)|^p \omega_0(x)\, dx\Big)^{\frac{1}{p}} < +\infty \tag{2.2}$$

In order to define weighted Sobolev space of $W^{1,p}(\Omega, \omega)$, as the space of all real-valued functions $u \in L^p(\Omega, \omega_0)$ such that the derivaties in the sense of distributions satisfy $\frac{\partial u}{\partial x_i} \in L^p(\Omega, \omega_i)$ for all $i = 1, \cdots, N$. This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,\omega} = \Big(\int_\Omega |u(x)|^p \omega_0(x)\, dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p \omega_i(x)\, dx\Big)^{\frac{1}{p}} \tag{2.3}$$

To deal with the Dirichlet problem, we use the space $X = W_0^{1,p}(\Omega, \omega)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{1,p,\omega}$. Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, \omega)$ and $(W_0^{1,p}(\Omega, \omega), \|.\|_{1,p,\omega})$ is a reflexive Banach space. Note that the expression

$$\|u\|_X = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p \omega_i(x)\, dx\Big)^{\frac{1}{p}} \tag{2.4}$$

is a norm defined on X and is equivalent to the norm (2.3). Moreover $(X, \|.\|_X)$ is a reflexive Banach space, and there exist a weight function σ on Ω and a parameter $1 < q < \infty$, with $\sigma^{1-q'} \in L^1_{loc}(\Omega)$ such that the Hardy inequality

$$\Big(\int_\Omega |u|^q \sigma(x)\, dx\Big)^{\frac{1}{q}} \leq C\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p \omega_i(x)\, dx\Big)^{\frac{1}{p}} \tag{2.5}$$

holds for every $u \in X$ with a constant $C > 0$ independent of u. Moreover, the imbedding $X \hookrightarrow\hookrightarrow L^q(\Omega, \sigma)$ is compact.

We recall that the dual of the weighted Sobolev spaces $W_0^{1,p}(\Omega, \omega)$ is equivalent to $W^{-1,p'}(\Omega, \omega^*)$, where $\omega^* = \Big\{\omega_i^* = \omega_i^{1-p'}\ ;\ i = 1 \cdots, N\Big\}$ and $p' = \dfrac{p}{p-1}$ is the conjugate of p. For more details we refer the reader to [15] (see also [16]).

3. Assumptions on the data and definition of an entropy solution

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $I\!R^N$, $N \geq 1$. Let $1 < p < \infty$, and let $\omega(x) = \{\omega_i(x)\}_{\{0 \leq i \leq N\}}$ is a vector of weight function.

Let now $-div\Big(a(x,u,\nabla u)\Big)$ be a Leray-Lions operator defined on $W_0^{1,p}(\Omega,\omega)$ into $W^{-1,p'}(\Omega,\omega^*)$ and where $a\ :\ \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function, such that for $i = 1, \cdots, N$

$$|a_i(x,s,\xi)| \leq \beta\omega_i^{1/p}(x)\Big(L(x) + \sigma^{1/p'}|s|^{q/p'} + \sum_{j=1}^{N}\omega_j^{1/p'}(x)|\xi_j|^{p-1}\Big), \tag{3.1}$$

for a.e. $x \in \Omega$, all $(s,\xi) \in I\!R \times I\!R^N$, some function $L \in L^{p'}(\Omega)$ and $\beta > 0$.

$$\Big(a(x,s,\xi) - a(x,s,\eta)\Big)(\xi - \eta) > 0 \quad \text{for all } (\xi,\eta) \in I\!R^N \times I\!R^N \text{ with } \xi \neq \eta, \tag{3.2}$$

$$a(x,s,\xi).\xi \geq \alpha\sum_{i=1}^{N} w_i|\xi_i|^p, \tag{3.3}$$

where α is a strictly positive constant.

Remark 3.1 Under assumptions (3.1)-(3.3), A is monotone and coercive differential operator in the space $W_0^{1,p}(\Omega,\omega)$ into its dual $W^{-1,p'}(\Omega,\omega^*)$, hence A is surjective.

4. Measures and (p,ω)-variational capacity.

In this section we will give some results concerning the measures and (p,ω)-capacity.

Let K be a compact subset of Ω. The (p,ω)-capacity of K is defined as

$$cap_{p,\omega}\Big(K,\Omega\Big) = inf\Big\{\int_\Omega \sum_{i=1}^{N}|\frac{\partial u}{\partial x_i}|^p\omega_i(x)\,dx,\ u \in C_0^\infty(\Omega),\ u \geq \chi_K\Big\}.$$

where χ_K is the characteristic function of K.
The (p,ω)-capacity of any open subset U of Ω, is then defined by:

$$cap_{p,\omega}\Big(U,\Omega\Big) = sup\Big\{cap_{p,\omega}\Big(K,\Omega\Big),\ K\ compact \subseteq U\Big\}.$$

and the (p,ω)-capacity of any Borelian set $B \subset \Omega$ by:

$$cap_{p,\omega}\Big(B,\Omega\Big) = inf\Big\{cap_{p,\omega}\Big(U,\Omega\Big),\ U\ open \supseteq B\Big\}.$$

Denote by $M_b(\Omega)$ the space of signed measures on Ω and:

$$\mathcal{M}_b^{p,\omega}(\Omega) = \Big\{\mu \in\ M_b(\Omega):\ \mu(E) = 0\ \forall\ E \subset \Omega\ \text{ such that }\ cap_{p,\omega}\Big(E,\Omega\Big) = 0\Big\}.$$

Proposition 4.1 *Let $\mu \in \mathcal{M}_b^{p,\omega}(\Omega)$, then there exists $\gamma \in W^{-1,p'}(\Omega,\omega^*)$ and $h \in L^1(\Omega,d\gamma)$ such that*

$$\mu = h\gamma$$

Proof. Let $u \in\ W_0^{1,p}(\Omega,\omega)$, and let $\widetilde{u}$ be a measurable representation $cap_{p,\omega}$-quasi-continuous of u. Let $F:\ W_0^{1,p}(\Omega,\omega) \to [0,+\infty], u \to F(u) = \displaystyle\int_\Omega \widetilde{u}^+ d\mu$. Remark that F is convex and lower semicontinuous in $W_0^{1,p}(\Omega,\omega)$. Indeed: let $u_n \to u$ in $W_0^{1,p}(\Omega,\omega)$ therefore $u_n \to u$ a.e. in Ω, and hence $\widetilde{u_n}^+ \to \widetilde{u}^+$ a.e. in Ω. Then by Fatou's lemma, we get:

$$F(u) = \int_\Omega \widetilde{u}^+ d\mu = \int_\Omega \liminf_n \widetilde{u_n}^+ d\mu \leq \liminf_n \int_\Omega \widetilde{u_n}^+ d\mu \leq \liminf_n F(u_n).$$

According to the separability of $W^{-1,p'}(\Omega,\omega^*)$, there exist $\lambda_n \in W^{-1,p'}(\Omega,\omega)$ and $a_n \in I\!R$ such that: $F(u) = \sup\limits_n \Big(\langle \lambda_n, u\rangle + a_n\Big)$, we have: $tF(u) = F(tu) \geq t\langle \lambda_n, u\rangle + a_n, \quad \forall t > 0, \forall n$, dividing by t and letting t tend to infinity we get $F(u) \geq \langle \lambda_n, u\rangle$, for all $u \in W_0^{1,p}(\Omega,\omega)$. Since $F(0) = \sup\limits_n a_n = 0$ then $a_n \leq 0$ and we have $F(u) \geq \sup\limits_n \Big(\langle \lambda_n, u\rangle\Big) \geq \sup\limits_n \Big(\langle \lambda_n, u\rangle + a_n\Big) = F(u)$, we conclude that $F(u) = \sup_n \Big(\langle \lambda_n, u\rangle\Big)$.
Let $\varphi \in C_c^\infty(\Omega)$, we have $\langle \lambda_n, \varphi\rangle \leq \displaystyle\int_\Omega \varphi^+ d\mu \leq \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}$, and

$$\langle \lambda_n, -\varphi\rangle \leq \int_\Omega (-\varphi)^+ d\mu = -\int_\Omega (\varphi)^+ d\mu \leq \int_\Omega (\varphi)^+ d\mu \leq \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)},$$

which implies $-\|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \leq \langle \lambda_n, \varphi\rangle$. Hence $|\langle \lambda_n, \varphi\rangle| \leq \|\mu\|_{M_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}$, then $\lambda_n \in W^{-1,p'}(\Omega,\omega^*) \cap M_b(\Omega)$.
Since $F(-\varphi) = 0, \ \forall \varphi \in C_c^\infty(\Omega), \ \varphi \geq 0$, then $0 \leq \langle \lambda_n, \varphi\rangle = \displaystyle\int_\Omega \varphi d\lambda_n$. Hence $\|\lambda_n\|_{M_b(\Omega)} \leq \|\mu\|_{\mathcal{M}_b(\Omega)}$. Let $\gamma = \displaystyle\sum_{n=1}^\infty \frac{\lambda_n}{2^n\Big(\|\lambda_n\|_{W^{-1,p'}} + 1\Big)}$, we have γ is absolutely converge in $W^{-1,p'}(\Omega,\omega^*)$. For all $\varphi \in C_c^\infty(\Omega)$, we have

$$\begin{aligned} |\langle \gamma, \varphi\rangle| = \left|\sum_{n=1}^\infty \frac{\langle \lambda_n, \varphi\rangle}{2^n\Big(\|\lambda_n\|_{W^{-1,p'}} + 1\Big)}\right| &\leq \sum_{n=1}^\infty \Big(\|\lambda_n\|_{\mathcal{M}_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}\Big)/2^n \\ &\leq \Big(\sum_{n=1}^\infty \frac{1}{2^n}\Big) \|\mu\|_{\mathcal{M}_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \\ &\leq \|\mu\|_{\mathcal{M}_b(\Omega)} \|\varphi\|_{L^\infty(\Omega)}. \end{aligned}$$

This shows that $\gamma \in W^{-1,p'}(\Omega,\omega^*)$. Let A be a Borel set in Ω, such that $\gamma(A) = 0$, we have

$$\sum_{n=1}^\infty \frac{\lambda_n(A)}{2^n\Big(\|\lambda_n\|_{W^{-1,p'}} + 1\Big)} = 0$$

and we deduce that $\lambda_n(A) = 0$ for all $n \geq 1$, from which $\lambda_n \ll \gamma$. Using the theorem of Radon-Nikodym, then there exists $f_n \in L^1(\Omega, d\gamma)$ such that, for all $A \subset \Omega$, we have $\lambda_n(A) = \displaystyle\int_A f_n d\gamma$, therefore $\lambda_n = f_n\gamma$.
Let $\varphi \in C_c^\infty(\Omega)$ and $\varphi \geq 0$, we have, for any Borel B and for all n, $F(\varphi) = \displaystyle\int_\Omega \varphi d\mu \geq \sup_n \int_\Omega \varphi d\lambda_n = \sup_n \int_\Omega \varphi f_n d\gamma$, then $f_n\gamma \leq \mu$ and $\displaystyle\int_B f_n d\gamma \leq \mu(B)$.
Therefore $\displaystyle\int_B \sup\Big(f_1, f_2, ..., f_k\Big) d\gamma \leq \mu(B)$, for all $k \geq 1$. For $k \to \infty$, we obtain, by theorem monotone convergence, $\displaystyle\int_B f d\gamma \leq \mu(B), \quad \forall B \subset \Omega$, where $f = \sup\limits_n f_n$. Remark that $\displaystyle\int_\Omega \varphi d\mu = \sup_n \int_\Omega \varphi f_n d\gamma \leq \int_\Omega \varphi f d\gamma \leq \int_\Omega \varphi d\mu, \forall \varphi \in C_c^\infty(\Omega)$, hence $\mu = f\gamma$, and since $\mu(\Omega) < \infty$, then $f \in L^1(\Omega, d\gamma)$. ■

Theorem 4.1 *Let $1 < p \leq N$ and $\mu \in \mathcal{M}_b(\Omega)$, then*

$$\mu \in \mathcal{M}_b^{p,\omega}(\Omega) \Longleftrightarrow \mu \in L^1(\Omega) + W^{-1,p'}(\Omega,\omega^*).$$

Proof. Let $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega,\omega^*)$, there exists $g \in L^1(\Omega)$ and $(g_\beta)_\beta \in \prod_{|\beta|\leq 1} L^{p'}(\Omega,\omega^*_\beta)$ such that

$$\mu = g + \sum_{|\beta|\leq 1} D^\beta g_\beta = g + \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}. \tag{4.1}$$

For any Borel set $B \subset \Omega$ such that $cap_{p,\omega}(B,\Omega) = 0$, we prove that $\mu(B) = 0$. Indeed, for all $\epsilon > 0$, there exists $u_\epsilon \in C_0^\infty(\Omega):\ u_\epsilon = 1$ in $B,\ u_\epsilon \geq 0$ and $\|u_\epsilon\|_{W_0^{1,p}(\Omega,\omega)} < \epsilon$.
Let $(g_n)_n$ sequence of $L^\infty(\Omega)$, such that $g_n \to g$ in $L^1(\Omega)$.
We set

$$\mu_n = g_n + \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i}.$$

We have $g_n \to g$ in $L^1(\Omega)$, then $\mu_n \to \mu$ in $M_b(\Omega)$. We have

$$\mu_n(B) = \int_B d\mu_n = \int_B (g_n + \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i})dx = \int_B g_n u_\epsilon dx + \int_B \sum_{i=1}^{N} \frac{\partial g_i}{\partial x_i} u_\epsilon dx = \int_B g_n u_\epsilon dx - \int_B \sum_{i=1}^{N} g_i \frac{\partial u_\epsilon}{\partial x_i} dx.$$

Hence

$$\begin{aligned}
|\mu_n(B)| &\leq \int_B |g_n| u_\epsilon dx + \int_B \sum_{i=1}^{N} |g_i \frac{\partial u_\epsilon}{\partial x_i}| dx.\\
&\leq \int_B |g_n| u_\epsilon dx + \int_B \sum_{i=1}^{N} |g_i| \omega_i^{\frac{-1}{p}} |\frac{\partial u_\epsilon}{\partial x_i}| \omega_i^{\frac{1}{p}} dx\\
&\leq \int_B |g_n| u_\epsilon dx + \Big(\int_B \sum_{i=1}^{N} |g_i|^{p'} \omega_i^*\Big)^{\frac{1}{p'}} \Big(\int_B \sum_{i=1}^{N} |\frac{\partial u_\epsilon}{\partial x_i}|^p \omega_i dx\Big)^{\frac{1}{p}}\\
&\leq \int_B |g_n| \sigma^{\frac{-1}{q}} u_\epsilon \sigma^{\frac{1}{q}} dx + \Big(\int_\Omega \sum_{i=1}^{N} |g_i|^{p'} \omega_i^*\Big)^{\frac{1}{p'}} \Big(\int_\Omega \sum_{i=1}^{N} |\frac{\partial u_\epsilon}{\partial x_i}|^p \omega_i dx\Big)^{\frac{1}{p}}\\
&\leq (\int_B |g_n|^{q'} \sigma^{\frac{-q'}{q}} dx)^{\frac{1}{q'}} (\int_B u_\epsilon^q \sigma dx)^{\frac{1}{q}} + \Big(\int_\Omega \sum_{i=1}^{N} |g_i|^{p'} \omega_i^*\Big)^{\frac{1}{p'}} \Big(\int_\Omega \sum_{i=1}^{N} |\frac{\partial u_\epsilon}{\partial x_i}|^p \omega_i dx\Big)^{\frac{1}{p}}\\
&\leq \Big(C(\int_B |g_n|^{q'} \sigma^{\frac{-q'}{q}} dx)^{\frac{1}{q'}} + (\int_\Omega \sum_{i=1}^{N} |g_i|^{p'} \omega_i^*)^{\frac{1}{p'}}\Big) \Big(\int_\Omega \sum_{i=1}^{N} |\frac{\partial u_\epsilon}{\partial x_i}|^p \omega_i dx\Big)^{\frac{1}{p}}\\
&\leq C_1 \epsilon.
\end{aligned}$$

Passing to limit as ϵ tends to zero, we conclude that $\mu_n(B) = 0$, and $\mu(B) = 0$.

Conversely, let $\mu \in \mathcal{M}_b^p(\Omega,\omega)$, we can always assume that $\mu \geq 0$, and by Proposition 4.1, we deduce that $\mu = h\gamma$ where $h \in L^1(\Omega, d\gamma),\ \gamma \in W^{-1,p'}(\Omega,\omega^*),\ \gamma \geq 0$ and $h \geq 0$.

Let $(K_n)_n$ be an increasing sequence compact of Ω such that $\cup_{n=1}^\infty K_n = \Omega$ and let $\mu_n^{(1)} = T_n(h\chi_{K_n})\gamma \in W^{-1,p'}(\Omega,\omega),\ (\mu_n^{(1)})_n$ be an increasing positive sequence in $W^{-1,p'}(\Omega,\omega^*)$, with compact supports in Ω. Let

$$\begin{cases} \mu_0 &= \mu_0^{(1)},\\ \mu_n &= \mu_n^{(1)} - \mu_{n-1}^{(1)}. \end{cases}$$

Let $\rho \in \mathcal{C}_0^\infty(I\!R^N),\ \rho(x) \geq 0$ a.e. and $\int_{I\!R^N} \rho(x)\,dx = 1$. Let $\rho_n(x) = n^N \rho(nx)$. We have $\mu_n * \rho_m \longrightarrow \mu_n$ in $W^{-1,p'}(\Omega,\omega^*)$ as $m \to \infty$ and $\mu_n * \rho_m \in \mathcal{C}_0^\infty(I\!R^N)$, for m large enough.
Choose $m = m_n$ such that $\|\mu_n * \rho_{m_n} - \mu_n\|_{W^{-1,p'}(\Omega,\omega^*)} \leq 2^{-n}$. Then $\mu_n = f_n + g_n$, where

$$\begin{cases} f_n &= \mu_n * \rho_{m_n},\\ g_n &= \mu_n - \mu_n * \rho_{m_n}. \end{cases}$$

We have $\sum g_n$ converges in $W^{-1,p'}(\Omega,\omega^*)$ then $g=\sum_{n=1}^{\infty} g_n \in W^{-1,p'}(\Omega,\omega)$.
Moreover $\|f_n\|_{L^1(\Omega)} = \|\mu_n * \rho_{m_n}\|_{L^1(\Omega)} \leq \|\mu_n\|_{\mathcal{M}_b(\Omega)}$ then $\sum f_n$ is absolutely convergent in $L^1(\Omega)$. Let $f=\sum f_n \in L^1(\Omega)$, we conclude that $\mu = \sum_{n=1}^{\infty}\mu_n = \sum_{n=1}^{\infty} f_n + \sum_{n=1}^{\infty} g_n = f+g$. ■

Definition 4.1 *Let μ be a Radon measure and E is a Borel set of Ω. The restriction of μ to E is the measure $\lambda=\mu_{|_E}$ defined by: $\lambda(B)=\mu_{|_E}(B)=\mu(E\cap B)$ for any Borel set B of Ω.*
The measure λ is concentrated on E if $\lambda=\lambda_{|_E}$.

Proposition 4.2 *Let $\mu\in\mathcal{M}_b(\Omega)$ and $1<p<N$. Then μ uniquely decomposes as $\mu=\mu_0+\lambda$, where:*

$$\begin{cases} \mu_0 & \in \mathcal{M}_b^{p,\omega}(\Omega) \\ \lambda=\mu_{|_E} & \text{and } \ cap_{p,\omega}\Big(E,\Omega\Big)=0. \end{cases}$$

Proof. We assume that μ is positive (if not, consider μ^+ and μ^-). We first show the uniqueness. Assume that $\mu=\mu_0+\lambda=\mu_0^{'}+\lambda^{'}$, where $(\mu_0,\mu_0^{'})\in(\mathcal{M}_b^{p,\omega}(\Omega))^2$ and $\lambda=\mu_{|_E}$ with $cap_{p,\omega}\Big(E,\Omega\Big)=0$ and $\lambda^{'}=\mu_{|_{E'}}$ with $cap_{p,\omega}\Big(E^{'},\Omega\Big)=0$. We have $\mu_0-\mu_0^{'}=\lambda^{'}-\lambda$.

Let B be a Borel subset of Ω, if $cap_{p,\omega}\Big(B,\Omega\Big)=0$, then $(\mu_0-\mu_0^{'})(B)=0$, therefore $\mu_0=\mu_0^{'}$ and $\lambda=\lambda^{'}$, if we consider $cap_{p,\omega}\Big(B,\Omega\Big)\neq 0$ and since $\lambda^{'}-\lambda$ is focused on $E\cap E'$ such that $cap_{p,\omega}\Big(E\cap E',\Omega\Big)=0$, then we have

$$(\lambda-\lambda^{'})(B)=(\lambda-\lambda^{'})(B\cap E\cap E^{'})=(\mu_0^{'}-\mu_0)(B\cap E\cap E^{'})\leq(\mu_0^{'}-\mu_0)(E\cap E^{'})=0.$$

Therefore $\mu_0=\mu_0^{'}$, which proves the uniqueness. Now we show the existence of μ. Let

$$\alpha=\sup\Big\{\mu(A):\ A \text{ is a Borel subset of } \Omega \text{ and } cap_{p,\omega}\Big(A,\Omega\Big)=0\Big\}<\infty,$$

there exists a sequence $(A_n)_n$ such that A_n is a Borel subset of Ω, $\lim_{n\to\infty}\mu(A_n)=\alpha$ and $cap_{p,\omega}\Big(A_n,\Omega\Big)=0$.
Let $E=\cup_n A_n$ is a Borel subset of Ω, we have $\mu(E)=\alpha$ and $cap_{p,\omega}\Big(E,\Omega\Big)=0$.
For $\lambda=\mu_{|_E}$ and $\mu_0=\mu-\lambda$, there exists $E\in B(\Omega): cap_{p,\omega}\Big(E,\Omega\Big)=0$ and $\lambda=\mu_{|_E}$.
It remains to show that $\mu_0\in M_b^{p,\omega}(\Omega)$. By contradiction, assume that A is a Borel subset of Ω such that $cap_{p,\omega}\Big(E,\Omega\Big)=0$ and $0<\mu_0(A)=\mu(A)-\lambda(A)=\mu(A)-\mu(A\cap E)=\mu(A\setminus E)$. We have $cap_{p,\omega}\Big(A\cup E,\Omega\Big)=0$ and $\mu(A\cup E)=\mu(E)+\mu(A\setminus E)=\alpha+\mu(A\setminus E)>\alpha$, which contradicts the definition of M. ■

Remark 4.1 Let μ be a Radon measure, we can write $\mu=\mu_0+\mu_1+\mu_2$, such that $\mu_0\in W^{-1,p'}(\Omega,\omega^*)$, $\mu_1\in L^1(\Omega)$ and $\mu_2(E)=\mu(E\cap N)$ with $cap_{p,\omega}(N,\Omega)=0$.

5. Existence of an entropy solution.

In this section, we study the existence of an entropy solution of problem (1.1). Throughout the paper, T_k denotes the truncation function at height $k\geq 0:\ T_k(r)=\min(k,\ \max(r,\ -k))$, and we define

$$\mathrm{T}_0^{p,\omega}(\Omega)=\Big\{u:\Omega\longrightarrow I\!R \text{ measurable}: T_k(u)\in W_0^{1,p}(\Omega,\omega),\ \forall k>0\Big\}.$$

We now give the definition of an entropy solution

Definition 5.1 *Let μ be a signed measure in $\mathcal{M}_b^{p,\omega}(\Omega)$. A measurable function $u \in T_0^{p,\omega}(\Omega)$ is an entropy solution to problem (1.1) if*

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)dx \leq \int_\Omega T_k(u-\varphi)\,d\mu \qquad \forall\ \varphi\ \in \mathcal{D}(\Omega). \tag{5.1}$$

Remark 5.1 For $p < 2 - \frac{1}{N}$ and $u \in T_0^{p,\omega}(\Omega)$, we will define the gradient of u as the function v such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ almost everywhere in Ω, for every $k > 0$.

Note that, each term in (5.1) is well defined thanks to the fact that $T_k(u-\varphi) \in W_0^{1,p}(\Omega,\omega)\bigcap L^\infty(\Omega)$(see [13]).

Our main result is

Theorem 5.1 *Under assumption (H_1) and we assume that μ is a signed measure in $\mathcal{M}_b(\Omega)$. Then, there exists at least an entropy solution of problem (1.1) if and only if $\mu \in \mathcal{M}_b^{p,\omega}(\Omega)$.*

Proof. Let μ be an entropy solution of problem (1.1). In view of (3.1) we have $a \in \prod_{i=1}^N L^{p'}(\Omega,\omega_i^\star)$, hence $-\mathrm{div}(a(x,u,\nabla u)) \in W^{-1,p'}(\Omega,\omega^\star)$ and $W^{-1,p'}(\Omega,\omega^\star) \subset L^1(\Omega) + W^{-1,p'}(\Omega,\omega^\star)$. Then $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega,\omega^\star)$ and by Theorem 4.1 we have $\mu \in \mathcal{M}_b^{p,\omega}(\Omega)$.

Conversely, let $\mu \in \mathcal{M}_b^{p,\omega}(\Omega)$, we have $\mu \in L^1(\Omega) + W^{-1,p'}(\Omega,\omega^*)$, we can write $\mu = f - \mathrm{div}F$, where $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega,\omega_i^\star)$. For prove that there exists at least an entropy solution of problem (1.1), we need five steps.

Step 1: The approximate problem.

In this step, we introduce a family of approximate problems and prove the existence of solutions to such problems.

Lemma 5.1 *Let $\mu = f - div(F)$ with $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'}(\Omega,\omega_i^\star)$. Let $(f_n)_n$ be a sequence in $L^\infty(\Omega)$ such that f_n converges to f strongly in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$, we consider the approximate problem:*

$$(\mathcal{P}_n)\quad \begin{cases} Au_n = f_n - div(F) & in\ \Omega, \\ u_n \in W_0^{1,p}(\Omega,\omega), \end{cases}$$

where $Au = div(a(x,u,\nabla u))$. Assume that (H_1) holds true, there exists one weak solution u_n for the approximate problem $(\mathcal{P}_n)$.

Proof. We first show that $\displaystyle\lim_{\|v\|_{W_0^{1,p}(\Omega,\omega)}\to\infty} \frac{\langle Av,v\rangle}{\|v\|_{W_0^{1,p}(\Omega,\omega)}} = +\infty$. Indeed, we have

$$\begin{aligned}\frac{\langle Av,v\rangle}{\|v\|_{W_0^{1,p}(\Omega,\omega)}} &\geq \frac{\alpha\displaystyle\sum_{i=1}^N\int_\Omega |\frac{\partial v}{\partial x_i}|^p\omega_i(x)\,dx}{\left(\displaystyle\sum_{i=1}^N\int_\Omega |\frac{\partial v}{\partial x_i}|^p\omega_i(x)\,dx\right)^{\frac{1}{p}}} \\ &\geq \alpha\|v\|^{p-\frac{1}{p}}_{W_0^{1,p}(\Omega,\omega)} \\ &\geq \alpha\|v\|^{p-1}_{W_0^{1,p}(\Omega,\omega)} \longrightarrow +\infty, \quad \text{since}\ \ p>1.\end{aligned}$$

Now we show that A is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,p}(\Omega,\omega)$ such that:

$$\begin{cases} u_k \rightharpoonup u \ \text{ weakly in }\ W_0^{1,p}(\Omega,\omega) \\ Au_k \rightharpoonup \chi \ \text{ weakly in }\ W^{-1,p'}(\Omega,\omega^*) \\ \limsup_k \langle Au_k,u_k\rangle \leq \langle\chi,u\rangle. \end{cases}$$

We prove that $\chi = Au$ and $\langle Au_k, u_k\rangle$ converges to $\langle \chi, u\rangle$ as k goes to $+\infty$. Indeed, the sequence $(u_k)_k$ is bounded in $W_0^{1,p}(\Omega,\omega)$, hence, in view of (3.1), we deduce that $(a(x,u_k,\nabla u_k))_k$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^*)$, and there exists $\varphi \in \prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^*)$, such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \text{ weakly in } \prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^*). \tag{5.2}$$

Let $v \in W_0^{1,p}(\Omega,\omega)$, we can write

$$\langle \chi, v\rangle = \lim_{k\to\infty}\langle Au_k, u_k\rangle = \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v dx = \int_\Omega \varphi \nabla v dx. \tag{5.3}$$

On the other hand, we have

$$\limsup_{k\to\infty}\langle Au_k, u_k\rangle = \limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \le \langle \chi, u\rangle \le \int_\Omega \varphi\nabla u dx. \tag{5.4}$$

Since $\displaystyle\int_\Omega \Big(a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u)\Big)\nabla(u_k-u)\,dx \ge 0$, then

$$\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \ge -\int_\Omega a(x,u_k,\nabla u)\nabla u dx + \int_\Omega a(x,u_k,\nabla u_k)\nabla u dx + \int_\Omega a(x,u_k,\nabla u)\nabla u_k dx.$$

In view of (5.2), we can deduce that

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx \ge \int_\Omega \varphi\nabla u dx,$$

By (5.3) and (5.4), we have

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k dx = \int_\Omega \varphi\nabla u dx \quad \text{and} \quad \lim_{k\to\infty}\langle Au_k,u_k\rangle = \langle\chi,u\rangle. \tag{5.5}$$

By (5.5), we can deduce that $\displaystyle\lim_{k\to\infty}\int_\Omega\Big(a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u)\Big)\nabla(u_k-u)dx = 0$, in view of [17], we conclude that ∇u_k converges to ∇u a.e. in Ω, in view of (3.1), we have $a(x,u_k,\nabla u_k)$ converges to $a(x,u,\nabla u)$ weakly in $\prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^*)$. Hence $\chi = Au$, and in view of [9], we conclude that there exists $u_n \in W_0^{1,p}(\Omega,\omega)$ solution of $(\mathcal{P}_n)$. ■

Step 2: A priori estimate.

Choose $T_k(u_n)$ as test function in (P_n), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx + \int_\Omega F\nabla T_k(u_n)dx,$$

we can write

$$\alpha\sum_{i=1}^{N}\int_\Omega \Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p\omega_i(x)dx \le k\|f\|_{L^1(\Omega)} + \sum_{i=1}^{N}\int_\Omega |F_i|\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|dx.$$

Hence

$$\alpha\sum_{i=1}^{N}\int_\Omega \Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p\omega_i(x)dx \le k\|f\|_{L^1(\Omega)} + \sum_{i=1}^{N}\int_\Omega |F_i|\omega_i^{\frac{-1}{p}}\Big(\frac{\alpha}{2}\Big)^{\frac{-1}{p}}\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|\omega_i^{\frac{1}{p}}\Big(\frac{\alpha}{2}\Big)^{\frac{1}{p}}dx,$$

then

$$\alpha\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_k(u_n)}{\partial x_i}|^p\omega_i(x)dx \leq k\|f\|_{L^1(\Omega)}+\frac{c(\alpha)}{p'}\|F\|_{\prod_{i=1}^{N}L^{p'}(\Omega,\omega_i^\star)}+\frac{\alpha/2}{p}\sum_{i=1}^{N}\int_{\Omega}\omega_i|\frac{\partial T_k(u_n)}{\partial x_i}|^p dx,$$

since $p>1$, we deduce

$$\alpha\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_k(u_n)}{\partial x_i}|^p\omega_i(x)dx \leq k\|f\|_{L^1(\Omega)}+\frac{c(\alpha)}{p'}\|F\|_{\prod_{i=1}^{N}L^{p'}(\Omega,\omega_i^\star)}+\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\Omega}\omega_i|\frac{\partial T_k(u_n)}{\partial x_i}|^p dx,$$

and

$$\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_k(u_n)}{\partial x_i}|^p\omega_i(x)dx \leq k\|f\|_{L^1(\Omega)}+\frac{c(\alpha)}{p'}\|F\|_{\prod_{i=1}^{N}L^{p'}(\Omega,\omega_i^\star)}.$$

By consequent

$$\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_k(u_n)}{\partial x_i}|^p\omega_i(x)dx \leq k\Big(\|f\|_{L^1(\Omega)}+\frac{c(\alpha)}{p'k}\|F\|_{\prod_{i=1}^{N}L^{p'}(\Omega,\omega_i)}\Big),$$

therefore

$$\Big(\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_k(u_n)}{\partial x_i}|^p\omega_i(x)dx\Big)^{\frac{1}{p}} \leq ck^{\frac{1}{p}},\quad \forall k>1. \tag{5.6}$$

Step 3: Local convergence in measure.

We will show that $(u_n)_n$ is a Cauchy sequence in measure. Let B_R be a ball, and let $k>0$ be large enough. By virtue of Hölder's inequality and Poincaré's inequality, one has

$$\begin{aligned}
k\ meas\Big((|u_n|>k)\cap B_R\Big) &= \int_{((|u_n|>k)\cap B_R)}|T_k(u_n)|dx \leq \int_{B_R}|T_k(u_n)|\sigma^{\frac{1}{q}}\sigma^{\frac{-1}{q}}dx\\
&\leq \Big(\int_{B_R}|T_k(u_n)|^q\sigma dx\Big)^{\frac{1}{q}}\Big(\int_{B_R}\sigma^{\frac{-q'}{q}}dx\Big)^{\frac{1}{q'}} \leq C_R\Big(\int_{\Omega}|T_k(u_n)|^p\sigma dx\Big)^{\frac{1}{p}}\\
&\leq C'_R\Big(\int_{\Omega}\sum_{i=1}^{N}|\frac{\partial T_k(u_n)}{\partial x_i}|^p\omega_i dx\Big)^{\frac{1}{p}} \leq C''_R k^{\frac{1}{p}},
\end{aligned}$$

Then

$$meas\Big((|u_n|>k)\cap B_R\Big) \leq \frac{C}{k^{1-\frac{1}{p}}} \qquad \forall k>1.$$

Let $\delta>0$, we have

$$\begin{aligned}
meas\Big((|u_n-u_m|>\delta)\cap B_R\Big) &\leq meas\Big((|u_n|>k)\cap B_R\Big)+\ meas\Big((|u_m|>k)\cap B_R\Big)\\
&+meas\Big(|T_k(u_n)-T_k(u_m)|>\delta\Big).
\end{aligned}$$

Since $(T_k(u_n))_n$ is bounded in $W_0^{1,p}(\Omega,\omega)$, then there exists $v_k\in W_0^{1,p}(\Omega,\omega)$ such that $T_k(u_n)$ converges to v_k weakly in $W_0^{1,p}(\Omega,\omega)$, we use $W_0^{1,p}(\Omega,\omega)\hookrightarrow\hookrightarrow L^q(\Omega,\sigma)$, and we obtain $T_k(u_n)$ converges to v_k strongly in $L^q(\Omega,\sigma)$ and a.e. in Ω, hence $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω.

Let $\epsilon > 0$, there exists $k(\epsilon) > 0$, such that $meas\Big((|u_n - u_m| > \delta) \cap B_R\Big) < \epsilon$, for $m, n > n_0(k(\epsilon), \delta, R)$, then $(u_n)_n$ is a Cauchy sequence in measure in B_R. Therefore, we conclude that $(u_n)_n$ converges to a measurable u a.e. in Ω. Hence, $T_k(u_n)$ converges to $T_k(u)$ weakly in $W_0^{1,p}(\Omega, \omega)$, strongly in $L^q(\Omega, \sigma)$ and a.e. in Ω.

Step 4: Intermediate inequalities.

We prove that for all φ in $W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x, u_n, \nabla\varphi)\nabla T_k(u_n - \varphi)dx \leq \int_\Omega f_n T_k(u_n - \varphi)dx + \int_\Omega F\nabla T_k(u_n - \varphi)dx. \tag{5.7}$$

For all $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$, we choose $T_k(u_n - \varphi)$ as test function in (P_n), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)dx = \int_\Omega f_n T_k(u_n - \varphi)dx + \int_\Omega F\nabla T_k(u_n - \varphi)dx. \tag{5.8}$$

We can write

$$\begin{aligned}\int_\Omega a(x, u_n, \nabla\varphi)\nabla T_k(u_n - \varphi)dx &= \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)dx \\ &- \int_\Omega \Big(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla\varphi)\Big)(\nabla u_n - \nabla\varphi)T_k'(u_n - \varphi)dx\end{aligned} \tag{5.9}$$

In view of (3.2), (5.8) and (5.9), we obtain (5.7).

Step 5: Passing to the limit.

For all $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$, we choose $T_k(u_n - \varphi)$ as test function in (P_n), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi)dx = \int_\Omega f_n T_k(u_n - \varphi)dx + \int_\Omega F\nabla T_k(u_n - \varphi)dx. \tag{5.10}$$

Let $M = k + \|\varphi\|_\infty$, since $T_M(u_n)$ converges to $T_M(u)$ weakly in $W_0^{1,p}(\Omega, \omega)$ and a.e. in Ω, then $T_k(u_n - \varphi)$ converges to $T_k(u - \varphi)$ weakly in $W_0^{1,p}(\Omega, \omega)$, a.e. in Ω, and by Lebesgue's theorem, we deduce that

$$\lim_{n\to+\infty} \int_\Omega f_n T_k(u_n - \varphi)dx = \int_\Omega f T_k(u - \varphi)dx, \tag{5.11}$$

and

$$\lim_{n\to+\infty} \int_\Omega F\nabla T_k(u_n - \varphi)dx = \int_\Omega F\nabla T_k(u - \varphi)dx. \tag{5.12}$$

We use (3.1), we deduce

$$\Big|a_i(x, T_M(u_n), \nabla\varphi)\Big|^{p'} \omega_i^* \leq \gamma\Big(k(x)^{p'} + |T_M(u_n)|^q\sigma + \sum_{i=1}^N |\frac{\partial\varphi}{\partial x_i}|^p\omega_i\Big),$$

and since $T_M(u_n)$ converges to $T_M(u)$ strongly in $L^q(\Omega, \sigma)$ and a.e. in Ω, then de obtain

$$\gamma\Big(k(x)^{p'} + |T_M(u_n)|^q\sigma + \sum_{i=1}^N |\frac{\partial\varphi}{\partial x_i}|^p\omega_i\Big) \longrightarrow \gamma\Big(k(x)^{p'} + |T_M(u)|^q\sigma + \sum_{i=1}^N |\frac{\partial\varphi}{\partial x_i}|^p\omega_i\Big), \quad \text{a.e. in } \Omega.$$

In view of Vitali theorem, we deduce that $a_i(x, T_M(u_n), \nabla\varphi)$ converges strongly to $a_i(x, T_M(u), \nabla\varphi)$ in $\prod_{i=1}^N L^{p'}(\Omega, \omega_i^*)$ for all $i = 1, ..., N$. Hence

$$\lim_{n\to\infty} \int_\Omega a(x, u_n, \nabla\varphi)\nabla T_k(u_n - \varphi)dx = \int_\Omega a(x, u, \nabla\varphi)\nabla T_k(u - \varphi)dx. \tag{5.13}$$

In view of (5.11), (5.12), (5.13) and passing to the limit in (5.7), we obtain

$$\int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)dx \le \int_\Omega fT_k(u-\varphi)dx + \int_\Omega F\nabla T_k(u-\varphi)dx. \tag{5.14}$$

Now, we can write

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)dx = \int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi)dx \tag{5.15}$$

$$+ \int_\Omega \Big(a(x,u,\nabla u)-a(x,u,\nabla\varphi)\Big)(\nabla u-\nabla\varphi)T_k'(u-\varphi)dx$$

In view of (3.2), (5.14) and (5.15), we obtain (5.1).

6. Uniqueness of an entropy solution.

In this section, we prove the following uniqueness theorem

Theorem 6.1 *Let μ be a signed measure in $L^1(\Omega)+W^{-1,p'}(\Omega,\omega^*)$. Then problem (1.1) has a unique an entropy solution.*

We prove the following lemma that will be a key point in the prove the uniqueness result

Lemma 6.1 *For any fixed $k>0$:*

$$\lim_{h\to\infty}\int_{\{h\le|u|\le h+k\}}\sum_{i=1}^N\Big|\frac{\partial u}{\partial x_i}\Big|^p\omega_i dx = 0. \tag{6.1}$$

Proof. Using $\varphi = T_h(u)$ as a test function in (1.1) leads to

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(u))dx \le \int_\Omega fT_k(u-T_h(u))dx + \int_\Omega F\nabla T_k(u-T_h(u))dx.$$

The definition of T_k permit to obtain that for any $h>0$ and any $k>0$

$$\int_{\{h\le|u|\le h+k\}} a(x,u,\nabla u)\nabla u\,dx \le k\int_{\{|u|\ge h\}}|f|\,dx + \int_{\{h\le|u|\le h+k)\}}F\nabla u\,dx.$$

The coercive character of a and the use of Young's inequality then make it possible to obtain

$$\frac{\alpha}{2}\int_{\{h\le|u|\le h+k\}}\sum_{i=1}^N\Big|\frac{\partial u}{\partial x_i}\Big|^p\omega_i dx \le k\int_{\{|u|\ge h\}}|f|dx + C\int_{\{h\le|u|\le h+k\}}\sum_{i=1}^N|F_i|^{p'}\omega_i^* dx,\ \ \forall h,k>0.$$

Now since $f\in L^1(\Omega)$ and $F\in\prod_{i=1}^N L^{p'}(\Omega,\omega_i^*)$,, it possible to conclude that

$$\lim_{h\to\infty}\int_{\{h\le|u|\le h+k\}}\sum_{i=1}^N\Big|\frac{\partial u}{\partial x_i}\Big|^p\omega_i dx = 0.$$

■

Proof of Theorem 6.1. Let $\mu = f - \operatorname{div} F$ with $f\in L^1(\Omega)$ and $F\in\prod_{i=1}^N L^{p'}(\Omega,\omega_i^*)$. We consider two entropy solutions u and v of (1.1) for the same that f and F. We plug the test function $T_k(u-T_h(v))$

in equations (1.1)(problem of u) and the test function $T_k(v-T_h(u))$ in equations (1.1) (problem of v), then, the sums of the two equations, lead to

$$\int_\Omega [a(x,u,\nabla u)-F]\nabla T_k(u-T_h(v))dx+\int_\Omega [a(x,v,\nabla v)-F]\nabla T_k(v-T_h(u))dx \le \int_\Omega f[T_k(u-T_h(v))+T_k(v-T_h(u))]\,dx. \tag{6.2}$$

It is easy to see that, if h tends to 0, we have $\int_\Omega f[T_k(u-T_h(v))+T_k(v-T_h(u))]\,dx \to 0$.
Let $C_0=\Big\{|u-v|\le k, |u|\le h, |v|\le h\Big\}$, $C_1=\Big\{|u-T_h(v)|\le k, |v|>h\Big\}$, $C_1^{'}=\Big\{|v-T_h(u)|\le k, |u|>h\Big\}$, $C_2=\Big\{|u-T_h(v)|\le k, |v|\le h, |u|>h\Big\}$ and $C_2^{'}=\Big\{|v-T_h(u)|\le k, |u|\le h, |v|>h\Big\}$, we have $\Omega=C_0\cup C_1\cup C_2=C_0\cup C_1^{'}\cup C_2^{'}$.
For C_0, we have

$$\begin{aligned}\int_{C_0} [a(x,u,\nabla u)-F]\nabla T_k(u-T_h(v))dx+\int_{C_0} [a(x,v,\nabla v)-F]\nabla T_k(v-T_h(u))dx \\ =\int_{C_0} [a(x,u,\nabla u)-F]\nabla(u-v)dx+\int_{C_0} [a(x,v,\nabla v)-F]\nabla(v-u)dx \\ =\int_{C_0} [a(x,u,\nabla u)-a(x,v,\nabla v)]\nabla(u-v)dx.\end{aligned} \tag{6.3}$$

For C_1 and $C_1^{'}$ we have

$$\int_{C_1} [a(x,u,\nabla u)-F]\nabla T_k(u-T_h(v))dx+\int_{C_1'} [a(x,v,\nabla v)-F]\nabla T_k(v-T_h(u))dx \ge -\int_{C_1} F\nabla u dx-\int_{C_1'} F\nabla v dx. \tag{6.4}$$

In view of the Hölder inequality, we obtain

$$\Big|\int_{C_1} F\nabla u dx\Big| = \Big|\int_\Omega \sum_{i=1}^N F_i\omega_i^{\frac{-1}{p}}\frac{\partial u}{\partial x_i}\omega_i^{\frac{-1}{p}}\,dx\Big| \le \Big(\int_{C_1}\sum_{i=1}^N |F_i|^p\omega_i^* dx\Big)^{\frac{1}{p'}}\Big(\int_{C_1}\sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^p\omega_i dx\Big)^{\frac{1}{p}}. \tag{6.5}$$

Due to the definition of T_k, and for any $0\le k\le h$, we have $C_1\subset\{h-k\le|u|\le h+k\}$. Since $F\in\prod_{i=1}^N L^{p'}(\Omega,\omega_i^*)$ and in view (6.1), we can deduce that $\lim_{h\to\infty}\int_{C_1} F\nabla u dx=0$, the same, we have $\lim_{h\to\infty}\int_{C_1'} F\nabla v dx=0$. In view of (6.4), we conclude that

$$\limsup_{h\to\infty}\Big\{\int_{C_1}[a(x,u,\nabla u)-F]\nabla u dx+\int_{C_1'}[a(x,v,\nabla v)-F]\nabla v dx\Big\}\ge 0. \tag{6.6}$$

For C_2 and $C_2^{'}$, we have

$$\begin{aligned}\int_{C_2} [a(x,u,\nabla u)-F]\nabla T_k(u-T_h(v))dx+\int_{C_2'} [a(x,v,\nabla v)-F]\nabla T_k(v-T_h(u))dx \\ =\int_{C_2} [a(x,u,\nabla u)-F]\nabla(u-v)dx+\int_{C_2'} [a(x,v,\nabla v)-F]\nabla(v-u)dx.\end{aligned} \tag{6.7}$$

In view of the coercive character, we obtain

$$\int_{C_2}[a(x,u,\nabla u)-F]\nabla(u-v)dx \geq -\int_{C_2}[a(x,u,\nabla u)-F]\nabla v dx-\int_{C_2}F\nabla u dx. \tag{6.8}$$

Due to the definition of T_k, we have $C_2 \subset \{h \leq |u| \leq h+k\} \cap \{h-k \leq |v| \leq h\}$ for any $0 \leq k \leq h$, in view of Hardy inequality and (3.1). It easy to see that

$$\begin{aligned}&\int_{C_2}[a(x,u,\nabla u)-F]\nabla(u-v)dx\\ &\geq -\Big(\int_{\{h\leq|u|\leq h+k\}}(\sum_{i=1}^{N}|\frac{\partial u}{\partial x_i}|^p\omega_i dx+|L|^{p'}+|F|^{p'})dx\Big)^{\frac{1}{p'}}\Big(\int_{\{h-k\leq|v|\leq h\}}\sum_{i=1}^{N}|\frac{\partial v}{\partial x_i}|^p\omega_i dx\Big)^{\frac{1}{p}}\\ &\quad-\int_{C_2}F\nabla v dx.\end{aligned}$$

Since $F \in \prod_{i=1}^{N} L^{p'}(\Omega,\omega_i^*)$ and in view of (6.1), (6.5), we deduce that

$$\lim_{h\to+\infty}\int_{C_2}[a(x,u,\nabla u)-F]\nabla(u-v)dx \geq 0, \tag{6.9}$$

and the same, we obtain

$$\lim_{h\to+\infty}\int_{C_2'}[a(x,v,\nabla v)-F]\nabla(v-u)dx \geq 0. \tag{6.10}$$

In view of (6.2), (6.3), (6.6), (6.7), (6.9) and (6.10), we conclude that

$$\lim_{h\to\infty}\int_{C_0}\Big(a(x,u,\nabla u)-a(x,v,\nabla v)\Big)\nabla(u-v)dx=0,$$

then

$$\int_{C_0}\Big(a(x,u,\nabla u)-a(x,v,\nabla v)\Big)\nabla(u-v)dx=0, \quad \forall k>0.$$

In view of (3.2), we obtain $\nabla u=\nabla v$, and then $u=v$ a.e. in Ω. ∎

References

1. A. DALL'AGLIO, *Approximated Solutions of Equations with L1 Data. Application to the H-convergence of quasilinear Parabolic Equations*, Ann. Mat. Pura Appl., 170 (1996) 207-240.

2. A. KUFNER, *Weighted Sobolev Spaces*, John Wiley and Sons, (1985).

3. B. O. TURESSON, *Nonlinear Potential, Theory and Weighted Sobolev Spaces*, Lecture Note in Mathematics, N 1736. Springer, 2000.

4. E. RAMI, A. BARBARA and E. AZROUL, *Existence of solutions for some quasilinear elliptic system with weight and measure-valued right hand side*, Afrika Matematika (2023) 34:74 https://doi.org/10.1007/s13370-023-01117-w.

5. E. RAMI, A. BARBARA and E. AZROUL, *Existence of T-p(x)-solution of a nonhomogeneous elliptic problem with right hand side measure*, J. Appl. Mathe. Phys. 9, 2717–2732 (2021).

6. G. STAMPACCHIA, *Le Problème de Direchlet pour les Equations Elliptiques du Second Ordre á Coefficients Discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1965), 189-258.

7. H. BREZIS, *Some variatinal problems of the Thomas-Fermi type, in variational inequalities and complementarity problems*, Cottle, Gianessi, and Lions eds., Wiley, New York (1980), 53-73.

8. H. BRESIZ and W. STRAUSS, *Semi-linear Second-Order Elliptic Equations in L1* J.Math. Soc. Japan, 25 N.4 (1973),565-590.

9. J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, Paris (1969).

10. L. BOCCARDO and T. GALLOUËT, *Nonlinear elliptic equations with right hand side measures*, Comm. Partial Differential Equations, 17, n.3,4 (1992), 641-655.

11. L. BOCCARDO and T. GALLOUËT, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal., 87(1989), 149-169.

12. L. BOCCARDO, T. GALLOUËT and L. ORSINA, *Existence and uniqueness of entropy solutions for non linear elliptic equations with measure data*, Ann. Inst. Henri Poincare Vol. 13,5 (1996), 539-551.

13. P. BÉNILIAN, L. BOCCARDO, T. GALLOUËT,R. GARIEPY,M. PIERRE, J.L. VAZQUEZ, *An L^1 theory for existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22(1995), 240-273.

14. P. DRABEK, A. KUFNER and F. NICOLOSI, *Non linear elliptic equations, singular and degenerated cases*, University of West Bohemia, (1996).

15. P. DRÁBEK, A. Ksc ufner and F. NICOLOSI, *Nonlinear elliptic equations, singular and degenerate cases*, University of West Bohemia, (1996).

16. P. DRÁBEK: *Strongly nonlinear degenerated and singular elliptic problems,* Nonlinear partial differential equations, Pitman Res. Notes Math. Longman, Harlow, **343**, (1996), pp. 112-146.

17. Y. AKDIM, E. AZROUL and A. BENKIRANE, *Existence solutions for Quasilinear Degenerated Elliptic Equations* , Electronic J. Diff. Equ., Vol. 2001, N 71, (2001) pp1-19.

18. Y. AKDIM, E. AZROUL and A. BENKIRANE, *Existence Results for Quasilinear Degenerated Equations via Strong Convergence Truncations*, Revista Matemtemática Complutence, Madrid,(2004),17;N2,359-379.

E. Rami (Corresponding author),

Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz,

Laboratory L2MASI, Department of Mathematics,

B.P 1796 Atlas, Fez, Morocco.

E-mail address: ramielhoucine@gmail.com

and

A. Barbara,

Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz,

Laboratory L2MASI, Department of Mathematics,

B.P 1796 Atlas, Fez, Morocco.

E-mail address: abdelkrim.barbara@usmba.ac.ma

and

E. Azroul,

Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz,

Laboratory L2MASI, Department of Mathematics,

B.P 1796 Atlas, Fez, Morocco.

E-mail address: elhoussine.azroul@usmba.ac.ma