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Existence and multiplicity of solutions for the Kirchhoff BVPs via genus theory on the half-line

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ABSTRACT: The aim of this paper is to establish the existence and multiplicity of solutions for a fourth-order boundary value problem of Kirchhoff type posed on the half-line via Krasnoselskii's genus theory.

Key Words: Genus theory, variational methods, Kirchhoff equation, (PS) condition, fourth-order BVPs.

Contents

1. Introduction

In this work, we are interested by the following problem:

$$\begin{cases} M\left(\|u\|^2\right)\left(u^{(4)}(x) - u''(x) + u(x)\right) = \lambda f(x, u(x)), \quad x \in [0, +\infty), \\ u(0) = u(+\infty) = 0, \\ u''(0) = u''(+\infty) = 0, \end{cases} \tag{1.1}$$

where f be a continuous function, $\lambda > 0$ is a numerical parameter, and the norm of u will be known later.

The problem (1) is a generalization of a model introduced by Kirchhoff [6] .

2. Preliminaries

In this section, we will present some definitions and necessary results used in the next of this work.

$L^2(0, +\infty)$ is defined by

$$L^2(0, +\infty) = \left\{u : [0, +\infty) \to \mathbb{R} \text{ is measurable}, \int_0^{+\infty} |u(t)|^2\, dt < +\infty\right\},$$

endowed with the norm

$$|u|_2 = \left(\int_0^{+\infty} |u(t)|^2\, dt\right)^{\frac{1}{2}}.$$

Proposition 2.1 *The space $(L^2([0, +\infty)), |.|_2)$ is separable, uniformly convex and reflexive; its conjugate space is L^2.*
For any $u \in L^2([0, +\infty))$ and $v \in L^2([0, +\infty))$, we have

$$\left|\int_0^{+\infty} uv\, dt\right| \le |u|_2 |v|_2.$$

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The Sobolev space $X = H_0^2(0,+\infty)$ is defined by

$$H_0^2(0,+\infty) = \Big\{u \in L^2([0,+\infty));\ u' \in L^2([0,+\infty)),\ u'' \in L^2([0,+\infty)),\ u(0)=0,\ u'(0)=0\Big\},$$

endowed with the natural norm

$$\|u\| = \left(\int_0^{+\infty} u''^2(x)\,dx + \int_0^{+\infty} u'^2(x)\,dx + \int_0^{+\infty} u^2(x)\,dx\right)^{\frac{1}{2}}.$$

Note that if $u \in H_0^2(0,+\infty)$, then $u(+\infty)=0,\ u'(+\infty)=0$, (see [1]). Let $r:[0,+\infty)\longrightarrow(0,+\infty)$ be a continuously differentiable and bounded function with $\max(|r|_{L^2},|r'|_{L^2}) < +\infty$.

We also consider the following space

$$C_{l,r}[0,+\infty) = \Big\{u \in C([0,+\infty),\mathbb{R}) : \lim_{x\to+\infty} r(x)u(x) \text{ exists}\Big\}$$

endowed with the norm

$$\|u\|_{\infty,r} = \sup_{x\in[0,+\infty)} r(x)u(x).$$

In the first stage of this section, we present some notions on the Krasnoselskii's genus theory (see [7], [3], [5], [4]) that we use in the proof of our main result. Let Y be a real Banach space, set

$$\Sigma = \{E \subset Y\setminus\{0\} : E \text{ is compact and } E=-E\}.$$

Definition 2.1 *Let $E\in\Sigma$ and $Y=\mathbb{R}$.*

The genus $\gamma(E)$ of E is defined by

$$\gamma(E) = \min\Big\{k\geqslant 1;\ \textit{there exists an odd continuous mapping } \phi : E\to\mathbb{R}^k\setminus\{0\}\Big\}, \tag{2.1}$$

If the mapping ϕ does not exist for any $k>0$, we set $\gamma(E)=\infty$.
Note also that if E is a subset, which consists of finitely many pairs of points, then $\gamma(E)=1$.
Moreover, from the Definition 2.1, $\gamma(\emptyset)=0$. A typical example of a set of genus k is a set, which is homeomorphic to a $(k-1)$ dimensional sphere S^{k-1} via an odd map.
Now, the following Kranoselskii's genus results are necessary throughout the present paper.

Theorem 2.1 *(see [5]) Let $Y=\mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega\subset\mathbb{R}^N$ with $0\in\Omega$. Then $\gamma(\partial\Omega)=N$.*

Corollary 2.1 *The genus of the unit sphere S^{N-1} of the space $\mathbb{R}^{N-1}$ is $\gamma(S^{N-1})=N$.*

Remark 2.1 *If Y is a separable infinite dimensional space with unit sphere S, then $\gamma(S)=\infty$.*

Proposition 2.2 *(See [5]) Let $A,B\in\Sigma$. Then, if there exists an odd map $f\in C(A,B)$, then $\gamma(A)\leq\gamma(B)$. Consequently, if there exists an odd homeomorphism $f:A\to B$, then $\gamma(A)=\gamma(B)$.*

Definition 2.2 *The functional I satisfies the Palais-Smale condition (PS) if for every sequence $(u_n)\subset Y$ such that*

$$|I(u_n)|\leqslant C \ \ \textit{and} \ \ I'(u_n)\to 0 \ \ \textit{as} \ \ n\to\infty,$$

then there is a subsequence of (u_n) which converges in the sense of the norm of Y.

The following result obtained by Clark in ([4]) is the main idea, which we use in the proof of Theorem 3.1.

Theorem 2.2 . *Let $I\in C^1(E,\mathbb{R})$ be a functional satisfying the Palais-Smale condition. Also suppose that:*

- *I is bounded from below and even.*
- *There is a compact set $K \in \Sigma$ such that $\gamma(K) = k$ and $\sup_{x\in K} I(x) < I(0)$.*

Then I possesses at least k pairs of distinct critical points and their corresponding critical values are less than $I(0)$.

Corollary 2.2 *([2]) The embedding $H_0^2(0,+\infty) \hookrightarrow C_{l,r}[0,+\infty)$ is continuous and compact.*

Proposition 2.3 *Let Λ be the function defined on the Banach space X by $\Lambda(u) = \|u\|^2$ and*

$$\langle \Lambda'(u), v\rangle = 2(u,v) = 2\Big(\int_0^{+\infty} u''v'' + \int_0^{+\infty} u'v' + \int_0^{+\infty} uv\Big).$$

Then we have:

1. *The functional Λ is convex.*
2. *The mapping $\Lambda' : X \longrightarrow X'$ is strictly monotone and bounded.*
3. *Λ' is of (S^+) type, namely if $u_n \rightharpoonup u$ and $\overline{\lim_{n\to+\infty}}\langle \Lambda'(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$.*
4. *Λ' is homeomorphism.*

Proof:

1. The functional Λ is convex.

   Let $u, v \in X$, and $t \in [0,1]$

   $$\begin{aligned}\Lambda\left(tu + (1-t)v\right) &= \|tu + (1-t)v\|^2\\ &\leqslant \|tu\|^2 + \|(1-t)v\|^2\\ &\leqslant t^2\|u\|^2 + (1-t)^2\|v\|^2\\ &\leqslant t\|u\|^2 + (1-t)\|v\|^2\\ &\leqslant t\Lambda(u) + (1-t)\Lambda(v),\end{aligned}$$

   it follows that Λ is convex.

2. The mapping $\Lambda' : X \longrightarrow X'$ is strictly monotone and bounded.

   Λ' is Frechet derivative of Λ, it follows that Λ' is continuous and bounded.

   For all $u, v \in X$ such that $u \neq v$

   $$\begin{aligned}\langle \Lambda'(u) - \Lambda'(v), u - v)\rangle &= 2(u, u-v) - 2(v, u-v)\\ &= 2(u-v, u-v)\\ &= 2\|u-v\|^2 > 0,\end{aligned}$$

   which mean that Λ' is strictly monotone.

3. Λ' is of (S^+) type.

   Let (u_n) be a sequence of X, such that $u_n \rightharpoonup u$ in X and $\overline{\lim_{n\to+\infty}}\langle \Lambda'(u_n), u_n - u\rangle \leqslant 0$, it suffices to show that $|(u_n, u_n - u)| \to 0$ as $n \to +\infty$ (since $u \in X'$ and $u_n - u \rightharpoonup 0$, then $|(u, u_n - u)| \to 0$, on the other hand, we have: $0 \leqslant \|u_n - u\| \leqslant |(u_n, u_n - u)| + |(u, u_n - u)|$. So, $u_n \to u$ in X).

   In view of monotonicity of Λ', we have $\langle \Lambda'(u_n) - \Lambda'(u), u_n - u\rangle \geqslant 0$, since $u_n \rightharpoonup u$ in X, it follows $\overline{\lim_{n\to+\infty}}\langle \Lambda'(u_n) - \Lambda'(u), u_n - u\rangle \to 0$ as $n \to +\infty$, so $u_n \to u$ in X.

4. Λ' is homeomorphism.

Note that the strict monotonicity of Λ' implies its injectivity. Moreover Λ' is a coercive operator. Indeed, let $u \in X$ and since X is a Hilbert space, we have:

$$\langle \Lambda'(u), u\rangle \quad = \quad 2(u,u) = 2\|u\|^2.$$

Consequently, thanks to Minty-Browder theorem ([8]), the operator Λ' is surjective and admits an inverse mapping. It suffices then to prove the continuity of $(\Lambda')^{-1}$. Let (x_n^*) be a sequence of X' such that $x_n^* \to x^*$ in X' as $n \to +\infty$, let $u_n, u \in X$ such that:

$$(\Lambda')^{-1}(x_n^*) = u_n \text{ and } (\Lambda')^{-1}(x^*) = u.$$

By coercivity of Λ', one deduce that the sequence (u_n) is bounded in the reflexive space X. For a subsequence, we have $u_n \rightharpoonup \widehat{u}$ in X, which implies that

$$\lim_{n\to+\infty} \langle \Lambda'(u_n) - \Lambda'(u), u_n - \widehat{u}\rangle = \lim_{n\to+\infty} \langle x_n^* - x^*, u_n - \widehat{u}\rangle = 0.$$

It follows from the fact that Λ' is of (S_+) type and the continuity of Λ' that $\Lambda'(u_n) \to \Lambda'(\widehat{u}) = \Lambda'(u)$ in X'. Moreover, since Λ' is an injection, we conclude that $u = \widehat{u}$.

□

Proposition 2.4 *A function $u \in X = H_0^2(0,+\infty)$ is said to be a weak solution of the problem (1.1) if*

$$M\left(\|u\|^2\right)\left(\int_0^{+\infty} (u''(x)\varphi''(x)\, dx + \int_0^{+\infty} u'(x)\varphi'(x)\, dx + \int_0^{+\infty} u(x)\varphi(x))\, dx\right) = \lambda \int_0^{+\infty} f(x,u(x))\varphi(x)\, dx, \quad \forall \varphi \in X.$$

This relation is called the weak variational formulation equivalent to the problem (1.1).

We associate to the problem (1.1) the energy functional defined by $I : X \longrightarrow \mathbb{R}$,

$$I(u) = \frac{1}{2}\widehat{M}\left(\|u\|^2\right) - \lambda \int_0^{+\infty} F(x,u)\, dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$ and $F(x,u) = \int_0^u f(x,t)\, dt$.

3. Main results and proofs

Theorem 3.1 *We assume that $M(t)$ and $f(x,t)$ satisfy the following assumptions:*

(M_1) *$M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function and satisfies the following condition:*

$$a_0 + a_1 t^\alpha \leqslant M(t) \leqslant b_0 + b_1 t^\alpha,$$

for all $t > 0$, where a_0, a_1, b_0, b_1 and α are positive constants.

(f_1) *$f : [0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that:*

$$|f(x,t)| \leqslant d_1(x) + d_2(x)|t|^{q-1},$$

for all $(x,t) \in]0,+\infty) \times \mathbb{R}$, where d_1, d_2 are positive functions such that $\frac{d_1}{r}, \frac{d_2}{r^q} \in L^1(0,+\infty)$ and $1 < q < 2\alpha + 2$.

(f_2) *f is an odd function with respect to the variable t,*

$$f(x,t) = -f(x,-t), \quad \forall (t,x) \in \mathbb{R} \times [0,+\infty),$$

and $F(x,u) = \int_0^u f(x,t)\,dt > 0$ *for every* $(x,u) \in [0,+\infty) \times \mathbb{R} - \{0\}$.

Then for any $k \in \mathbb{N}$, there exists λ_k such that when $\lambda > \lambda_k$, Problem (1.1) has at least k distinct pairs of nontrivial solutions.

Proof:

Claim 1 *: $I \in C^1(X,\mathbb{R})$.*

Step 1 *: I is well defined.*

Indeed, let $u \in X$, it follows from (f_1), (M_1) and the fact that X embed continuously in $C_{l,r}$ that

$$\begin{aligned}
|I(u)| &= \left|\frac{1}{2}\widehat{M}(\|u\|^2) - \lambda\int_0^{+\infty} F(x,u)\,dx\right| \\
&\leqslant \frac{1}{2}\widehat{M}(\|u\|^2) + \lambda\int_0^{+\infty} |F(x,u)|\,dx \\
&\leqslant \frac{1}{2}\int_0^{\|u\|^2} M(t)\,dt + \lambda\int_0^{+\infty}\int_0^u |f(x,t)|\,dt\,dx \\
&\leqslant \frac{1}{2}\int_0^{\|u\|^2}\Big(b_0 + b_1 t^\alpha\Big)\,dt + \lambda\int_0^{+\infty}\int_0^u d_1(x) + d_2(x)|t|^{q-1}\,dt\,dx \\
&\leqslant \frac{1}{2}\left[b_0\|u\|^2 + \frac{b_1}{\alpha+1}\|u\|^{2\alpha+2}\right] + \lambda\int_0^{+\infty}\left(d_1(x)u(x) + \frac{d_2(x)}{q}|u|^q\right)dx \\
&\leqslant \frac{1}{2}\left[b_0\|u\|^2 + b_1\|u\|^{2\alpha+2}\right] + \lambda\int_0^{+\infty}\frac{d_1(x)}{r(x)}r(x)|u(x)|\,dx + \lambda\int_0^{+\infty}\frac{d_2(x)}{r^q}|u|^q r^q\,dx \\
&\leqslant \frac{1}{2}\left[b_0\|u\|^2 + b_1\|u\|^{2\alpha+2}\right] + \lambda\|u\|_{\infty,r}\int_0^{+\infty}\frac{d_1(x)}{r(x)}\,dx + \lambda\|u\|^q_{\infty,r}\int_0^{+\infty}\frac{d_2(x)}{r^q}\,dx \\
&\leqslant \frac{1}{2}\left[b_0\|u\|^2 + b_1\|u\|^{2\alpha+2}\right] + \lambda c\|u\|\left|\frac{d_1}{r}\right|_{L^1} + \lambda c^q\|u\|^q\left|\frac{d_2}{r^q}\right|_{L^1} < \infty.
\end{aligned}$$

Step 2 *: I is Gâteaux-differentiable.*

Indeed, for all $v \in X$, for any small $s > 0$, we have:

$$\begin{aligned}
I(u+sv) - I(u) &= \frac{1}{2}\widehat{M}(\|u+sv\|^2) - \lambda\int_0^{+\infty} F(x,u+sv)\,dx - \frac{1}{2}\widehat{M}(\|u\|^2) + \lambda\int_0^{+\infty} F(x,u)\,dx \\
&= \frac{1}{2}\left[\widehat{M}(\|u+sv\|^2) - \widehat{M}(\|u\|^2)\right] - \lambda\int_0^{+\infty}\Big(F(x,u+sv) - F(x,u)\Big)\,dx.
\end{aligned}$$

From the mean value theorem, we obtain

$$F(x,u(x)+sv(x)) - F(x,u(x)) = sv(x)f(x,u(x)+s\theta v(x)), \quad \theta \in (0,1),$$

$$\begin{aligned}
\widehat{M}(\|u+sv\|^2) - \widehat{M}(\|u\|^2) &= \Big(\|u+sv\|^2 - \|u\|^2\Big)M(\|u+s\theta v\|^2) \\
&= \Big(2s(u,v) + s^2\|v\|^2\Big)M(\|u+s\theta v\|^2),
\end{aligned}$$

then,

$$I(u+sv)-I(u) = \frac{1}{2}\Big(2s(u,v)+s^2\|v\|^2\Big)M(\|u+s\theta v\|^2)-\lambda\int_0^{+\infty} svf(x,u+s\theta v)\,dx.$$

So,

$$\frac{I(u+sv)-I(u)}{s} = \Big((u,v)+\frac{1}{2}s\|v\|^2\Big)M(\|u+s\theta v\|^2)-\lambda\int_0^{+\infty} vf(x,u+s\theta v)\,dx.$$

For all $u,v\in X$, and for all small s, and $\theta\in(0,1)$:

$$\begin{aligned}|u(x)+s\theta v(x)|r(x) &\leqslant |u(x)|r(x)+s\theta|v(x)|r(x)\\ &\leqslant \|u\|_{\infty,r}+\|v\|_{\infty,r}\\ &\leqslant c\|u\|+c\|v\|\\ &\leqslant c\Big(\|u\|+\|v\|\Big)=R_{u,v}.\end{aligned}$$

Moreover, from (f_1) and $X\hookrightarrow C_{l,r}$ we have:

$$\begin{aligned}|f(x,u+s\theta v)v(x)| &\leqslant d_1(x)|v(x)|+d_2(x)|u+s\theta v|^{q-1}|v(x)|\\ &\leqslant \frac{d_1(x)}{r(x)}r(x)|v(x)|+\frac{d_2(x)}{(r(x))^q}|u+s\theta v|^{q-1}(r(x))^{q-1}|v(x)|r(x)\\ &\leqslant \frac{d_1(x)}{r(x)}\|v\|_{\infty,r}+\frac{d_2(x)}{(r(x))^q}R_{u,v}^{q-1}\|v\|_{\infty,r}\\ &\leqslant c\Big(\frac{d_1(x)}{r(x)}+R_{u,v}^{q-1}\frac{d_2(x)}{(r(x))^q}\Big)\|v\|\in L^1(0,+\infty).\end{aligned}$$

From the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_0^{+\infty} f(x,u+s\theta v)v\,dx=\int_0^{+\infty} f(x,u)v(x)\,dx.$$

Since M is a continuous function, then

$$\lim_{n\to+\infty}\frac{I(u+sv)-I(u)}{s} = (u,v)M(\|u\|^2)-\lambda\int_0^{+\infty} f(x,u)v(x)\,dx,$$

then,

$$I'(u).v=M(\|u\|^2)\Big(\int_0^{+\infty}u''(x)v''(x)\,dx+\int_0^{+\infty}u'(x)v'(x)\,dx+\int_0^{+\infty}u(x)v(x)\,dx\Big)-\lambda\int_0^{+\infty}f(x,u)v(x)\,dx.$$

***Step 3** : I' is continuous.*

Indeed, let $(u_n)\subset X$ such that $u_n\to u$ as $n\to+\infty$ in X; since $X\hookrightarrow C_{l,r}$ then $u_n\to u$ as $n\to+\infty$ in $C_{l,r}$, so (u_n) is bounded in $C_{l,r}$, then there exists $L>0$ such that $\|u_n\|_{\infty,r}\leqslant L$, $\forall n$. We have then

$$\begin{aligned}\left|\frac{f(x,u_n)}{r(x)}\right| &\leqslant \frac{d_1(x)}{r(x)}+\frac{d_2(x)}{(r(x))^q}|u_n(x)|^{q-1}(r(x))^{q-1}\\ &\leqslant \frac{d_1(x)}{r(x)}+\frac{d_2(x)}{(r(x))^q}\Big(r(x)|u_n(x)|\Big)^{q-1}\\ &\leqslant \frac{d_1(x)}{r(x)}+\frac{d_2(x)}{(r(x))^q}\|u_n\|_{\infty,r}^{q-1}\\ &\leqslant \frac{d_1(x)}{r(x)}+L^{q-1}\frac{d_2(x)}{(r(x))^q}\in L^1(0,+\infty).\end{aligned}$$

From the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_0^{+\infty}\frac{f(x,u_n(x))}{r(x)}\,dx=\int_0^{+\infty}\frac{f(x,u(x))}{r(x)}\,dx. \tag{3.1}$$

For all $v\in X$, we have

$$\begin{aligned}
(I'(u_n)-I'(u),v) &= M(\|u_n\|^2)(u_n,v)-\lambda\int_0^{+\infty}f(x,u_n)v(x)\,dx-M(\|u|^2)(u,v)+\lambda\int_0^{+\infty}f(x,u)v(x)\,dx\\
&= M(\|u_n\|^2)\Big(\int_0^{+\infty}u_n''(x)v''(x)\,dx+\int_0^{+\infty}u_n'(x)v'(x)\,dx+\int_0^{+\infty}u_n(x)v(x)\,dx\Big]\\
&\quad -M(\|u\|^2)\Big[\int_0^{+\infty}u''(x)v''(x)\,dx+\int_0^{+\infty}u'(x)v'(x)\,dx+\int_0^{+\infty}u(x)v(x)\,dx\Big]\\
&\quad -\lambda\int_0^{+\infty}f(x,u_n)v(x)\,dx+\lambda\int_0^{+\infty}f(x,u)v(x)\,dx\\
&= M(\|u_n\|^2)\Big[\int_0^{+\infty}u_n''(x)v''(x)\,dx+\int_0^{+\infty}u_n'(x)v'(x)\,dx+\int_0^{+\infty}u_n(x)v(x)\,dx\Big]\\
&\quad -M(\|u_n\|^2)\Big[\int_0^{+\infty}u''(x)v''(x)\,dx+\int_0^{+\infty}u'(x)v'(x)\,dx+\int_0^{+\infty}u(x)v(x)\,dx\Big]\\
&\quad +M(\|u_n\|^2)\Big[\int_0^{+\infty}u''(x)v''(x)\,dx+\int_0^{+\infty}u'(x)v'(x)\,dx+\int_0^{+\infty}u(x)v(x)\,dx\Big]\\
&\quad -M(\|u\|^2)\Big[\int_0^{+\infty}u''(x)v''(x)\,dx+\int_0^{+\infty}u'(x)v'(x)\,dx+\int_0^{+\infty}u(x)v(x)\,dx\Big]\\
&\quad -\lambda\int_0^{+\infty}\Big(f(x,u_n(x)-f(x,u(x))\Big)v(x)\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
(I'(u_n)-I'(u),v) &= M(\|u_n\|^2)\Big[\int_0^{+\infty}(u_n''-u'')v''\,dx+\int_0^{+\infty}(u_n'-u')v'\,dx+\int_0^{+\infty}(u_n-u)v\,dx\Big]\\
&\quad +\Big[M(\|u_n\|^2)-M(\|u\|^2)\Big]\Big[\int_0^{+\infty}u''v''\,dx+\int_0^{+\infty}u'v'\,dx+\int_0^{+\infty}uv\,dx\Big]\\
&\quad -\lambda\int_0^{+\infty}\frac{\Big(f(x,u_n(x)-f(x,u(x))\Big)}{r(x)}r(x)v(x)\,dx.
\end{aligned}$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Big|(I'(u_n)-I'(u),v)\Big| &\leqslant M(\|u_n\|^2)\Big[\int_0^{+\infty}\Big((u_n''-u'')^2\Big)^{\frac12}\Big(\int_0^{+\infty}v''^2\Big)^{\frac12}\\
&\quad+\int_0^{+\infty}\Big((u_n'-u')^2\Big)^{\frac12}\Big(\int_0^{+\infty}v'^2\Big)^{\frac12}+\int_0^{+\infty}\Big((u_n-u)^2\Big)^{\frac12}\Big(\int_0^{+\infty}v^2\Big)^{\frac12}\Big]\\
&\quad+\Big[M(\|u_n\|^2)-M(\|u\|^2)\Big](u,v)+\lambda\int_0^{+\infty}\frac{|f(x,u_n(x)-f(x,u(x))|}{r(x)}r(x)|v(x)|\,dx\\
&\leqslant M(\|u_n\|^2)\Big[\int_0^{+\infty}\Big((u_n''-u'')^2\Big)^{\frac12}\|v\|_{X'}+\int_0^{+\infty}\Big((u_n'-u')^2\Big)^{\frac12}\|v\|_{X'}\\
&\quad+\int_0^{+\infty}\Big((u_n-u)^2\Big)^{\frac12}\|v\|_{X'}\Big]+\Big[M(\|u_n\|^2)-M(\|u\|^2)\Big]\|u\|_X\|v\|_{X'}\\
&\quad+\lambda\int_0^{+\infty}\frac{|f(x,u_n(x)-f(x,u(x))|}{r(x)}\|v\|_{\infty,r}\,dx\\
&\leqslant M(\|u_n\|^2)\|u_n-u\|\|v\|_{X'}+\Big[M(\|u_n\|^2)-M(\|u\|^2)\Big]\|u\|_X\|v\|_{X'}\\
&\quad+\lambda c\|v\|\int_0^{+\infty}\frac{|f(x,u_n(x)-f(x,u(x))|}{r(x)}\,dx.
\end{aligned}$$

On the other side

$$\begin{aligned}
\|I'(u_n)-I'(u)\|_{X'} &= \sup_{\|v\|\leqslant 1}|(I'(u_n)-I'(u),v)|\\
&\leqslant \sup_{\|v\|\leqslant 1}M(\|u_n\|^2)\|u_n-u\|\|v\|_{X'}+\sup_{\|v\|\leqslant 1}\Big(M(\|u_n\|^2)-M(\|u\|^2)\Big)\|u\|_X\|v\|_{X'}\\
&\quad+\sup_{\|v\|\leqslant 1}\lambda c\|v\|\int_0^{+\infty}\frac{|f(x,u_n(x)-f(x,u(x))|}{r(x)}\,dx\\
&\leqslant M(\|u_n\|^2)\|u_n-u\|+\Big(M(\|u_n\|^2)-M(\|u\|^2)\Big)\|u\|\\
&\quad+\lambda c\int_0^{+\infty}\frac{|f(x,u_n(x)-f(x,u(x))|}{r(x)}\,dx.
\end{aligned}$$

Passing to the limit when $n\to+\infty$, and from (3.1), we obtain that $I''(u_n)\to I''(u)$ as $n\to+\infty$, hence I' is continuous.

Claim 2 *: I is bounded from below.*

Indeed, from (M_1) and (f_1), we have:

$$\begin{aligned}
I(u) &= \frac12\widehat{M}(\|u\|^2)-\int_0^{+\infty}F(x,u(x))\,dx\\
&\geqslant \frac12\int_0^{\|u\|^2}(a_0+a_1t^\alpha)\,dt-\lambda\int_0^{+\infty}\Big(d_1(x)|u(x)|+d_2(x)\frac{|u(x)|^q}{q}\,dx\\
&\geqslant \frac12\Big[a_0\|u\|^2+\frac{a_1}{\alpha+1}\|u\|^{2\alpha+2}\Big]-\lambda\int_0^{+\infty}\frac{d_1(x)}{r(x)}r(x)|u(x)|-\frac{\lambda}{q}\int_0^{+\infty}\frac{d_2(x)}{r^q(x)}|r(x)u(x)|^q\,dx\\
&\geqslant \frac12\Big[a_0\|u\|^2+\frac{a_1}{\alpha+1}\|u\|^{2\alpha+2}\Big]-\lambda\|u\|_{\infty,r}\int_0^{+\infty}\frac{d_1(x)}{r(x)}\,dx-\lambda\frac{\|u\|^q_{\infty,r}}{q}\int_0^{+\infty}\frac{d_2(x)}{r^q(x)}\,dx\\
&\geqslant \frac12\Big[a_0\|u\|^2+\frac{a_1}{\alpha+1}\|u\|^{2\alpha+2}\Big]-\lambda c\Big|\frac{d_1}{r}\Big|_{L^1}\|u\|-\lambda\frac{c^q}{q}\Big|\frac{d_2}{r^q}\Big|_{L^1}\|u\|^q. \qquad (3.2)
\end{aligned}$$

As $2\alpha+2>q$, I is bounded from below.

Claim 3 *: I satisfies the (PS) condition.*

Indeed, assume that $(u_n) \subset X$ is a sequence such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad as \quad n \to +\infty. \tag{3.3}$$

From (3.3), we have $|I(u_n)| \leqslant d_3$. This fact, combined with (3.2), imply that

$$d_3 \geqslant I(u_n) \geqslant \frac{1}{2}\Big[a_0\|u\|^2 + \frac{a_1}{\alpha+1}\|u\|^{2\alpha+2}\Big] - \lambda c\Big|\frac{d_1}{r}\Big|_{L^1}\|u\| - \lambda\frac{c^q}{q}\Big|\frac{d_2}{r^q}\Big|_{L^1}\|u\|^q.$$

Since $2\alpha+2 > q$ and I is coercive, then (u_n) is bounded in X. Hence, there exists a subsequence of (u_n) still denoted by (u_n) such that (u_n) converges weakly to some u in X. Since $X \hookrightarrow\hookrightarrow C_{l,r}$, then (u_n) converges to u in $C_{l,r}[0,+\infty)$ i.e. $u_n \to u$ in $C_{l,r}[0,+\infty)$. Then by (3.3), we have $\langle I'(u_n), u_n - u\rangle \to 0$ as $n \to +\infty$. Thus,

$$\begin{aligned}\langle I'(u_n), u_n - u\rangle &= M(\|u_n\|^2)\Big(\int_0^{+\infty} u_n''(u_n'' - u'') + \int_0^{+\infty} u_n'(u_n' - u') + \int_0^{+\infty} u_n(u_n - u)\Big)\\ &\quad -\lambda\int_0^{+\infty} f(x,u_n)(u_n - u)\,dx \to 0.\end{aligned}$$

By (f_1), we get

$$\begin{aligned}\Big|\int_0^{+\infty} f(x,u_n)(u_n-u)\,dx\Big| &\leqslant \int_0^{+\infty}\Big(d_1(x)|u_n-u| + d_2(x)|u_n|^{q-1}|u_n-u|\Big)\,dx\\ &\leqslant \int_0^{+\infty}\Big(\frac{d_1(x)}{r(x)}r(x)|u_n-u| + \frac{d_2(x)}{(r(x))^q}|r(x)u_n(x)|^{q-1}|r(x)(u_n-u)|\Big)\,dx\\ &\leqslant \Big(\Big|\frac{d_1}{r}\Big|_{L^1} + \|u_n\|_{\infty,r}^{q-1}\Big|\frac{d_2}{r^q}\Big|_{L^1}\Big)\|u_n-u\|_{\infty,r}\\ &\leqslant \Big(\Big|\frac{d_1}{r}\Big|_{L^1} + c^{q-1}\|u_n\|^{q-1}\Big|\frac{d_2}{r^q}\Big|_{L^1}\Big)\|u_n-u\|_{\infty,r}.\end{aligned}$$

Since (u_n) is bounded in X and (u_n) converges strongly to u in $C_{l,r}[0,+\infty)$, we obtain

$$\int_0^{+\infty} f(x,u_n)(u_n-u)\,dx \to 0 \quad as \quad n \to +\infty.$$

Hence

$$M(\|u_n\|^2)\Big(\int_0^{+\infty} u_n''(u_n''-u'') + \int_0^{+\infty} u_n'(u_n'-u') + \int_0^{+\infty} u_n(u_n-u)\Big) \to 0,$$

as $n \longrightarrow +\infty$. From (M_1), it follows

$$\int_0^{+\infty} u_n''(u_n''-u'') + \int_0^{+\infty} u_n'(u_n'-u') + \int_0^{+\infty} u_n(u_n-u) \longrightarrow 0.$$

By Proposition (2.3), we get that $u_n \to u$ in X.

Claim 4 *: Consider $\{e_1, e_2, ...\}$, a Schauder basis of the space $H_0^2(0,+\infty)$ (see [1]), and for each $k \in \mathbb{N}$, consider X_k, the subset of $H_0^2(0,+\infty)$ generated by the k vectors $\{e_1, e_2, ..., e_k\}$. Clearly X_k is a subspace of $H_0^2(0,+\infty)$.*

Consider, for $\rho > 0$

$$K_k(\rho) = \Big\{u \in X_k : \|u\|_X^2 = \sum_{i=1}^k \xi_i^2 = \rho^2\Big\}.$$

For any $\rho > 0$, we consider the odd homeomorphism $\chi : K_k(\rho) \to S^{k-1}$ defined by $\chi(u) = (\xi_1, \xi_2, ..., \xi_k)$, where S^{k-1} is the sphere in $\mathbb{R}^k$. From Theorem 2.1 and Proposition 2.2, we conclude that $\gamma(K_k(\rho)) = k$.

It follows from hypothesis (f_1) that $\int_0^{+\infty} F(x, u(x))\, dx > 0$ for any $u \in K_k(\rho)$. Then $\mu_k = \inf_{u \in K_k(\rho)} \int_0^{+\infty} F(x, u(x))\, dx$ is strictly positive.

Let $\lambda_k = \frac{1}{2\mu_k}\left(b_0\rho^2 + b_1\rho^{2\alpha+2}\right)$, and note that $\lambda_k > 0$. When $\lambda > \lambda_k$ then for any $u \in K_k(\rho)$ we have

$$\begin{aligned} I(u) &= \frac{1}{2}\widehat{M}(\|u\|^2) - \lambda \int_0^{+\infty} F(x, u(x))\, dx \\ &= \frac{1}{2}\int_0^{\|u\|^2} M(s)\, ds - \lambda \int_0^{+\infty} F(x, u(x))\, dx \\ &\leqslant \frac{1}{2}\left[b_0\|u\|^2 + b_1\|u\|^{2\alpha+2}\right] - \lambda\mu_k \\ &< \frac{1}{2}\left[b_0\rho^2 + b_1\rho^{2\alpha+2}\right] - \lambda_k\mu_k = 0, \end{aligned}$$

which implies that

$$\sup_{K_k(\rho)} I(u) < 0 = I(0).$$

Thanks to Theorem 2.2, I has at least k distinct pairs of nontrivial solutions, then, Problem (1.1) has infinitely of nontrivial solutions.

□

4. Example

We consider the problem

$$\begin{cases} M\left(\|u\|^2\right)\left(u^{(4)}(x) - u''(x) + u(x)\right) = \dfrac{e^{-x}}{x^2+1} u|u|^{q-2}, \quad x \in [0, +\infty), \\ u(0) = u(+\infty) = 0, \\ u''(0) = u''(+\infty) = 0, \end{cases} \tag{4.1}$$

where $M(t) = 1 + t^\alpha$, $\lambda = 1$, and $q < 2\alpha + 2$. We put $f(x,t) = \frac{e^{-x}}{x^2+1} t|t|^{q-2}$, $d_1(x) = \frac{1}{1+x^4}$, $d_2(x) = e^{-x}$ and $r(x) = \frac{1}{1+x}$.

It is easy to see that conditions $(f_1), (f_2)$ and (M_1) hold. So by Theorem 3.1, (4.1) has infinitely of nontrivial solutions.

References

1. H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.
2. M. Briki, T. Moussaoui, D. Oregan, *Existence of solutions for a fourth-order boundary value problem on the half-line via critical point theory*, Electronic Journal of Qualitative Theory of Differential Equations, 24(2016), 1-11.
3. K.C. Chang. *Critical Point Theory and Applications, Shanghai Scientific and Technology Press*, Shanghai, (1986).
4. D.C. Clarke, *A variant of the Lusternik-Schnirelman theory*, Indiana Univ. Math. J., (22)1972, 65-74.
5. O. Kavian, *Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques*, Springer-Verlag, 1993.
6. G. Kirchhoff, *Mechanik*, Teubner, Leipzig., 1883.
7. M.A. Krasnoselskii, *Methods in the Theory of Nonlinear Integral equations*, MacMillan, New York, 1964.

8. E. Zeidler, *Nonlinear Function Analysis and its Applications*, vol. II/B: Nonlinear Monotone Operators, Springer, New York, 1990.

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