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Contraction Principle in Bicomplex Valued G-Metric Spaces And Its Application in the System of Linear Equations

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ABSTRACT: In this paper, we established bicomplex valued G-metric spaces and introduced the notion of G-Banach Contraction mapping, G-Kannan mapping, G-Bianchini Contraction mapping and prove fixed point theorems in such spaces. We also provide some non-trivial examples and applications of this theory part to demonstrate the validity of our proven results.

Key Words: Fixed point, Bicomplex valued metric spaces, G-Banach contraction, G-Kannan mapping, G-Bianchini mapping.

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1. Introduction and Preliminaries

Fixed point theory has been an intriguing area of study over the past five decades. The Banach contraction principle, formulated by Banach in 1922 [24], can be articulated as follows:

(i) In a metric space (X, d), a mapping $T : X \to X$ is termed as a Banach contraction if there is a value $k \in [0, 1)$ such that the inequality

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$.

In the year 1969, Kannan proposed the concept of Kannan mapping in the following way:

(ii) A mapping T is referred to as a Kannan mapping if there is a value r in the range of 0 to $\frac{1}{2}$ such that the inequality

$$d(Tx, Ty) \leq r\,(d(x, Tx) + d(y, Ty))$$

holds for all $x, y \in X$.

With (X, d) being complete metric spaces and at atleast one of (i) and (ii) holding, T has a unique fixed point(UFP in shortly) on X.

Next, we engage in the evolution of spaces. The rigid research has focused on the investigation of fixed points that meet particular contraction mapping requirements. At first Metric space was generalized and given a new name, b-metric space, by Bakhtin [14]. In summary, there exist numerous other generalized b-metric spaces, including dislocated quasi-b-metric spaces, quasi-b-metric spaces, b-metric like spaces, and quasi-b-metric-like spaces. The concept of metric space was later expanded upon and a 2-metric space was introduced by Ghaler [26]. The concept of the Ghaler results was generalized in 1992 by Dhage [3], who introduced a new class of generalized metrics known as D-metrics. Mustafa and Sims [30], [29] have demonstrated that the majority of the findings pertaining to Dhage's D-metric spaces(

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[3]- [6]) are unreasonable, so they developed what they termed G-metric spaces, an enhanced form of the generalized metric space structure. The following is how Mustafa and Sims [29] presented the idea of G-metric spaces in 2006:

Definition 1.1 [29] Assume that $X \neq \emptyset$ be any random set, and let $\wp_0 : X^3 \to \mathbb{R}^+$ be a function that fulfills the following axioms:

(G_1) $\wp_0(p, q, r) = 0$ iff $p = q = r$,

(G_2) $0 < \wp_0(x, x, y)$, for all $p, q \in X$, with $p \neq q$,

(G_3) $\wp_0(p, p, q) \leq \wp_0(p, q, r)$, for all $p, q, r \in X$, with $r \neq q$,

(G_4) $\wp_0(p, q, r) = \wp_0(p, r, q) = \wp_0(q, r, p) = \cdots$ (symmetry in all three variables),

(G_5) $\wp_0(p, q, r) \leq \wp_0(p, a, a) + \wp_0(a, q, r)$, for all $p, q, r, a \in X$, (rectangle inequality).

Then the mapping $\wp_0$ is referred to as a generalized metric, or more specifically a G-metric on X and the pair $(X, \wp_0)$ is called a G-metric space.

Furthermore, the authors discuss D-metric spaces and provide instances that refute many of the fundamental assertions made about the topological structure of these spaces, thereby invalidating a large number of results made about them. Additionally, a large number of authors investigated numerous fixed point theorems for mappings on complete G-metric spaces that satisfy different contractive conditions; see ([17]- [20]).

Initially, Azam et al. [2] introduced the idea of complex metric spaces ($CVMS$ in shortly) to capitalize on the concepts of complex valued normed spaces and complex valued Hilbert spaces. The following is how Kang et al. [25] introduced the idea of complex-valued G-metric spaces($CVGMS$ in shortly):

Definition 1.2 [25] Assume that $X \neq \emptyset$ be any random set, and let $\wp_1 : X^3 \to \mathbb{C}_{\not{k}}$ be a function satisfying the following axioms:

(CG_1) $\wp_1(p, q, r) = 0$ iff $p = q = r$,

(CG_2) $0 \prec \wp_1(p, p, q)$, for all $p, q \in X$, with $p \neq q$,

(CG_3) $\wp_1(p, p, q) \preceq \wp_1(p, q, r)$, for all $p, q, r \in X$, with $r \neq q$,

(CG_4) $\wp_1(p, q, r) = \wp_1(p, r, q) = \wp_1(q, r, p) = \cdots$ (symmetry in all three variables),

(CG_5) $\wp_1(p, q, r) \preceq \wp_1(p, a, a) + \wp_1(a, q, r)$, for all $p, q, r, a \in X$.

The mapping $\wp_1$ is referred to as a complex valued generalized metric, or more precisely, a $CVGM$ on X. The pair $(X, \wp_1)$ is referred to as $CVGMS$. This definition is more inclusive than the findings of Mustafa and Sims and Azam et al.: see ([2], [30]).

Segre [7] initiated an attempt to construct special algebras in a novel way towards the end of the 18th century. As elements of an infinite set of algebras, he proposed bicomplex numbers, tricomplex numbers, and other commutative generalizations of complex numbers. Later, in the 1930s, other academics made significant developments in this area (refer to references [13], [21], [15]). Subsequently, Price [13] pioneered bicomplex algebra and function theory. This subject is highly relevant in a number of mathematical science fields, in addition to other fields in science and technology that have seen a recent upsurge in interest. A noteworthy investigation into the basic functions of bicomplex numbers was carried out by Luna-Elizarrarás et al. [21].

Subsequently, we delineate fundamental concepts and symbols for subsequent use. Here we define $\mathbb{C}_0, \mathbb{C}_1$, and $\mathbb{C}_2$, respectively, to represent the set of real, complex, and bicomplex numbers.

Research Motivation behind this work: We discovered after the literature review that more research was necessary for the bicomplex metric space theory. Our findings extend the scope of these investigations. It came out that there was no previous literature describing the $BVGMS$. Therefore, we first establish the $BVGMS$ and then show some fixed point results for popular contraction mappings. Additionally, we solve some unique fixed point (UFP in short) theorems in this new bicomplex valued generalized metric space.

Bicomplex Number: The bicomplex number, as stated by Segre [7], is:

$$\kappa = t_1 + t_2 i_1 + t_3 i_2 + t_4 i_1 i_2,$$

where $t_1, t_2, t_3, t_4 \in \mathbb{C}_0$, and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$, we denote the collection of bicomplex numbers $\mathbb{C}_2$ as follows:

$$\mathbb{C}_2 = \{\kappa : \kappa = t_1 + t_2 i_1 + t_3 i_2 + t_4 i_1 i_2, t_1, t_2, t_3, t_4 \in \mathbb{C}_0\},$$

i.e.,

$$\mathbb{C}_2 = \{\kappa : \kappa = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = t_1 + t_2 i_1 \in \mathbb{C}_1$ and $z_2 = t_3 + t_4 i_1 \in \mathbb{C}_1$.
In $\mathbb{C}_2$, there are four idempotent elements: $0, 1; \delta_1 = \frac{1+i_1 i_2}{2}$; and $\delta_2 = \frac{1-i_1 i_2}{2}$ out of which the nontrivial components δ_1 and δ_2, such that $\delta_1 + \delta_2 = 1$ and $\delta_1 \delta_2 = 0$. One unique way to express every bicomplex number $z_1 + i_2 z_2$ is as the sum of δ_1 and δ_2, specifically

$$\kappa = z_1 + i_2 z_2 = (z_1 - i_1 z_2)\,\delta_1 + (z_1 + i_1 z_2)\,\delta_2.$$

The idempotent illustration of bicomplex numbers and the complex coefficients $\kappa_1 = (z_1 - i_1 z_2)$ are the names given to this representation of κ. The bicomplex numbers κ have idempotent components, denoted by and $\kappa_2 = (z_1 + i_1 z_2)$.
The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a non-negative real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$is defined in [16], by

$$\begin{aligned}\|\kappa\| = \|z_1 + i_2 z_2\| &= \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(t_1^2 + t_2^2 + t_3^2 + t_4^2\right)^{\frac{1}{2}},\end{aligned}$$

where $\kappa = t_1 + t_2 i_1 + t_3 i_2 + t_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.
Since $\mathbb{C}_2$ is complete and the linear space $\mathbb{C}_2$ is a norm linear space with respect to the specified norm, $\mathbb{C}_2$ is the Banach space.
The definition of $\precsim_{i_2}$, the partial order relation on $\mathbb{C}_2$, is given by Choi et al. [16], as follows:
If $\mathbb{C}_2$ is the collection of bicomplex numbers, and $\kappa = h_1 + i_2 h_2, \aleph = k_1 + i_2 k_2 \in \mathbb{C}_2$, then $\kappa \precsim_{i_2} \aleph$ iff $h_1 \precsim k_1$ and $h_2 \precsim k_2$,
that is, $\kappa \precsim_{i_2} \aleph$, if any of the subsequent circumstances holds true:

(1) $h_1 = k_1, h_2 = k_2$,
(2) $h_1 \prec k_1, h_2 = k_2$,
(3) $h_1 = k_1, h_2 \prec k_2$, and
(4) $h_1 \prec k_1, h_2 \prec k_2$.

Specifically, if $\kappa \precsim_{i_2} \aleph$ and $\kappa \neq \aleph$, that is, if one of (2), (3), and (4) is satisfied, It is possible to write $\kappa \lesssim_{i_2} \aleph$; if only (4) is satisfied, It is possible to write $\kappa \prec_{i_2} \aleph$.
According to the Choi et al. [16], for any two bicomplex numbers $\kappa, \aleph \in \mathbb{C}_2$ the following characteristics of the norm on the bicomplex algrebra must be observed:

(i) $\kappa \precsim_{i_2} \aleph \Rightarrow \|\kappa\| \leq \|\aleph\|$,
(ii) $\|\kappa + \aleph\| \leq \|\kappa\| + \|\aleph\|$,
(iii) $\|a\kappa\| = a\|\kappa\|$, where $a > 0$ is any real number,
(iv) $\|\kappa\aleph\| \leq \sqrt{2}\|\kappa\|\|\aleph\|$, and the equality is only true in the case that $\aleph$ and κ are at least partially degenerated,
(v) $\left\|\kappa^{-1}\right\| = \|\kappa\|^{-1}$ With $0 \prec \kappa$, where κ is a degenerated bicomplex number,
(vi) $\left\|\frac{\kappa}{\aleph}\right\| = \frac{\|\kappa\|}{\|\aleph\|}$, if the bicomplex number $\aleph$ is degenerated.

In the year 2009 Choi et al. [16] defined the bicomplex valued metric spaces as:

Definition 1.3 [16] Assume $\bar{P} \neq \emptyset$ be any set. Consider the mapping $\wp_2 : \bar{P} \times \bar{P} \to \mathbb{C}_2$ satisfies the following conditions:

$(BC1)$ $0 \precsim_{i_2} \wp_2(p,q)$ for all $p,q \in \bar{P}$ (positivity),

$(BC2)$ $\wp_2(p,q) = 0$ iff $p = q$,

$(BC3)$ $\wp_2(p,q) = \wp_2(q,p)$ for all $p,q \in \bar{P}$ (symmetry), and

$(BC4)$ $\wp_2(p,q) \precsim_{i_2} \wp_2(p,z) + \wp_2(z,q)$ for all $p,q,z \in \bar{P}$ (triangle inequility).

Then the pair $(\bar{P}, \wp_2)$ is called the bicomplex valued metric spaces.

Example 1.1 *[15] Consider $\bar{P} = \left\{0, \frac{1}{2}, 2\right\}$, define a bicomplex valued metric $\wp_2 : \bar{P} \times \bar{P} \to \mathbb{C}_2$ by $\wp_2(p,q) = (1+i_2)\,|p-q|, \forall p,q \in \bar{P}$.*
Mentioned the definition (1.3) of $\wp_2$ given above, it is simple to ensure that $(\bar{P}, \wp_2)$ is a bicomplex valued metric space.

2. Main results

In this part, we start with presenting the idea of a bicomplex valued G-metric space. Afterward, we establish well-known fixed point results in $BVGMS$ and provide a few non-trivial instances.

Definition 2.1 Consider a non-empty set P, and let $\wp_2 : P^3 \to \mathbb{C}_2$ be a function that fulfills the following axioms:

(BCG_1) $\wp_2(p,q,r) = 0$ iff $p = q = r$,

(BCG_2) $0 \prec_{i_2} \wp_2(p,p,q)$, for all $p,q \in P$, with $p \neq q$,

(BCG_3) $\wp_2(p,p,q) \precsim_{i_2} \wp_2(p,q,r)$, for all $p,q,r \in P$, with $r \neq q$,

(BCG_4) $\wp_2(p,q,r) = \wp_2(p,r,q) = \wp_2(q,r,p) = \cdots$ (symmetry in all three variables),

(BCG) $\wp_2(p,q,r) \precsim_{i_2} \wp_2(p,a,a) + \wp_2(a,q,r)$, for all $p,q,r,a \in P$, (rectangle inequality).

Then the mapping $\wp_2$ is referred to as a bicomplex valued generalized metric or, more specifically a bicomplex valued G-metric on P and the pair $(P, \wp_2)$ is called a bicomplex valued G-metric space.

Lemma 2.1 *[10] Let α, β be positive real numbers. Then*

$$(\alpha + \beta)^s \leq 2^{s-1}\left(\alpha^s + \beta^s\right)$$

for all $s \in \mathbb{N}$.

Example 2.1 *Consider $P = \mathbb{Q}^+ \cup \{0\}$ be a real valud set. Defined a metric $\wp_2 : P^3 \to \mathbb{C}_2$ such that*

$$\wp_2(\Theta_1, \Theta_2, \Theta_3) = (|\Theta_1 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_1 - \Theta_3|) + i_2(|\Theta_1 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_1 - \Theta_3|).$$

Now we aim to demonstrate that $(P, \wp_2)$ forms a bicomplex valued G-metric space. For every $\Theta_1, \Theta_2, \Theta_3$, $\alpha \in P$, we get
$(BCG1)$ *If $\wp_2(\Theta_1, \Theta_2, \Theta_3) = 0$, than*

$$\wp_2(\Theta_1, \Theta_2, \Theta_3) = (|\Theta_1 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_1 - \Theta_3|) + i_2(|\Theta_1 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_1 - \Theta_3|) = 0$$

Thus $(|\Theta_1 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_1 - \Theta_3|) = 0$.Hence $\Theta_1 = \Theta_2 = \Theta_3$.
Again, suppose that $\Theta_1 = \Theta_2 = \Theta_3$, then

$$\wp_2(\Theta_1, \Theta_2, \Theta_3) = (|\Theta_1 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_1 - \Theta_3|) + i_2(|\Theta_1 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_1 - \Theta_3|) = 0.$$

$(BCG2)$ *Assumes that $\Theta_2 \neq \Theta_3$. Then, we have*

$$\begin{aligned} 0 &\prec_{i_2} |\Theta_2 - \Theta_3| + i_2|\Theta_2 - \Theta_3| \\ &\prec_{i_2} (|\Theta_2 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_3 - \Theta_2|) + i_2(|\Theta_2 - \Theta_2| + |\Theta_2 - \Theta_3| + |\Theta_3 - \Theta_2|) \\ &= \wp_2(\Theta_2, \Theta_2, \Theta_3) \end{aligned}$$

$(BCG3)$ *Since* $|\Theta_1-\Theta_2| \leq |\Theta_1-\Theta_3|+|\Theta_3-\Theta_2|$, *we have*

$$\begin{aligned}\wp_2(\Theta_1,\Theta_1,y) &= (|\Theta_1-\Theta_1|+|\Theta_1-\Theta_2|+|\Theta_1-\Theta_2|)+i_2(|\Theta_1-\Theta_1|+|\Theta_1-\Theta_2|+|\Theta_1-\Theta_2|)\\ &= (|\Theta_1-\Theta_2|+|\Theta_1-\Theta_2|)+i_2(|\Theta_1-\Theta_2|+|\Theta_1-\Theta_2|)\\ &\precsim_{i_2} (|\Theta_1-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_1-\Theta_3|)+i_2(|\Theta_1-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_1-\Theta_3|)\\ &= \wp_2(\Theta_1,\Theta_2,\Theta_3).\end{aligned}$$

$(BCG4)$ *It is easy to see that* $\wp_2(\Theta_1,\Theta_2,\Theta_3)=\wp_2(\pi\{\Theta_1,\Theta_3,y\})$, *where* π *is a permutation.*
$(BCG5)$ *By Lemma(2.1), we get*

$$\begin{aligned}&\wp_2(\Theta_1,\Theta_2,\Theta_3)\\ =&(|\Theta_1-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_1-\Theta_3|)+i_2(|\Theta_1-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_1-\Theta_3|)\\ \precsim_{i_2}&(|\Theta_1-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_1-\Theta_3|)^s+i_2(|\Theta_1-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_1-\Theta_3|)^s\\ \precsim_{i_2}&(|\Theta_1-\alpha|+|\alpha-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_3-\alpha|+|\alpha-\Theta_3|)^s\\ &+i_2(|\Theta_1-\alpha|+|\alpha-\Theta_2|+|\Theta_2-\Theta_3|+|\Theta_3-\alpha|+|\alpha-\Theta_3|)^s\\ \precsim_{i_2}&2^{\Theta_1-1}\left((|\Theta_1-\alpha|+|\Theta_1-\alpha|)^s+(|\alpha-\Theta_2|+|\Theta_2-\Theta_3|+|\alpha-\Theta_3|)^s\right)\\ &+2^{\Theta_1-1}i_2\left((|\Theta_1-\alpha|+|\Theta_1-\alpha|)^s+(|\alpha-\Theta_2|+|\Theta_2-\Theta_3|+|\alpha-\Theta_3|)^s\right)\\ =&2^{\Theta_1-1}\left((|\Theta_1-\alpha|+|\alpha-\alpha|+|\Theta_1-\alpha|)^s+i_2(|\Theta_1-\alpha|+|\alpha-\alpha|+|\Theta_1-\alpha|)^s\right)\\ &+2^{\Theta_1-1}\left((|\alpha-\Theta_2|+|\Theta_2-\Theta_3|+|\alpha-\Theta_3|)^s+\alpha i_2(|\alpha-\Theta_2|+|\Theta_2-\Theta_3|+|\alpha-\Theta_3|)^s\right)\\ =&2^{s-1}(\wp_2(\Theta_1,\alpha,\alpha)+\wp_2(\alpha,\Theta_2,\Theta_3))\end{aligned}$$

Now taking $s=1$, *then we get*

$$\wp_2(\Theta_1,\Theta_2,\Theta_3)\precsim_{i_2}\wp_2(\Theta_1,\alpha,\alpha)+\wp_2(\alpha,\Theta_2,\Theta_3)\ \ \textit{for all}\ \ \Theta_1,\Theta_2,\Theta_3,\alpha\in P.$$

Hence $(P,\wp_2)$ *is a* $BVGMS$.

Proposition 2.1 *Assume that* $(X,\wp_2)$ *be a bicomplex valued G-metric space. Then for any* $p,q,r\in X$
(1) $\wp_2(p,q,r)\precsim_{i_2}\wp_2(p,p,q)+\wp_2(p,p,r)$,
(2) $\wp_2(p,q,q)\precsim_{i_2}2\wp_2(q,p,q)$.

Definition 2.2 Consider $(X,\wp_2)$ be a $BVGMS$. let $\{\Theta_n\}$ be a sequence in X, we say that $\{\Theta_n\}$ is referred to as G-convergent and converges to z if for every $c\in\mathbb{C}_2$ with $0\precsim_{i_2}c$, $\exists$ a natural number V such that $\wp_2(z,\Theta_n,\Theta_m)\precsim_{i_2}c$ for all $n,m\geq V$. $\Theta_n\to z$ is written mathematically, and z is referred to as the limit of the sequence $\{\Theta_n\}$.

Lemma 2.2 *Assume that* $(X,\wp_2)$ *be a* $BVGMS$ *and* $\{\Theta_n\}$ *be a sequence in* X. *Then* $\{\Theta_n\}$ *is G-convergent to* z *iff* $\|\wp_2(z,\Theta_n,\Theta_m)\|\to 0$ *as* $n,m\to\infty$.

Definition 2.3 Assume that $(X,\wp_2)$ be a $BVGMS$, Then a sequence $\{\Theta_n\}$ is called G-Cauchy if for every $0\precsim_{i_2}c\in\mathbb{C}_2$, $\exists\ k\in\mathbb{N}$ such that $\wp_2(\Theta_n,\Theta_m,\Theta_l)\precsim_{i_2}$ c for all $n,m,l\geq k$.

Lemma 2.3 *Assume that* $(X,\wp_2)$ *be a* $BVGMS$ *and* $\{\Theta_n\}$ *be a sequence in* X. *Then* $\{\Theta_n\}$ *is G-Cauchy sequence iff*

$$\|\wp_2(\Theta_n,\Theta_m,\Theta_l)\|\to 0\ \textit{as}\ n,m,l\to\infty.$$

Definition 2.4 A $BVGMS$ $(X,\wp_2)$ is referred to as G-complete if every G-Cauchy sequence is G-convergent in X.

Definition 2.5 Assume that $X\neq\emptyset$ be a subsets of G-complete $BVGMS$ $(X,\wp_2)$. A self-mapping $\eth$ on X, is referred to as a G-Banach Contraction mappings and if $\exists$ a real valued number $k\in[0,1)$ such that

$$\wp_2(\eth p,\eth q,\eth r)\precsim_{i_2}k\wp_2(p,q,r)$$

for all $p,q,r\in X$ and $s\geq 1$ and $sk\leq 1$.

Theorem 2.1 *Assume that $(X, \wp_2)$ be a G-complete BVGMS. Assume $\eth : X \to X$ be a G-Banach contraction mappings, i.e.,*

$$\wp_2(\eth p, \eth q, \eth r) \precsim_{i_2} k\wp_2(p, q, r) \tag{2.1}$$

for all $p, q, r \in X$, where $k \in [0, 1)$. Then $\eth$ has a UFP on X.

Proof: Assume that $\eth$ satisfies condition (2.1). Let $\Theta_0 \in X$ be any arbitrary point, and define the sequence $\{\Theta_n\}$ by $\Theta_n = \eth^n \Theta_0$. Then by (2.1), we have

$$\wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1}) \precsim_{i_2} k\wp_2(\Theta_{n-1}, \Theta_n, \Theta_n) \tag{2.2}$$

Again by (2.1), we have

$$\wp_2(\Theta_{n-1}, \Theta_n, \Theta_n) \precsim_{i_2} k\wp_2(\Theta_{n-2}, \Theta_{n-1}, \Theta_{n-1})$$

Then from (2.2), we have

$$\wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1}) \precsim_{i_2} k^2\wp_2(\Theta_{n-2}, \Theta_{n-1}, \Theta_{n-1})$$

Continuing in the same manner, we obtain

$$\wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1}) \precsim_{i_2} k^n\wp_2(\Theta_0, \Theta_1, \Theta_1) \tag{2.3}$$

For any natural numbers n and m where $m > n$, we can use equation (BCG5) and (2.3) to obtain the following:

$$\begin{aligned}\wp_2(\Theta_n, \Theta_m, \Theta_m) &\precsim_{i_2} \wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1}) + \wp_2(\Theta_{n+1}, \Theta_{n+2}, \Theta_{n+2}) \\ &+ \wp_2(\Theta_{n+2}, \Theta_{n+3}, \Theta_{n+3}) + \cdots + \wp_2(\Theta_{m-1}, \Theta_m, \Theta_m) \\ &\precsim_{i_2} \left(k^n + k^{n+1} + k^{n+2} + \cdots + k^{m-1}\right)\wp_2(\Theta_0, \Theta_1, \Theta_1) \\ &\precsim_{i_2} \frac{k^n}{1-k}\wp_2(\Theta_0, \Theta_1, \Theta_1).\end{aligned}$$

Therefore, we have

$$\|\wp_2(\Theta_n, \Theta_m, \Theta_m)\| \leq \frac{k^n}{1-k}\|\wp_2(\Theta_0, \Theta_1, \Theta_1)\|$$

Since $k \in [0, 1)$ if we take limits as $n \to \infty$, then $\frac{k^n}{1-k}\|\wp_2(\Theta_0, \Theta_1, \Theta_1)\| \to 0$, i.e., $\|\wp_2(\Theta_n, \Theta_m, \Theta_m)\| \to 0$. For $n, m, l \in \mathbb{N}$, From lemma (1.1), we obtain

$$\wp_2(\Theta_n, \Theta_m, \Theta_l) \precsim_{i_2} \wp_2(\Theta_n, \Theta_m, \Theta_m) + \wp_2(\Theta_l, \Theta_m, \Theta_m).$$

Therefore

$$\|\wp_2(\Theta_n, \Theta_m, \Theta_l)\| \leq \|\wp_2(\Theta_n, \Theta_m, \Theta_m)\| + \|\wp_2(\Theta_l, \Theta_m, \Theta_m)\|.$$

Taking limit as $n, m, l \to \infty$, we get $\|\wp_2(\Theta_n, \Theta_m, \Theta_l)\| \to 0$. So by lemma (2.3) , $\{\Theta_n\}$ is G-Cauchy sequence. By G-completeness of $(X, \wp_2)$, $\exists\ u \in X$ such that $\{\Theta_n\}$ is G-convergent and converages to u. Next, we prove that $\eth u = u$. Contrarily, assuming that $\eth u \neq u$. Then by

$$\wp_2(\Theta_{n+1}, \eth u, \eth u) \precsim_{i_2} k\wp_2(\Theta_n, u, u) \tag{2.4}$$

and hence

$$\|\wp_2(\Theta_{n+1}, \eth u, \eth u)\| \leq k\|\wp_2(\Theta_n, u, u)\|$$

Considering $n \to \infty$ as the limit. Since G is continuous on its variables according to Proposition 2.4, we have

$$\|\wp_2(u, \eth u, \eth u)\| \leq k\|\wp_2(u, u, u)\|$$

which is contradictory as $k \in [0, 1)$. Thus $\eth u = u$.
Lastly, to demonstrate uniqueness, assume that $\bar{w}(\neq u)$ is another fixed point of $\eth$. Thus $\eth\bar{w} = \bar{w}$. Then by (2.1),

$$\wp_2(u, \bar{w}, \bar{w}) = \wp_2(\eth u, \eth\bar{w}, \eth\bar{w}) \precsim_{i_2} k\wp_2(u, \bar{w}, \bar{w})$$

Therefore

$$\|\wp_2(u,\bar{w},\bar{w})\| \leq k\|\wp_2(u,\bar{w},\bar{w})\|$$

Since $k \in [0,1)$, we have $\|\wp_2(u,\bar{w},\bar{w})\| \leq 0$. Consequently, $u=\bar{w}$, and as a result, u is the UFP of $\eth$. That brings the proof to a close.

Example 2.2 *Assume that $X=\{0,\frac{1}{2},2\}$ and $\wp_2 : X^3 \to \mathbb{C}_2$ be $BVGMS$ defined as follows:*

$$\wp_2(p,q,r) = (|p-q|+|q-r|+|p-r|) + i_2(|p-q|+|q-r|+|p-r|).$$

for all $p,q,r \in X$. Define a mapping $\eth : X \to X$ defined as

$$\eth(p) = \begin{cases} 0, & \text{if } p=0 \\ 0, & \text{if } p=\frac{1}{2} \\ \frac{1}{2}, & \text{if } p=2 \end{cases}.$$

Let us choose any positive real number k such that $\frac{1}{4} \leq k < 1$. Then $\eth$ satisfy $\wp_2(\eth p,\eth q,\eth r) \precsim_{i_2} k\wp_2(p,q,r)$ holds for all $p,q,r \in X$, where $\frac{1}{4} \leq k < 1$. Hence $p=0$ is the UFP of $\eth$.

Definition 2.6 Assume that $X \neq \emptyset$ be any subset of G-complete $BVGMS$ $(X,\wp_2)$. A map $\eth : X \to X$ is referred to as a G-Kannan mapping if $\exists\ r \in \left[0,\frac{1}{2}\right)$ such that

$$\wp_2(\eth x,\eth y,\eth z) \precsim_{i_2} r\left(\wp_2(x,\eth x,\eth x) + \wp_2(y,\eth y,\eth y) + \wp_2(z,\eth z,\eth z)\right)$$

for all $x,y,z \in X$.

Theorem 2.2 *Assume that $(X,\wp_2)$ be a G-complete $BVGMS$. Let $\eth$ be a G-Kannan mapping on X, i.e.,*

$$\wp_2(\eth x,\eth y,\eth z) \precsim_{i_2} r(\wp_2(x,\eth x,\eth x) + \wp_2(y,\eth y,\eth y) + \wp_2(z,\eth z,\eth z))$$

for all $x,y,z \in X$, where $r \in [0,\frac{1}{2})$. Then $\eth$ has a UFP on X.

Proof:Let $\eth$ be a mapping that satisfies G-Kannan. Choose an arbitrary point $\Theta_0 \in X$, and establish the sequence $\{\Theta_n\}$ using the recurrence relation $\Theta_{n+1} = \eth\Theta_n$ for all $n \in \mathbb{N}\cup\{0\}$. Then, due to $\eth$ satisfying G-Kannan mapping, we have

$$\begin{aligned} \wp_2\left(\Theta_n,\Theta_{n+1},\Theta_{n+1}\right) &\precsim_{i_2} \wp_2\left(\eth\Theta_{n-1},\eth\Theta_n,\eth\Theta_n\right) \\ &\precsim_{i_2} r\left(\wp_2\left(\Theta_{n-1},\eth\Theta_{n-1},\eth\Theta_{n-1}\right) + \wp_2\left(\Theta_n,\eth\Theta_n,\eth\Theta_n\right) + \wp_2\left(\Theta_n,\eth\Theta_n,\eth\Theta_n\right)\right). \\ &\precsim_{i_2} r\left(\wp_2\left(\Theta_{n-1},\Theta_n,\Theta_n\right) r + \wp_2\left(\Theta_n,\Theta_{n+1},\Theta_{n+1}\right) + \wp_2\left(\Theta_n,\Theta_{n+1},\Theta_{n+1}\right)\right). \end{aligned}$$

So, we get

$$\wp_2\left(\Theta_n,\Theta_{n+1},\Theta_{n+1}\right) \precsim_{i_2} \left(\frac{r}{1-2r}\right) \wp_2\left(\Theta_{n-1},\Theta_n,\Theta_n\right)$$

Again by $\eth$ satisfies G-Kannan mapping, we get

$$\wp_2\left(\Theta_{n-1},\Theta_n,\Theta_n\right) \precsim_{i_2} \left(\frac{r}{1-2r}\right) \wp_2\left(\Theta_{n-2},\Theta_{n-1},\Theta_{n-1}\right)$$

Continuing with the same approach, we obtain

$$\wp_2\left(\Theta_n,\Theta_{n+1},\Theta_{n+1}\right) \precsim_{i_2} \left(\frac{r}{1-2r}\right)^n \wp_2\left(\Theta_0,\Theta_1,\Theta_1\right). \tag{2.5}$$

Let $k=\frac{r}{1-2r}$. Clearly, we can claim that $k<1$ as $r<\frac{1}{2}$. From (2.5), we have

$$\wp_2\left(\Theta_n,\Theta_{n+1},\Theta_{n+1}\right) \precsim_{i_2} (k)^n \wp_2\left(\Theta_0,\Theta_1,\Theta_1\right). \tag{2.6}$$

For any natural numbers n and m where $m > n$, we can use equation $(BCG5)$ and (2.6) to obtain the following:

$$\begin{aligned}
\wp_2(\Theta_n, \Theta_m, \Theta_m) \precsim_{i_2} & \wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1}) + \wp_2(\Theta_{n+1}, \Theta_{n+2}, \Theta_{n+2}) \\
& + \wp_2(\Theta_{n+2}, \Theta_{n+3}, \Theta_{n+3}) + \cdots + \wp_2(\Theta_{m-1}, \Theta_m, \Theta_m) \\
\precsim_{i_2} & k^n \wp_2(\Theta_0, \Theta_1, \Theta_1) + k^{n+1} \wp_2(\Theta_0, \Theta_1, \Theta_1) + k^{n+2} \wp_2(\Theta_0, \Theta_1, \Theta_1) + \cdots \\
& + k^{m-1} \wp_2(\Theta_0, \Theta_1, \Theta_1) \\
\precsim_{i_2} & \left(k^n + k^{n+1} + k^{n+2} + \ldots + k^{m-1}\right) \wp_2(\Theta_0, \Theta_1, \Theta_1) \\
= & (k^n)\left(1 + k^n + k^{n+1} + \ldots + k^{m-n-1}\right) \wp_2(\Theta_0, \Theta_1, \Theta_1) \\
\precsim_{i_2} & \frac{k^n}{1-k} \wp_2(\Theta_0, \Theta_1, \Theta_1).
\end{aligned}$$

Therefore, we obtain

$$\|\wp_2(\Theta_n, \Theta_m, \Theta_m)\| \leq \frac{k^n}{1-k} \|\wp_2(\Theta_0, \Theta_1, \Theta_1)\| \tag{2.7}$$

Take $n \to \infty$, we get $\|\wp_2(\Theta_n, \Theta_m, \Theta_m)\| \to 0$. For $n, m, l \in \mathbb{N}$, we obtain

$$\wp_2(\Theta_n, \Theta_m, \Theta_l) \precsim_{i_2} \wp_2(\Theta_n, \Theta_m, \Theta_m) + \wp_2(\Theta_l, \Theta_m, \Theta_m)$$

Therefore

$$\|\wp_2(\Theta_n, \Theta_m, \Theta_l)\| \leq \|\wp_2(\Theta_n, \Theta_m, \Theta_m)\| + \|\wp_2(\Theta_l, \Theta_m, \Theta_m)\|$$

Taking limit as $n, m, l \to \infty$, we get $\|\wp_2(\Theta_n, \Theta_m, \Theta_l)\| \to 0$. So by lemma (2.3), $\{\Theta_n\}$ is bicomplex valued G-Cauchy sequence.
Since $(X, \wp_2)$ is G-complete. So we have, $\{\Theta_n\}$ is bicomplex valued G-convergent and converges to some $\bar{v} \in X$. Now, we will show that $\eth\bar{v} = \bar{v}$. Assume that $\eth\bar{v} \neq \bar{v}$. Then, we get

$$\begin{aligned}
\wp_2(\Theta_{n+1}, \eth\bar{v}, \eth\bar{v}) \precsim_{i_2} & \wp_2(\eth\Theta_n, \eth\bar{v}, \eth\bar{v}) \\
\precsim_{i_2} & r\left(\wp_2(\Theta_n, \eth\Theta_n, \eth\Theta_n) + \wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v}) + \wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v})\right) \\
\precsim_{i_2} & r\left(\wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1}) + \wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v}) + \wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v})\right) \\
\precsim_{i_2} & r\wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1}) + 2r\wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v})
\end{aligned}$$

and so,

$$\|\wp_2(\Theta_{n+1}, \eth\bar{v}, \eth\bar{v})\| \leq r\|\wp_2(\Theta_n, \Theta_{n+1}, \Theta_{n+1})\| + 2r\|\wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v})\|$$

Taking $n \to \infty$, we get

$$\|\wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v})\| \leq k\|\wp_2(\bar{v}, \bar{v}, \bar{v})\|, \text{ where } k = \frac{r}{1-2r}$$

which is contradictory as $k \in [0, 1)$. Thus we conclude that $\eth\bar{v} = \bar{v}$.
Lastly, consider $\bar{u} \in X$ be another fixed point of $\eth$ in order to demonstrate the uniqueness of the fixed point. Thus $\eth\bar{u} = \bar{u}$. Then by $\eth$ satisfies G-Kannan mapping, we obtain

$$\wp_2(\bar{v}, \bar{u}, \bar{u}) = \wp_2(\eth\bar{v}, \eth\bar{u}, \eth\bar{u}) \precsim_{i_2} r\left(\wp_2(\bar{v}, \eth\bar{v}, \eth\bar{v}) + \wp_2(\bar{u}, \eth\bar{u}, \eth\bar{u}) + \wp_2(\bar{u}, \eth\bar{u}, \eth\bar{u})\right)$$

Hence

$$0 \leq \|\wp_2(\bar{v}, \bar{u}, \bar{u})\| \leq r\|\wp_2(\bar{v}, \bar{v}, \bar{v})\| + r\|\wp_2(\bar{u}, \bar{u}, \bar{u})\| + r\|\wp_2(\bar{u}, \bar{u}, \bar{u})\| = 0$$

Thus, $\|\wp_2(\bar{v}, \bar{u}, \bar{u})\| = 0$, and then $\bar{u} = \bar{v}$. Therefore $\bar{v}$ is a UFP of $\eth$.

Example 2.3 *Let $X = \{0, \frac{1}{2}, 2\}$ and $\wp_2 : X^3 \to \mathbb{C}_2$ be $BVGMS$ defined as follows:*

$$\wp_2(p,q,r) = (|p-q| + |q-r| + |p-r|) + i_2(|p-q| + |q-r| + |p-r|)$$

for all $p, q, r \in X$. Then clearly, we can verify that $(X, \wp_2)$ is a G-complete $BVGMS$. Let us define a mapping $\eth : X \to X$ as

$$\eth(0) = 0, \quad T\left(\frac{1}{2}\right) = 0 \quad and \quad T(2) = \frac{1}{2}$$

. Let us choose any positive real number k such that $0.29 \leq k < 0.5$. Then $\eth$ satisfy

$$\wp_2(\eth p, \eth q, \eth r) \precsim_{i_2} r(\wp_2(p, \eth p, \eth p) + \wp_2(q, \eth q, \eth q) + \wp_2(r, \eth r, \eth r))$$

holds for all $p, q, r \in X$, where $0.29 \leq r < 0.5$. Hence $p = 0$ is the UFP of $\eth$.

Definition 2.7 Let $X \neq \emptyset$ be subsets of a G-complete $BVGMS$ $(X, \wp_2)$. A function $\eth : X \to X$ is referred to as a G-Bianchini mapping if $\exists$ a real valued number $h \in [0, 1)$ such that

$$\wp_2(\eth p, \eth q, \eth r) \precsim_{i_2} h \ \max\left(\wp_2(p, \eth p, \eth p), \wp_2(q, \eth q, \eth q), \wp_2(r, \eth r, \eth r)\right)$$

for all $p, q, r \in X$.

Theorem 2.3 *Suppose that $(X, \wp_2)$ be a G-complete $BVGMS$. Let $\eth$ be a G-Bianchini mapping on X, i.e.,*

$$\wp_2(\eth\alpha, \eth\beta, \eth\bar{u}) \precsim_{i_2} h \ \max\{\wp_2(\alpha, \eth\alpha, \eth\alpha), \wp_2(\beta, \eth\beta, \eth\beta), \wp_2(\bar{u}, \eth\bar{u}, \eth\bar{u})\} \tag{2.8}$$

for all $\alpha, \beta, \bar{u} \in X$. Then $\eth$ has a UFP on X.

Proof: Suppose $\eth$ satisfies the property of being a G-Bianchini mapping. Let's take an arbitrary point $\Theta_0 \in X$, and then we can define the sequence $\{\Theta_n\}$ as $\Theta_{n+1} = \eth\Theta_n$ for all $n \in \mathbb{N} \cup \{0\}$. As a result of $\eth$ satisfying the property of being a G-Bianchini mapping, we obtain

$$\begin{aligned}
& \wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right) \\
\precsim_{i_2} \quad & \wp_2\left(\eth\Theta_{n-1}, \eth\Theta_n, \eth\Theta_n\right) \\
\precsim_{i_2} \quad & h \ \max\left\{\wp_2\left(\Theta_{n-1}, \eth\Theta_{n-1}, \eth\Theta_{n-1},\right), \wp_2\left(\Theta_n, \eth\Theta_n, \eth\Theta_n,\right), \wp_2\left(\Theta_n, \eth\Theta_n, \eth\Theta_n\right)\right\} \\
\precsim_{i_2} \quad & h \ \max\left\{\wp_2\left(\Theta_{n-1}, \Theta_n, \Theta_n\right), \wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right), \wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right)\right\}.
\end{aligned} \tag{2.9}$$

for all $n \in N$. If for some $n, \max\left\{\wp_2\left(\Theta_{n-1}, \Theta_n, \Theta_n\right), \wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right), \wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right)\right\} = \wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right)$, then by Equation (2.9), it follows that

$$(1-h)\wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right) \precsim_{i_2} 0 \implies \wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right) = 0$$

Thus,

$$\wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right) \precsim_{i_2} h \ \wp_2\left(\Theta_{n-1}, \Theta_n, \Theta_n\right), n \in \mathbb{N}.$$

Again by $\eth$ satisfies G-Bianchini mapping, we get

$$\wp_2\left(\Theta_{n-1}, \Theta_n, \Theta_n\right) \precsim_{i_2} (h) \wp_2\left(\Theta_{n-2}, \Theta_{n-1}, \Theta_{n-1}\right)$$

Continuing with the same approach, we obtain

$$\wp_2\left(\Theta_n, \Theta_{n+1}, \Theta_{n+1}\right) \precsim_{i_2} (h)^n \wp_2\left(\Theta_0, \Theta_1, \Theta_1\right). \tag{2.10}$$

For any natural numbers n and m where $m > n$, we can use equation ($BCG5$) and (2.10) to obtain the following:

$$\begin{aligned}
\wp_2(\Theta_n,\Theta_m,\Theta_m) \precsim_{i_2} & \wp_2(\Theta_n,\Theta_{n+1},\Theta_{n+1})+\wp_2(\Theta_{n+1},\Theta_{n+2},\Theta_{n+2}) \\
& +\wp_2(\Theta_{n+2},\Theta_{n+3},\Theta_{n+3})+\cdots+\wp_2(\Theta_{m-1},\Theta_m,\Theta_m) \\
\precsim_{i_2} & h^n\wp_2(\Theta_0,\Theta_1,\Theta_1)+h^{n+1}\wp_2(\Theta_0,\Theta_1,\Theta_1)+h^{n+2}\wp_2(\Theta_0,\Theta_1,\Theta_1)+\cdots \\
& ++h^{m-1}\wp_2(\Theta_0,\Theta_1,\Theta_1) \\
\precsim_{i_2} & \left(h^n+h^{n+1}+h^{n+2}+\cdots+h^{m-1}\right)\wp_2(\Theta_0,\Theta_1,\Theta_1) \\
= & (h^n)\left(1+h^n+h^{n+1}+\ldots+h^{m-n-1}\right)\wp_2(\Theta_0,\Theta_1,\Theta_1) \\
\precsim_{i_2} & \frac{h^n}{1-h}\wp_2(\Theta_0,\Theta_1,\Theta_1).
\end{aligned}$$

Therefore, we obtain

$$\|\wp_2(\Theta_n,\Theta_m,\Theta_m)\| \le \frac{h^n}{1-h}\|\wp_2(\Theta_0,\Theta_1,\Theta_1)\| \tag{2.11}$$

Take $n\to\infty$, we get $\|\wp_2(\Theta_n,\Theta_m,\Theta_m)\|\to 0$. For $n,m,l\in\mathbb{N}$, we obtain

$$\wp_2(\Theta_n,\Theta_m,\Theta_l) \precsim_{i_2} \wp_2(\Theta_n,\Theta_m,\Theta_m)+\wp_2(\Theta_l,\Theta_m,\Theta_m)$$

Therefore

$$\|\wp_2(\Theta_n,\Theta_m,\Theta_l)\| \le \|\wp_2(\Theta_n,\Theta_m,\Theta_m)\|+\|\wp_2(\Theta_l,\Theta_m,\Theta_m)\|$$

*Taking limit as $n,m,l\to\infty$, we get $\|\wp_2(\Theta_n,\Theta_m,\Theta_l)\|\to 0$. So by lemma (**??**), $\{\Theta_n\}$ is bicomplex valued G-Cauchy sequence.*
Since $(X,\wp_2)$ is G-complete. Therefore $\{\Theta_n\}$ is bicomplex valued G-convergent sequence and converges to some $\bar{u}\in X$. Now, we will to show that $\eth\bar{u}=\bar{u}$. Then, we get

$$\begin{aligned}
& \wp_2(\bar{u},\eth\bar{u},\eth\bar{u}) \\
\precsim_{i_2} & \wp_2(\bar{u},\Theta_n,\Theta_n)+\wp_2(\Theta_n,\eth\bar{u},\eth\bar{u}) \\
\precsim_{i_2} & \wp_2(\bar{u},\Theta_n,\Theta_n)+h\ \max\{\wp_2(\Theta_{n-1},\Theta_n,\Theta_n),\wp_2(\bar{u},\eth\bar{u},\eth\bar{u}),\wp_2(\bar{u},\eth\bar{u},T\bar{u})\}
\end{aligned}$$

Since $\wp_2(\bar{u},\Theta_n,\Theta_n)\to 0$ and $\wp_2(\Theta_{n-1},\Theta_n,\Theta_n)\to 0$ as $n\to\infty$, we obtain that

$$(1-h)\wp_2(\bar{u},\eth\bar{u},\eth\bar{u}) \precsim_{i_2} 0.$$

Therefore, we get

$$(1-h)\|\wp_2(\bar{u},\eth\bar{u},\eth\bar{u})\| \le 0.$$

This holds unless $\|\wp_2(\bar{u},\eth\bar{u},\eth\bar{u})\|=0$, thus $\eth\bar{u}=\bar{u}$. Let's consider $p\in X$ as another fixed point of $\eth$. By utilizing equation (2.8), we obtain the following result:

$$\wp_2(\eth\bar{u},\eth\bar{u},\eth p) \precsim_{i_2} h\max\{\wp_2(\bar{u},\eth\bar{u},\eth\bar{u}),\wp_2(\bar{u},\eth\bar{u},\eth\bar{u}),\wp_2(p,\eth p,\eth p)\}=0$$

which is contradictory. So we conclude that $\wp_2(\eth\bar{u},\eth\bar{u},\eth p)=0$ and thus $\bar{u}=p$. Hence $\eth$ has a UFP on X.

3. An application to System of Linear Equations

We provide an application utilizing theorem (2.1) in this section.

Theorem 3.1 *Assume that the BVGMS $X=\mathbb{C}^n$ has the metric*

$$\wp_2(x,y,z)=\max_{1\le k\le n}(|x_k-y_k|+|y_k-z_k|+|x_k-z_k|)+i_2(|x_k-y_k|+|y_k-z_k|+|x_k-z_k|).$$

where $x, y, z \in X$. If

$$\sum_{j=1}^{n} |\alpha_{kj}| \leq \alpha < 1 \quad \textit{for all } i = 1, 2, \ldots, n$$

then the linear system

$$\begin{cases} \alpha_{11}z_1 + \alpha_{12}z_2 + \ldots + \alpha_{1n}z_n = p_1 \\ \alpha_{21}z_1 + \alpha_{22}z_2 + \ldots + \alpha_{2n}z_n = p_2 \\ \vdots \\ \alpha_{n1}z_1 + \alpha_{n2}z_2 + \ldots + \alpha_{nn}z_n = p_n \end{cases} \tag{3.1}$$

of n linear equations in n unknowns has a unique solution.

Proof: Since every bicomplex valued metric space is a bicomplex valued G- metric space, it is simple to illustrate that $(X, \wp_2)$ is a complete $BVGMS$. Consequently, we must demonstrate that the mapping $\eth : X \to X$ supplied by

$$\eth(x) = Ax + b$$

where $x = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n, b = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{pmatrix}$$

is a contraction. Since

$$\begin{aligned} & \wp_2(\eth x, \eth y, Tz) \\ = \; & \max_{1\leq k\leq n} (\sum_{j=1}^{n} |\alpha_{kj}| \left(|x_k - y_k| + |y_k - z_k| + |x_k - z_k|\right) \\ & \qquad + i_2 \sum_{j=1}^{n} |\alpha_{kj}| \left(|x_k - y_k| + |y_k - z_k| + |x_k - z_k|\right)) \\ = \; & \max_{1\leq k\leq n} \sum_{j=1}^{n} |\alpha_{kj}| \left(|x_k - y_k| + |y_k - z_k| + |x_k - z_k| + i_2|x_k - y_k| + |y_k - z_k| + |x_k - z_k|\right) \\ = \; & \sum_{j=1}^{n} |\alpha_{kj}| \max_{1\leq k\leq n} \left(|x_k - y_k| + |y_k - z_k| + |x_k - z_k| + i_2|x_k - y_k| + |y_k - z_k| + |x_k - z_k|\right) \\ = \; & \sum_{j=1}^{n} |\alpha_{kj}| \, \wp_2(x, y, z) \\ \precsim_{i_2} \; & \alpha \wp_2(x, y, z) \end{aligned}$$

We come to the conclusion that the mapping $\eth$ is a Banach contraction. The system of linear equations (3.1) has a unique solution according to theorem (2.1).

4. Conclusion

This paper presents a novel concept of a complete $BVGMS$, where we have updated the general background of bicomplex valued metric space and demonstrated some well-known fixed point results. We also apply the results of this work in the system of linear equations which indicates the importance of this work. Our research results illustrate the existence and uniqueness of a fixed point for different contraction scenarios. We believe that these findings will be a significant contribution to this specific area of future research. We may find interesting results if we apply the ideas laid out in this paper to further research on alternative metric spaces, like bicomplex valued control metric space and bicomplex valued cone metric space.

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