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A Study on Metrizability in M-Topology

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ABSTRACT: Metrization is one of the most vital topological property since every metric space induces a topology on it but the converse may not be true always. So its natural to raise the question that when a topological space is metrizable in the multiset context. This paper primarily aims to give the answer for the query along with relavent notions. Also, the version of Nagata-Smirnov Metrization Theorem which gives an equivalent condition to metrizability of a topological space in general topology is studied in this environment by introducing some metrics in $\mathbb{R}^m_M$.

Key Words: Multiset topology, countability, product topology, embedding, metrizability.

Contents

1. Introduction

As multiset(or m-set) is a generalization of classial set theory to use this tool for modelling different real world problems with enormous repetition and due to this reason a number of researchers are engaged in the studies on multisets and their applications to physics, computer science, relational database, logic, different fields of mathematics and others. Against these applications of multisets, the theoretical considerations on multiset theory are also earnestly studied by Yager [27], Miyamoto [16] and Blizard [1,2,3]. In addition to these studies, Girish and John [9,10] were the first to explain the remarkable ideas on multiset topology. Various multiset topological notions were also introduced, including M-compactness [15,13], subspace M-topology [13], generalized closed sets [20], M-connectedness [14], countability [19] and separation axioms [6]. Shravan and Tripathy [21,23] studied the concept of ideal in M-topological space. Evenmore, the notion of multipoint was proposed by them [24] which plays a significant role in the multiset topology. Also, in this paper they have introduced the idea of a metric on a colletion of multipoints and shown that every multiset metric space generates a topology and proved the Urysohn lemma in this environment.

It is true that, a topological space is metrizable if the associated metrics are induced on it. In this proposal, we use embedding of m-sets to investigate the circumstances in which a multiset topological space induces the relevant metric.

In structuring this research output, the section 2 consists some preliminary information on M-topological space. The section 3 introduces countability and embedding in multiset context and combinig these ideas we find the prerequisite conditions for an M-topological space to be metrizable. Nonetheless, we looked into some more stringent requirements for metrizability which is explained in the section 4.

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2. Preliminaries

Here, we shall briefly recollect some of the primary definitions and results relating to multiset topological space that are necessary to make the subsequent study self-contained.

Definition 1 *[8] A multiset (m-set, for short) M is a collection of elements drawn from a set X, represented by a count function $C_M : X \longrightarrow N$ where N is the set of non-negative integers.*

$C_M(x)$ denotes how many times each element from X appears in the multiset M. If an element from the set X is not included in the multiset M, the count function assigns a count of zero to that element.

Let X be a universal set. Then the notation $[X]^m$ represents the collection of all m-sets from X in which no element in the m-set repeats more than m times.

On the other hand, $[X]^\infty$ represents the collection of all m-sets over X such that there is no restriction on the number of times an element can appear in an m-set.

Let A be an m-set. The power m-set $P(A)$ of M is the set of all the sub-m-sets of A.

The power set of an m-set is the support set of the power m-set and is denoted by $P^*(A)$.

Definition 2 *[9] Let M be an m-set drawn from X and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology (M-topology, in short) if τ satisfies the following properties :*

(i) $\emptyset$ and M are in τ,

(ii) The union of the elements of any sub collection of τ is in τ,

(iii) The intersection of the elements of any finite sub collection of τ is in τ.

Then (M, τ) is called a multiset topological space.

An M-basis for an M-topology on M is a collection $\mathscr{B}$ of partial whole sub-m-sets of M (called M-basis element) such that

(i) for each $x \in M$ and for some $m > 0$, $\exists$ at least one M-basis element $B \in \mathscr{B}$ containing m/x,

(ii) if $m/x \subseteq M_1 \cap M_2$ then $\exists$ an M-basis element M_3 containing m/x s.t. $M_3 \subseteq M_1 \cap M_2$.

Definition 3 *[13] Let N be a sub-m-set of an M-topological space (M, τ). Then the collection $\tau_N = \{N \cap A; A \in \tau\}$ is an M-topology on N, called the subspace M-topology on N.*

Definition 4 *[8] A multiset function $f : (M, \tau) \longrightarrow (N, \sigma)$ is called*

(i) continuous if $f^{-1}(U) \in \tau$ for all $U \in \sigma$,

(ii) onto if Ran f is equal to co-dom f and $C_1(x, y) \geq C_2(x, y)$ for all (x, y) in f,

(iii) one–one if no two elements in Dom f have the same image under f with $C_1(x, y) \leq C_2(x, y)$ for all (x, y) in f,

(iv) bijective if f is both injective and surjective,
where $C_1(x, y)$ and $C_2(x, y)$ denote the count of the first and second co-ordinate in the ordered pair (x, y) respectively.

Definition 5 *[8] Let M_1, M_2 be two m-sets then the cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m_1/x, m_2/y)/m_1m_2 : x \in M_1, y \in M_2\}$.*

Definition 6 *[24] Let $[X]^m$ be a space of m-sets. A multipoint is an m-set M in X such that*

$$C_M(x) = \begin{cases} m, & \text{for } x \in X; \\ 0, & \text{for } y \neq x, y \in X. \end{cases}$$

A multipoint m/x is a subset of an m-set M or $m/x \in M$ if $m \leq C_M(x)$.

An m-set N in an M-Topological space (M, τ) is said to be a neighborhood (or, nbd) of m/x if and only if there exists an open m-set P such that, $m/x \in P \subset N$.

Definition 7 *An M-topological space (M, τ) is called*

(i) M-regular space [6] if for all $F \in \tau^c$ and for all $\{m/x\} \nsubseteq F$, then there exist $G, H \in \tau$ such that $G \cap H = \phi$ and $F \subseteq G$, $\{m/x\} \subseteq H$,

(ii) M-normal space [24] if for any two disjoint closed m-sets F_1, F_2 there exist two disjoint open m-sets G, H such that $F_1 \subseteq G$, $F_2 \subseteq H$.

Definition 8 *[24] Let $M = \{m_1/x_1, m_2/x_2, ..., m_k/x_k\}$ be a collection of multipoints in $[X]^m$. We define a function $d : M \times M \longrightarrow [0, \infty)$ with the following properties:*

(i) $d(m_1/x_1, m_2/x_2) \geq 0$, for all $m_1/x_1, m_2/x_2 \in M$,

(ii) $d(m_1/x_1, m_2/x_2) = 0$ if $x_1 = x_2$ and $m_1 = m_2$,

(iii) $d(m_1/x_1, m_2/x_2) = d(m_2/x_2, m_1/x_1)$,

(iv) $d(m_1/x_1, m_3/x_3) \leq d(m_1/x_1, m_2/x_2) + d(m_2/x_2, m_3/x_3)$.

Then d is referred as a metric on M and (M, d) forms a multiset metric space.
The topology on M generated by the M-sphere $S_r(m/x)$ (which are defined based on the metric d), is referred to as the d-metric M-topology. Very often we use this M-topology as metric M-topology.
(M, τ) is metrizable if it can be equipped with a metric that induces the same topology τ.

Definition 9 *[15] A collection $\mathscr{A} = \{A_\lambda : \lambda \in \Lambda\}$ of m-sets of an M-topological space (M, τ) is said to be a cover of M if $C_M(x) \leq C_{\cup_\lambda A_\lambda}(x)$, for all $x \in X$. $\mathscr{A}$ is called an open cover of M if $\mathscr{A} \subset \tau$.*

Definition 10 *[17] Let (M, τ) be an M-topological space. A collection $\mathscr{P} \subseteq P^*(M)$ is called locally finite if each $m/x \in M$ has an open neighborhood U (which intersects only finitely many m-sets in $\mathscr{P}$) such that, for every m-set V in only a finite subcollection $\mathscr{Q}$ of $\mathscr{P}$,*

$$C_{U \cap V}(y) > 0, \ \forall y \in X.$$

Result 1 *[17] If a collection $\mathscr{P}$ of sub-m-sets is locally finite, then*

$$\bigcup \overline{P} = \overline{\bigcup P} \text{ for all } P \in \mathscr{P}.$$

Definition 11 *[19] Let τ be an M-topology on M. Then the space is called*

(i) first countable iff every multipoint m/x in M has a countable neighbourhood M-basis;

(ii) second countable whenever $\exists$ a countable M-basis for τ.

3. Metrization cCnditions for M-Topological Spaces

The primary objective of this section is to establish under which conditions an M-topological space is generating a metric therein. We introduce product of topological spaces and embedding in multiset context.

An m-set on the universal set $\mathbb{R}$(the set of real numbers) is denoted by $\mathbb{R}_M$ whereas $\mathbb{R}_M^m$ denotes the m-set $\mathbb{R}_M$ with highest multiplicity m. Also, $(\mathbb{R}_M^m)^\omega$ and $(\mathbb{R}_M^m)^J$ represent the countably infinite and arbitrary product of $\mathbb{R}_M^m$ with itself respectively for the subsequent studies.

Definition 12 *Let M and N be two M-topological spaces and $f : M \longrightarrow N$ be a bijective m-set function. Then f is called an M-homeomorphism if both f and f^{-1} are continuous.*
If such M-homeomorphism exists then M is said to be homeomorphic to N.

Definition 13 *Let $\{(M_\alpha, \tau_\alpha); \alpha \in I\}$ be a family of M-topological spaces. Let $\mathscr{B}$ be the collection of all m-sets of the form $\prod_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \tau_\alpha$. Then $\mathscr{B}$ forms an M-basis for some M-topology on $M (= \prod_{\alpha \in \Lambda} M_\alpha)$ and is called the product M-topology.*

Note 1 *Let (M, τ) be a product M-topological space of a family of M-topological spaces $\{(M_\alpha, \tau_\alpha); \alpha = 1, 2, ..., n\}$ and $p_\alpha : M \longrightarrow M_\alpha$ denote the α^{th} projection m-set function. Then*

(i) for each α, p_α is an open m-set function.

(ii) the product M-topology τ is the smallest M-topology on M such that each projection map p_α is a continuous m-set function.

Theorem 1 *Let $f : (M, \tau) \longrightarrow (N, \sigma)$ and $g : (M, \tau) \longrightarrow (P, \delta)$ are continuous m-set functions, then $f \times g : (M, \tau) \longrightarrow (N \times P, \sigma \times \delta)$ defined by*
$(f \times g)(m/x) = (f(m/x), g(m/x))$ is a continuous m-set function.

Proof 1 *Since*
$$p_1 \circ (f \times g)(m/x) = p_1(f(m/x), g(m/x)) = f(m/x)$$
and
$$p_2 \circ (f \times g)(m/x) = p_2(f(m/x), g(m/x)) = g(m/x),$$
so $p_1 \circ (f \times g)$ and $p_2 \circ (f \times g)$ are continuous m-set functions. Consequently, $(f \times g)$ is continuous m-set function.

Definition 14 *Let (M, τ_1) and (N, τ_2) be two M-topological spaces. An embedding $f : M \longrightarrow N$ is an m-set function which is a homeomorphism when it is regarded as an m-set function from M onto $(f(M), \tau_2/f(M))$.*

It is convenient to illustrate some metrics on $\mathbb{R}^m_M$ to generate an M-topology on the product space of the same.

Proposition 1 *Let $\mathbb{R}^m_M$ be an m-set with highest multiplicity m. Then an m-set function*
$d : \mathbb{R}^m_M \times \mathbb{R}^m_M \longrightarrow [0, \infty)$ defined by $d(m_1/x_1, m_2/x_2) = |m_1 - m_2| + |x_1 - x_2|$ is also a metric on $\mathbb{R}^m_M$.

Remark 1 *The above metric d gives the usual M-topology τ_d on $\mathbb{R}^m_M$.*

The following propositions express a number of way in which the above metric d can be utilized.

Proposition 2 *Let $\mathbb{R}^m_M$ be an m-set with highest multiplicity m and $d^* : \mathbb{R}^m_M \times \mathbb{R}^m_M \longrightarrow [0, \infty)$ be defined by $d^*(m_1/x_1, m_2/x_2) = min\{d(m_1/x_1, m_2/x_2), 1\}$ where $d(m_1/x_1, m_2/x_2) = |m_1 - m_2| + |x_1 - x_2|$ then d^* is a metric on $\mathbb{R}^m_M$.*

Proof 2 *Since $d(m_1/x_1, m_2/x_2)$ is a metric on $\mathbb{R}^m_M$ so first three properties of a metric easily hold.*
For triangle inequality:
Let $m_1/x_1, m_2/x_2, m_3/x_3 \in \mathbb{R}^m_M$ and $d(m_1/x_1, m_2/x_2) \geq 1$ or $d(m_2/x_2, m_3/x_3) \geq 1$.
Then $d^*(m_1/x_1, m_2/x_2) + d^*(m_2/x_2, m_3/x_3) \geq 1$ while
$d^*(m_1/x_1, m_3/x_3) = min\{d(m_1/x_1, m_3/x_3), 1\} \leq 1$.
Thus, $d^*(m_1/x_1, m_3/x_3) \leq d^*(m_1/x_1, m_2/x_2) + d^*(m_2/x_2, m_3/x_3)$.
Again, if $d(m_1/x_1, m_2/x_2) < 1$ and $d(m_2/x_2, m_3/x_3) < 1$ then,
$d(m_1/x_1, m_3/x_3) \leq d(m_1/x_1, m_2/x_2) + d(m_2/x_2, m_3/x_3)$
$\leq d^*(m_1/x_1, m_2/x_2) + d^*(m_2/x_2, m_3/x_3)$
So, $d^*(m_1/x_1, m_3/x_3) \leq d^*(m_1/x_1, m_2/x_2) + d^*(m_2/x_2, m_3/x_3)$.
Consequently, d^* is a metric on $\mathbb{R}^m_M$.

Proposition 3 *Let $(\mathbb{R}^m_M)^\omega$ be the countably infinite product of $\mathbb{R}^m_M$ with itself and let $m/x = (m_1/x_1, m_2/x_2, ...), n/y = (n_1/y_1, n_2/y_2, ...)$ be two elements of $(\mathbb{R}^m_M)^\omega$. Then d^*_ω defined by*
$$d^*_\omega(m/x, n/y) = sup\{d^*((m/x)_i, (n/y)_i)/i\}$$
is a metric on $(\mathbb{R}^m_M)^\omega$ where $i \in \mathbb{N}$ and $(m/x)_i$ is the i^{th} component of m/x.

Proof 3 *It is readily verified that d^* has all the properties required of a metric on $\mathbb{R}_M^m$ except for triangular inequality.*
To prove triangular inequality:
let $m/x, n/y, p/z \in \mathbb{R}_M^m$ and for all $i \in \mathbb{N}$, we can write
$d^*((m/x)_i,(n/y)_i) \le d^*((m/x)_i,(p/z)_i) + d^*((p/z)_i,(n/y)_i)$
$\implies d^*((m/x)_i,(n/y)_i)/i \le \{d^*((m/x)_i,(p/z)_i) + d^*((p/z)_i,(n/y)_i)\}/i$
$\le \{d^*((m/x)_i,(p/z)_i)/i\} + \{d^*((p/z)_i,(n/y)_i)/i\}$
$\le d^*_\omega(m/x,p/z) + d^*_\omega(p/z,n/y)$
$\implies d^*_\omega(m/x,n/y) \le d^*_\omega(m/x,p/z) + d^*_\omega(p/z,n/y)$
Eventually, d^*_ω is a metric on $(\mathbb{R}_M^m)^\omega$.

Theorem 2 *The metric d^*_ω defined by $d^*_\omega(m/x,p/y) = sup\{d^*((m/x)_i,(p/y)_i)/i : i \in \mathbb{N}\}$ induces the product M-topology on $(\mathbb{R}_M^m)^\omega$.*

Proof 4 *First observed that d^*_ω is a metric on the multiset $(\mathbb{R}_M^m)^\omega$. Second task is to show that d^*_ω generates a product M-topology on $(\mathbb{R}_M^m)^\omega$.*
Let, P be an open m-set in the metric M-topology and let $m/x \in P$. It is required to find an open m set Q in the product M-topology such that $m/x \in Q \subset P$.
Choose an ε-ball $B_{d^*_\omega}(m/x,\varepsilon) \subset P$ and an n large enough such that $1/n < \varepsilon$.
Suppose Q be an M-basis for the product M-topology such that
$Q = (m_1/x_1 - \varepsilon, m_1/x_1 + \varepsilon) \times ... \times (m_n/x_n - \varepsilon, m_n/x_n + \varepsilon) \times \mathbb{R}_M^m \times \mathbb{R}_M^m \times ....$
We claim that $Q \subset B_{d^*_\omega}(m/x,\varepsilon)$.
Since, for given any $p/y \in (\mathbb{R}_M^m)^\omega$, $d^*(m_i/x_i, p_i/y_i)/i \le 1/n$ for $i \ge n$, then

$$d^*_\omega(m/x,p/y) \le max\{d^*(m_1/x_1,p_1/y_1)/1, ..., d^*(m_n/x_n,p_n/y_n)/n, 1/n\}.$$

If $p/y \in Q$ then $d^*_\omega(m/x,p/y) < \varepsilon$, hence $Q \subset B_{d^*_\omega}(m/x,\varepsilon)$.

Conversely, let $B = \Pi_{i\in\mathbb{N}} B_i$ be an M-basis for product M-topology where each B_i is open in $\mathbb{R}_M^m$ for $i = 1,2,...,n$ and $B_i = \mathbb{R}_M^m$ for all others.
Let, $m/x \in B$ and our target is to find an open m-set A in the metric M-topology such that $m/x \in A \subset B$.
Let us choose $(m_i/x_i - \varepsilon_i, m_i/x_i + \varepsilon_i) \in \mathbb{R}_M^m$ such that $(m_i/x_i - \varepsilon_i, m_i/x_i + \varepsilon_i) \subset B_i$ for $i = 1,2,...,n$ and each $\varepsilon_i \le 1$.
Now, we take $\varepsilon = min\{\varepsilon_i/i : i = 1,2,...,n\}$ and asserts that $m/x \in B_{d^*_\omega}(m/x,\varepsilon) \subset B$.
Let $p/y \in B_{d^*_\omega}(m/x,\varepsilon)$ then $d^*(m_i/x_i,p_i/y_i)/i \le d^*_\omega(m/x,p/y) < \varepsilon$ for all i.
If $i = 1,2,...,n$ then $\varepsilon \le \varepsilon_i/i$, so $d^*(m_i/x_i,p_i/y_i) < \varepsilon_i \le 1$. Consequently, $p/y \in B$.

Definition 15 *Let $(\mathbb{R}_M^m)^J$ be the arbitrary product of $\mathbb{R}_M^m$ with itself and $(m/x)_j$ and $(n/y)_j)$ be two multipoints of $(\mathbb{R}_M^m)^J$. Let us define $\rho_M(m/x,n/y) = sup\{d^*((m/x)_j,(n/y)_j)\}$ for arbitrary index set J and $j \in J$ where $d^*(m/x,n/y) = min\{d(m/x,n/y),1\}$. Then ρ_M is a metric on $(\mathbb{R}_M^m)^J$. This metric is called uniform metric on M and the M-topology induced by this metric is called uniform M-topology.*

One can compare the uniform M-topology with product M-topology on $(\mathbb{R}_M^m)^J$.

Theorem 3 *The uniform M-topology on $(\mathbb{R}_M^m)^J$ is finer than the product M-topology on it.*

Proof 5 *Let us consider a multipoint $m/x = (m_\beta/x_\beta)_{\beta\in J}$ and $\mathscr{A} = \{\prod A_\beta : \beta \in J\}$ be an M-basis for product M-topology on $(\mathbb{R}_M^m)^J$. We take an M-basis element $\prod A_\beta$ such that $A_\beta \ne \mathbb{R}_M^m$ for $\beta_1, \beta_2, ..., \beta_r$.*
Let us take $\varepsilon_r > 0$ such that $B^*_d(m_{\beta_r}/x_{\beta_r}, \varepsilon_r) \subset A_{\beta_r}$, for each r.
Let, $\varepsilon = min\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_r\}$ then $B_{\rho_M}(m/x,\varepsilon) \subset \prod A_\beta$.
For, if any $n/y \in (\mathbb{R}_M^m)^J$ such that $\rho_M(m/x,n/y) < \varepsilon$ then $d^*(m_\beta/x_\beta, n_\beta/y_\beta) < \varepsilon$ for all β.
This implies that $n/y \in \prod A_\beta$.
Thus, the uniform M-topology is finer than the product M-topology.

Theorem 4 *[24] Let (M,τ) be an M-normal space. Then for every pair of disjoint non-empty whole sub-m-sets $F_1, F_2 \subseteq M$, there is a continuous m-set function $f : M \longrightarrow [0,1]$ such that $f(m/x) = 0$ for all $m/x \in F_1$ and $f(m/x) = 1$ for all $m/x \in F_2$.*

The following statement is an immediate consequence of the above theorem.

Lemma 1 *For any M-normal space, there exists a countable collection of continuous m-set functions* $f_r : M \longrightarrow [0,1]$ *such that for given* $m/x \in M$ *and any nbd* A *of* m/x*, there must be an index* r *such that* $f_r(m/x) = 1$ *and* $f_r(M \ominus A) = 0$.

Proof 6 *Let us consider a countable M-basis* $\{B_r\}$ *for* M*. Applying the above theorem for every pair of indices* r, s *for which* $\bar{B}_r \subset B_s$*, we get continuous m-set functions* $g_{r,s} : M \longrightarrow [0,1]$ *such that* $g_{r,s}(\bar{B}_r) = 1$ *and* $g_{r,s}(M \ominus B_s) = 0$.
Then $\{g_{r,s}\}$ *is a countable collection of continuous m-set functions.*
For, let m/x *and a nbd* A *of* m/x *and consider an M-basis* B_s *such that* $m/x \in B_s \subset A$.
Then from M-regularity, we take B_r *such that* $m/x \in B_r$ *and* $\bar{B}_r \subset B_s$*. Thus the m-set function* $g_{r,s}$ *is defined for every pair* r, s *and* $g_{r,s}(m/x) = 1$ *and* $g_{r,s}(M \ominus B_s) = 0$.
Eventually, $\{g_{r,s}\}$ *is countable as it is indexed with subset of* $\mathbb{N} \times \mathbb{N}$*. Reindexing it, we are having the collection* $\{f_r\}$ *such that* $f_r(m/x) = 1$ *and* $f_r(M \ominus A) = 0$.

Finally, we reach in the stage of study's primary goal which is the set of conditions that an M-topological space must satisfy in order to generate a metric therein.

We conclude this section with the contributing result which serves as the M-set analogue of Urysohn's Metrization Theorem from point set topology, followed by the embedding on $(\mathbb{R}^m_M)^J$.

Theorem 5 *An M-regular M-topological space having a countable M-basis is metrizable.*

Proof 7 *Let* M *be an M-regular space. Then from the above result, there is a countable collection* $\{f_r\}$ *of continuous m-set functions* $f_r : M \longrightarrow [0,1]$*. We are proving this theorem by embedding* M *into the subspace* $[0,1]^\omega$ *of* $(\mathbb{R}^m_M)^\omega$.

Let us assume $(\mathbb{R}^m_M)^\omega$ *with the product M-topology, we define an m-set function* $F : M \longrightarrow (\mathbb{R}^m_M)^\omega$ *by* $F(m/x) = (f_1(m_1/x_1), f_2(m_2/x_2), ...)$.
We shall show that, F *is an embedding.*
Each f_n *is continous m-set function and* $(\mathbb{R}^m_M)^\omega$ *has the product M-topology, hence* F *is continuous.*
Let, $m_1/x_1 \neq m_2/x_2$*, then there exists an* n *such that* $f_r(m_1/x_1) = 1$ *and* $f_r(m_2/x_2) = 0$.
Hence $F(m_1/x_1) \neq F(m_2/x_2)$*. So,* F *is injective.*
Now, we establish that F *is an M-homeomorphism of* M *onto* $F(M) = N$*, a subspace of* $(\mathbb{R}^m_M)^\omega$.
Since F *is continuous and injective hence is a bijective m-set function. So to prove* F *is an M-homeomorphism it is sufficient to show that for each open m-set* A *in* M*, the m-set* $F(A)$ *is open in* N.
Let, $n_1/z_1 \in F(A)$.
We shall find an open m-set B *of* N *such that* $n_1/z_1 \in B \subset F(A)$.
Let, $m_3/x_3 \in A$ *such that* $F(m_3/x_3) = n_1/z_1$.
Then there exists r *such that* $f_r(m_3/x_3) = 1$ *and* $f_r(M \ominus A) = 0$.
Let us take, $(0,\infty)_M = \{(n_i/r_i) : n_i \in \mathbb{N}, r_i \in (0,\infty), i \in I\}$ *which is open in* $\mathbb{R}^m_M$ *, then* $C = \Pi_r^{-1}\{(0,\infty)_M\}$ *is open in* $(\mathbb{R}^m_M)^\omega$.
Again, consider $B = C \cap N$*, then* B *is open in* N.
We claim that, $n_1/z_1 \in B \subset F(A)$.
Since, $\Pi_r(n_1/z_1) = \Pi_r(F(m_3/x_3)) = f_r(m_3/x_3) = 1$ *so* $n_1/z_1 \in B$.
Now, if $n_2/z_2 \in B$ *then* $n_2/z_2 = F(m_4/x_4)$ *for some* $m_4/x_4 \in M$ *and* $\Pi_r(n_2/z_2) \in (0,\infty)_M$.
Since, $\Pi_r(n_2/z_2) = \Pi_r(F(m_4/x_4)) = f_r(m_4/x_4)$ *which is equal to* 0 *outside* A *so* $m_4/x_4 \in A$*. Then* $n_2/z_2 = F(m_4/z_4) \subset F(A)$.
Eventually, F *is an embedding. Hence the proof is completed.*

The following proposition is a direct significance of the previous result.

Proposition 4 *Let* M *be an M-normal space and* $\{f_j\}_{j\in J}$ *is a family of continuous m-set functions from* M *to* $\mathbb{R}^m_M$ *such that for each multipoint* $m_0/x_0 \in M$ *and each neighbourhood* O *containing* m_0/x_0*, there exists an index* j *such that* $f_j(m_0/x_0) = 1$ *when* $m_0/x_0 \in O$ *and* $f_\alpha(m_0/x_0) = 0$ *outside* O*. Then the m-set function* $F : M \longrightarrow (\mathbb{R}^m_M)^J$ *defined by* $F(x) = (f_j(x))_{j\in J}$ *is an embedding of* M *in* $(\mathbb{R}^m_M)^J$*. Moreover, if* $f_j : M \longrightarrow [0,1]$ *for each* j*, then* F *is an embedding from* M *in* $[0,1]^J$.

4. A Sufficient Condition for Metrizability

In this article, we are looking for the circumstances in which an M-topological space can be metrizable. Already in the previous section we have found the required conditions but these are not sufficient. To make the requirement sufficient, we impose another weaker condition in this section.

Definition 16 *A collection $\mathscr{P}$ of sub-m-sets of an M-topological space (M, τ) is said to be a countably locally finite if*
$\mathscr{P} = \bigcup_{r\in\mathbb{Z}_+} \mathscr{P}_r$
where each $\mathscr{P}_r$ is locally finite.

Theorem 6 *Let M be a metrizable M-topological space. If $\mathscr{C}$ is an open covering of M then there is a countably locally finite open covering $\mathscr{D}$ of M which refines $\mathscr{C}$.*

Proof 8 *Let $\mathscr{C}$ be an open covering of M and the elements of $\mathscr{C}$ are $P, Q, R, \ldots$.*
Let us choose a metric for the m-set M. For given any element P of $\mathscr{C}$, let us define $N_r(P) = \{m/x : B(m/x, 1/r) \subset P\}$.
Now, using an odering for each P in $\mathscr{C}$, we define $O_r(P) = N_r(P) \ominus \bigcup_{Q\prec P} Q$.
We claim that $O_r(P), O_r(Q)$ and $O_r(R)$ are disjoint and separated when $P \prec Q \prec R$.

For this, let us take $Q \prec R$. Since, $m/x \in O_r(Q)$ so, $m/x \in N_r(Q)$ i.e, $B(m/x, 1/r) \subset Q$. Again, since $n/y \in O_r(R)$ so $n/y \notin Q$. Hence, $n/y \notin B(m/x, 1/r)$.
That means $O_r(P)$ is not an open m-set. But we are actually finding open m-sets that covers M.
For, let $D_r(P) = \{m/x \in O_r(P) : B(m/x, 1/3r) \subset P\}$.
The m-sets $D_r(P), D_r(Q), D_r(R)$ are now disjoint when $P \prec Q \prec R$.
For, if Q and R are distinct elements of $\mathscr{C}$, then $d(m/x, n/y) \geq 1/3r$ when $m/x \in D_r(Q)$ and $n/y \in D_r(R)$. Clearly, $D_r(Q) \subset Q$.
Now, we define $\mathscr{D}_r = \{D_r(Q) : Q \in \mathscr{C}\}$. It needs to show that $\{\mathscr{D}_r\}$ is locally finite collection of open m-sets that refines $\mathscr{C}$.
Since, $D_r(Q) \subset Q$ for each $Q \in \mathscr{C}$, so $\{\mathscr{D}_r\}$ refines $\mathscr{C}$. Also, since any neighbourhood of radius $1/6r$ of any $m/x \in M$ can intersect atmost one element of $\{\mathscr{D}_r\}$, so $\{\mathscr{D}_r\}$ is locally finite.
We claim that $\mathscr{D} = \bigcup_{r\in\mathbb{Z}_+} \mathscr{D}_r$ is a covering of M.
Let, $m/x \in M$ and $m/x \in P \subset \mathscr{C}$. Since, P is open so we can choose r such that $B(m/x, 1/r) \subset P$. Then $m/x \in N_r(P)$. Consequently, $m/x \in O_r(P)$.
Therefore, $m/x \in D_r(P) \in \mathscr{D}_r$. Eventually, $\mathscr{D}$ is an open covering of M.

Definition 17 *A sub-m-set P of an M-topological space (M, τ) is called a G_δ m-set in M if*
$P = \bigcap_{r\in\mathbb{Z}_+} O_r$
where each O_r is open sub-m-set of M.

Example 1 *Let (M, τ) be an M-topological space then every open m-set of M is a G_δ m-set.*

Example 2 *Let (M, τ) be an M-topological space where $M = \{2/a, 2/b\}$ and $\tau = \{\emptyset, M, \{1/a\}, \{2/a\}, \{1/b\}, \{2/b\}, \{1/a, 1/b\}, \{1/a, 2/b\}, \{2/a, 1/b\}\}$. Then every sub-m-set is a G_δ m-set.*

Theorem 7 *Let M be an M-regular space with a countably locally finite basis $\mathscr{A}$. Then M is M-normal and any closed set in M is a G_δ m-set in M.*

Proof 9 *Let, O be an open m-set in M. We shall show that there is a countable collection $\{P_r\}$ of open m-sets of M such that $O = \bigcup P_r = \bigcup \bar{P}_r$.*

Since, $\mathscr{A}$ is countably locally finite so $\mathscr{A} = \bigcup \mathscr{A}_r$, where each $\mathscr{A}_n$ is locally finite.

Let, $\mathscr{D}_r = \{A \in \mathscr{A}_r : \bar{A} \subset O\}$. Then $\mathscr{D}_r$ is locally finite, being a subcollection of $\mathscr{A}_r$.
Let us consider $P_r = \bigcup_{A\in\mathscr{D}_r} A$.

Then each P_r is an open m-set and from Result 1 we have, $\bar{P}_r = \bigcup_{A\in\mathscr{D}_r} \bar{A}$.

Now, we will show that, $\bigcup \bar{P}_r \subset O$.

Since, each $\mathscr{D}_r \subset \mathscr{A}_r$ *and each* $\bar{A} \subset O$*, so* $\bar{P}_r = \bigcup_{A \in \mathscr{D}_r} \bar{A} \subset O$*. It implies that* $\bigcup \bar{P}_r \subset O$ *and* $\bigcup P_r \subset \bigcup \bar{P}_r \subset O$*.*

We now show that, $O \subset \bigcup P_r$*.*

For, let $m/x \in O$*. By M-regularity there is an M-basis element* $A \in \mathscr{A}$ *such that* $m/x \in A$ *and* $\bar{A} \subset O$*. Since,* $A \in \mathscr{A}_r$ *for some* r*, so* $A \in \mathscr{D}_r$*. Hence,* $A \subset P_r$ *and* $m/x \in A \subset P_r \subset \bigcup P_r$*, which implies that* $O \subset \bigcup P_r$*. Thus, we have* $\bigcup P_r \subset \bigcup \bar{P}_r \subset O \subset \bigcup P_r$*.*

Therefore, $O = \bigcup P_r = \bigcup \bar{P}_r$*.*

Again, we will show that every closed m-set B *in* M *is a* G_δ *m-set in* M*. For, let* $O = M \ominus B$*, then there are m-sets* P_r *such that* $O = \bigcup \bar{P}_r$*. Thus,* $B = \bigcap (M \ominus \bar{P}_r)$ *and hence* B *is a countable intersection of open m-sets in* M*.*

We are to show that M *is M-normal. For let* R *and* S *be disjoint closed m-sets in* M*. We construct a countable collection of open m-sets* $\{P_r\}$ *such that* $\bigcup P_r = \bigcup \bar{P}_r = M \ominus S$ *and then* $\{P_r\}$ *is disjoint from* S *and covers* R*. Similarly, there is a countable covering* $\{Q_r\}$ *of* S *by open m-sets such that* $\bigcup Q_r = \bigcup \bar{Q}_r = M \ominus R$*, so* $\bar{Q}_r$ *is disjoint from* R*. Hence the proof is completed.*

Lemma 2 *Let* M *be an M-normal space and* P *be a closed* G_δ *m-set in* M*. Then there exists a continuous m-set function* $g : M \longrightarrow [0,1]$ *such that* $g(m/x) = 0$ *where* $m/x \in P$ *and* $g(m/x) = 1$ *for* $m/x \notin P$*.*

Proof 10 *Since,* P *is a* G_δ *m-set, so* $P = \bigcap_{r \in \mathbb{Z}_+} O_r$*, where each* O_r *is open m-set.*
Now for each r*, let us choose a continuous m-set function* $g_r : M \longrightarrow [0,1]$ *such that* $g(m/x) = 0$*,* $m/x \in P$ *and* $g(m/x) = 1$ *for* $m/x \in M \ominus O_r$*. Let us construct* $g(x) = \sum g_r(m/x)/2^r$ *and since, the series is uniformly convergent so* g *is continuous. Also,* $g(m/x) = 0$ *for* $m/x \in P$ *and* $g(m/x) = 1$ *for* $m/x \notin P$*.*

Example 3 *Let* (M, τ) *be an* M*-topological space where* M *is the set of real numbers from* $[0,1]$ *with atmost repetition of elements* 2*, taken with the discrete* M*-topology. Then* M *is metrizable but it has no countable M-basis.*

The aforementioned illustration demonstrates that a countable M-basis is not sufficient for an M-topological space to be metrizable. This compel us to look at something more that provides both the necessary as well as sufficient conditions for an M-topological space to be metrizable which reminds us the so-called Nagata-Smirnov Metrization Theorem in general topology. Owing the same, we are establishing the result as follows:

Theorem 8 *A multiset topological space* M *is metrizable if and only if it is M-regular with a countably locally finite M-basis.*

Proof 11 *Let us consider an M-regular multiset topological space* M *with a countably locally finite basis* $\mathscr{A}$*. Then by theorem 11,* M *is M-normal and every closed m-set in* M *is a* G_δ *m-set.*

Our aim is to show that M *is metrizable. We shall prove this by embedding* M *into the space* $((\mathbb{R}^m_M)^J, \rho_M)$ *for some* J*.*
Let $\mathscr{A} = \bigcup \mathscr{A}_r$*, where each* $\mathscr{A}_r$ *is locally finite. For each positive integer* r *and each element* $A \in \mathscr{A}_r$*, let us choose a continuous m-set function* $g_{r,A} : M \longrightarrow [0, 1/r]$ *with* $g_{r,A}(m/x) > 0$ *for* $m/x \in A$ *and* $g_{r,A}(m/x) = 0$ *for* $m/x \notin A$*.*
This collection $\{g_{r,A}\}$ *of functions isolates closed m-sets and multi points in* M*.*
For, let us consider a multipoint m_0/x_0 *and a neighbourhood* P *containing* m_0/x_0*, there exists an M-basis element* Q *such that* $m_0/x_0 \in Q \subset P$*. Then* $Q \in A_r$ *for some* r *such that* $g_{r,A}(m_0/x_0) > 0$ *and* $g_{r,A}$ *vanishes outside* P*.*
Let J *be the subset consisting pairs of* (r, A) *where* $r \in Z_+$ *and* $A \in \mathscr{A}_r$*.*
We define $G : M \longrightarrow [0,1]^J$ *by* $G(m/x) = (g_{r,A}(m/x))$*.*
We need to show that G *is an embedding. Since uniform* M*-topology is finer than the product* M*-topology, so* G *is injective and carries open m-sets from* M *to* $G(M)$*. Now its remain to prove the continuity of* G*.*
Let us consider a multipoint $m_0/x_0 \in M$ *and* $\varepsilon > 0$*. We have to find a neighbourhood* Q *such that*

$m_0/x_0 \in M \Longrightarrow \rho_M(G(m/x), G(m_0/x_0)) < \varepsilon$.
For fixed r, let us choose a neighbourhood P_r of m_0/x_0 which intersects only finitely many elements of $\mathscr{A}_r$. Since each m-set function $g_{r,A}$ is continuous, so $g_{r,A}$ are identically zero on P_r.
Let us consider a neighbourhood T_r such that $m_0/x_0 \in T_r \subset P_r$ where each of the remaining functions $g_{r,A}$ varies by atmost $\varepsilon/2$.
Now, we assume such a neighbourhood T_r containing m_0/x_0 for each r and an arbitrary number N such that $1/N \leq \varepsilon/2$. Also define $Q = T_1 \cap T_2 \cap ... \cap T_N$.
We claim that Q is the desired neighbourhood of m_0/x_0.
Let $m/x \in Q$. Then
(i) if $r \leq N$, then $\mid g_{r,A}(m/x) - g_{r,A}(m_0/x_0) \mid \leq \varepsilon/2$, since $g_{r,A}$ either vanishes identically or varies by atmost $\varepsilon/2$ on Q.
(ii) if $r > N$, then $\mid g_{r,A}(m/x) - g_{r,A}(m_0/x_0) \mid \leq 1/r < \varepsilon/2$ as $g_{r,A} : M \longrightarrow [0, 1/r]$.
Therefore, $\rho(G(m/x), G(m_0/x_0)) \leq \varepsilon/2 < \varepsilon$.

Conversely, let M is metrizable and M-regular. We have to show that M has a countably locally finite M-basis.
Let $\mathscr{A}_r = \{B(m/x, 1/r) : m/x \in M\}$ be a covering of M. Then by theorem 10, there must be a countably locally finite open covering $\mathscr{C}_r$ of M that refines $\mathscr{A}_r$.
Consider $\mathscr{C} = \bigcup_{r \in \mathbb{Z}_+} \mathscr{C}_r$. Then $\mathscr{C}$ is countably locally finite. We shall show that $\mathscr{C}$ is an M-basis for M.
Let $m/x \in M$ and $\varepsilon > 0$. We will prove that there is an element C of $\mathscr{C}$ such that $m/x \in C \subset B(m/x, \varepsilon)$.
Let us choose r such that $1/r < \varepsilon/2$. Since $\mathscr{C}_r$ covers M, we can choose $C \in \mathscr{C}_r$ such that $m/x \in C$ and $C \subset B(m/x, 2/r) \subset B(m/x, \varepsilon)$. Hence $\mathscr{C}$ is an M-basis for M.

5. Conclusions

The M-topology generated from m-set is being studied extensively since the inception of the same. Also, a metric is introduced in this context. This paper gives a meticulous study of metrization theory and its extension upto some extent by introducing a generalized metric on $\mathbb{R}$. This work may be upgraded to the direction of fuzzy multiset topology.

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