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# On the translation surfaces generated by spherical indicatrices of regular curves in Euclidean 3-space and their characterizations

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ABSTRACT: In the present paper, we investigate the translation surfaces generated by the tangent and normal indicatrices of two regular curves in three-dimensional Euclidean space. We establish the necessary and sufficient conditions for the generating curves of these translation surfaces to be geodesic lines, asymptotic lines, and lines of curvature. Furthermore, we identify the essential conditions for these translation surfaces to be developable or minimal. We conclude this work by generalizing the results obtained for spherical $k$-indicatrices.

Key Words: Translation surfaces, Geodesic line, Asymptotic line, line of curvature, Gaussian curvature, Mean curvature, Spherical k-indicatrices.

## Contents



## 1. Introduction

A Darboux surface is a surface composed of identical curves, known as generators, that are symmetrical through space isometries. A parameterization of this surface is defined by

$$\psi(u, v) = A(v).\alpha(u) + \beta(v),$$

where $\alpha$ and $\beta$ are two spatial curves and $A(v)$ is an orthogonal matrix.
In differential geometry, translation surfaces are well known as a special case of Darboux surfaces, with, the orthogonal matrix $A$ being an identity matrix and both curves intersect each other. A translation surface refers to a surface generated by translating one curve, denoted as $\alpha(u)$, parallel to itself along another curve, represented as $\beta(v)$, therefore, the parametric representation for this type of surface is given as

$$\psi(u, v) = \alpha(u) + \beta(v).$$

The theory of translation surfaces has always been an interesting topic in Euclidean space. Various differential geometers have previously explored the properties and characteristics of translation surfaces. Verstraelen et al. have investigated minimal translation surfaces of plane type in n-dimensional Euclidean spaces [17]. Liu obtained some characterizations of translation surfaces with constant mean curvature or constant Gauss curvature in Euclidean 3-space $\mathbb{E}^3$ and Minkowski 3-space $\mathbb{E}^3_1$ [11]. In [2] Ali et al. gave some results on curvatures of some special points of the translation surfaces in $\mathbb{E}^3$, in the same regard, Muntenau and Nistor studied the second fundamental form of the translation surfaces in Euclidean 3-space and they obtained some characterizations by using the second Gaussian curvature $K_{II}$ of the translation surfaces [13]. Recently, in [1] Neriman Acar et al. studied translation surfaces generated by

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the spherical indicatrices of space curves in $\mathbb{E}^3$, and obtained some characterizations based on the fact that these surfaces are developable or minimal. In [5] Cetin et al. have investigated geometric properties of surfaces that are parallel to translation surfaces in Euclidean 3-space. In [6,7] Cetin et al. studied translation surfaces in Euclidean 3-space generated by two space curves, and using non-planar space curves he expressed some properties of translation surfaces according to Frenet frames in Minkowski 3-space. Also lately In [18] A. Yadav and A. Yadav, delved into the translation surfaces generated by spherical indicatrices of timelike curves within Minkowski 3-space. Their study focused on examining the minimality and developability of these surfaces, as well as investigating specific properties of the generating curves.

In [10], we studied the translation surface generated by the principal normals of two regular curves provided with their alternative frames. Our aim was to determine the characteristics related to the minimality and the developability of this surface. Following this, we generalized the study to confirm the results obtained.

This paper delves into the characteristics of translation surfaces generated by the tangent and normal indicatrices of two regular curves, denoted as $\alpha$ and $\beta$, within three-dimensional Euclidean space. Building upon this investigation, and with the aim of enhancing the achieved outcomes, we equip the two space curves, $\alpha$ and $\beta$, with their respective Frenet frames to analyze the translation surface generated by their tangent indicatrices. Furthermore, we provide them with their alternative frames to explore the translation surface generated by their normal indicatrices. Subsequently, we extend these findings to generalize the results obtained. Our work is centered on determining the decisive conditions required for the generating curves of these translation surfaces, to become a geodesic line, asymptotic line, and line of curvature. Additionally, we identify the necessary and sufficient conditions that dictate whether these translation surfaces assume a developable or minimal surface configuration. Subsequently, we extended our investigation to the translation surfaces generated by the spherical k-indicatrices, to validate the achieved outcomes.

## 2. Preliminaries

We denote by $\mathbb{E}^3$ a three-dimensional Euclidean space, and by $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ $(s \to \alpha(s))$ a regular curve in $\mathbb{E}^3$, parameterized by arc length.

The Serret-Frenet frame along the curve $\alpha$ is the orthonormal frame, denoted as

$$(T(s), N(s), B(s)),$$

where

$$T(s) = \alpha'(s), \qquad N(s) = \frac{T'(s)}{\| T'(s) \|} \quad \text{and} \quad B(s) = T(s) \wedge N(s).$$

The Serret-Frenet derivation formulas are given by the following matrix representation:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix},$$

where $\kappa(s)$ and $\tau(s)$ are respectively the curvature and torsion of the curve at the point $\alpha(s)$.

**Definition 2.1** *[16] The curve $\alpha$ is called a general helix if the tangent vector at each point makes a constant angle with a fixed direction.*

**Proposition 2.1** *[16] The curve $\alpha$ is a general helix if and only if the ratio $\frac{\tau}{\kappa}$ is constant.*

The alternative frame of the curve $\alpha = \alpha(s)$ is the orthonormal frame representing the Serret-Frenet frame of the curve $s \in I \mapsto T(s)$, denoted as:

$$(N(s), C(s), W(s)),$$

where

$$C(s) = \frac{N'(s)}{\| N'(s) \|} \qquad \text{and} \qquad W(s) = N(s) \wedge C(s).$$

The derivative formulas of the alternative frame are defined as follows:

$$\begin{pmatrix} N'(s) \\ C'(s) \\ W'(s) \end{pmatrix} = \begin{pmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{pmatrix} \begin{pmatrix} N(s) \\ C(s) \\ W(s) \end{pmatrix},$$

where

$$f = \sqrt{\kappa^2 + \tau^2}, \qquad g = \sigma f \qquad \text{and} \qquad \sigma = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)'. \tag{2.1}$$

**Definition 2.2** *[14] The curve $\alpha$ is said to be a slant helix if the normal vector at each point makes a constant angle with a fixed direction.*

**Proposition 2.2** *[14] The curve $\alpha$ is a slant helix if and only if the function $\sigma = \frac{g}{f}$ is constant.*

Let $S$ be a regular surface in $\mathbb{E}^3$ defined by $X = X(u, v)$. We call the **unit normal vector** to the surface $S$, the vector:

$$U(u, v) = \frac{X_u \wedge X_v}{\| X_u \wedge X_v \|},$$

where $X_u = \frac{\partial X(u,v)}{\partial u}$, $X_v = \frac{\partial X(u,v)}{\partial v}$.
The coefficients of the first fundamental form and the second fundamental form of the surface $X = X(u, v)$ are given respectively by:

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,$$
$$l = \langle X_{uu}, N \rangle, \quad m = \langle X_{uv}, N \rangle, \quad n = \langle X_{vv}, N \rangle.$$

The Gaussian curvature $K$ and the mean curvature $H$ of the surface $S$ are expressed as follows:

$$K = \frac{ln - m^2}{EG - F^2},$$
$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)}.$$

**Definition 2.3** *Let $S$ be a regular surface. It is said to be developable if its Gaussian curvature $K$ is zero at any point, and $S$ is said to be minimal if its mean curvature $H$ is zero at any point.*

**Definition 2.4** *A surface of constant angle in $\mathbb{E}^3$ is a surface whose unit normal vector makes a constant angle with a field of fixed direction.*

**Definition 2.5** *For a curve $\alpha$ lying on a regular surface $S$, we have the following*:

- *The normal curvature of $\alpha$ is given by*: $\kappa_n^\alpha = \langle T', U \rangle$,
- *The geodesic curvature of $\alpha$ is given by*: $\kappa_g^\alpha = \langle T', U \wedge T \rangle$,
- *The geodesic torsion of $\alpha$ is given by*: $\tau_g^\alpha = - \langle U', U \wedge T \rangle$,

*where $U$ is the unit normal vector to $S$ and $T$ is the tangent vector to $\alpha$.*

**Definition 2.6** *For a curve $\alpha$ lying on a regular surface $S$, we have the following definitions*:

- *$\alpha$ is an asymptotic line if its normal curvature is zero, i.e.* $\kappa_n^\alpha = 0$.
- *$\alpha$ is a geodesic line if its geodesic curvature is zero, i.e.* $\kappa_g^\alpha = 0$.
- *$\alpha$ is a line of curvature if its geodesic torsion is zero, i.e.* $\tau_g^\alpha = 0$.

## 3. Translation surfaces generated by the tangent indicatrices of regular curves in $\mathbb{E}^3$

Let $u \to \alpha(u)$ and $v \to \beta(v)$ two non-degenerate curves of class $C^3$ of $\mathbb{E}^3$. Denote by $(T_\alpha, N_\alpha, B_\alpha, \kappa_\alpha, \tau_\alpha)$ and $(T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta)$ the Serret-Frenet frames of the curves $\alpha$ and $\beta$, respectively.
The translation surface generated by the tangent indicatrices of the curves $\alpha$ and $\beta$ is defined as follows:

$$M_T: \ X(u,v) \ = \ T_\alpha(u) \ + \ T_\beta(v).$$

The unit normal vector of the translation surface $M_T$ is given by:

$$U(u,v) = \frac{N_\alpha \wedge N_\beta}{\sin[\phi_T(u,v)]},$$

where $\phi_T = \phi_T(u,v)$ is the angle between the vectors $N_\alpha$ and $N_\beta$.
We have $\langle U, N_\alpha\rangle = \langle U, N_\beta\rangle = 0$, consequently, the unit normal vector $U$ of the surface $M_T$ can be expressed in the frames $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ as follows:

$$U = U_1 = \cos\theta_{T_\alpha} T_\alpha + \sin\theta_{T_\alpha} B_\alpha,$$
$$U = U_2 = \cos\theta_{T_\beta} T_\beta + \sin\theta_{T_\beta} B_\beta,$$

where $\theta_{T_\alpha}$ and $\theta_{T_\beta}$ are the angles between the vectors $T_\alpha$, $U_1$ and $T_\beta$, $U_2$ respectively.
Following that, we can express the tangent vector of the curve $u \to T_\alpha(u)$ as:

$$\frac{T'_\alpha(u)}{\parallel T'_\alpha(u) \parallel} = N_\alpha(u).$$

The normal curvature, geodesic curvature, and geodesic torsion of the curve $u \to T_\alpha(u)$ lying on $M_T$, denoted by $\kappa_n^{T_\alpha}$, $\kappa_g^{T_\alpha}$, and $\tau_g^{T_\alpha}$ respectively, are given as follows:

$$\begin{aligned} \kappa_n^{T_\alpha} &= -\,\kappa_\alpha \cos\theta_{T_\alpha} + \tau_\alpha \sin\theta_{T_\alpha}, \\ \kappa_g^{T_\alpha} &= \kappa_\alpha \sin\theta_{T_\alpha} + \tau_\alpha \cos\theta_{T_\alpha}, \\ \tau_g^{T_\alpha} &= -\,\theta'_{T_\alpha}, \end{aligned}$$

where $\theta'_{T\alpha}$ is the derivative of $\theta_{T_\alpha}$ with respect to $u$.

**Theorem 3.1**

- *The curve $u \to T_\alpha(u)$ is an asymptotic line if and only if* $\frac{\tau_\alpha}{\kappa_\alpha} = \cot\theta_{T_\alpha}$.
- *The curve $u \to T_\alpha(u)$ is a geodesic line if and only if* $\frac{\tau_\alpha}{\kappa_\alpha} = -\tan\theta_{T_\alpha}$.
- *The curve $u \to T_\alpha(u)$ is a line of curvature if and only if* $\frac{\partial\theta_{T_\alpha}}{\partial u} = 0$.

This leads to the following corollaries:

**Corollary 3.1** *If the generating curve $u \to T_\alpha(u)$ is an asymptotic line (resp. geodesic), then the angle $\theta_{T_\alpha}$ does not depend on $v$.*

**Corollary 3.2** *The generating curve $u \to T_\alpha(u)$ is a line of curvature if and only if the angle $\theta_{T_\alpha}$ does not depend on $u$.*

**Corollary 3.3** *If the generating curve $u \to T_\alpha(u)$ is an asymptotic line (resp. geodesic), then $u \to T_\alpha(u)$ is a line of curvature if and only if $\alpha$ is a general helix.*

**Proof:** Suppose that $u \to T_\alpha(u)$ is an asymptotic line, according to Theorem 3.1, we have:

$$\frac{\tau_\alpha}{\kappa_\alpha} \ = \ \cot\theta_{T_\alpha}.$$

Using Corollary 3.1, it follows that, $\frac{\tau_\alpha}{\kappa_\alpha}$ is constant if and only if $\theta_{T_\alpha}$ does not depend on $u$. and we conclude with Proposition 2.1 and Theorem 3.1.
The proof is the same for a geodesic line. □

**Corollary 3.4** *If the generating curve $u \to T_\alpha(u)$ is an asymptotic line (resp. geodesic), then the curve $\alpha$ is flat (plane) if and only if the angle $\theta_{T_\alpha} = \frac{\pi}{2} + k\pi$ (resp. $\theta_{T_\alpha} = k\pi$).*

The components of the first fundamental form of the surface $M_T$ are given by:

$$E = \kappa_\alpha^2, \qquad F = \kappa_\alpha \kappa_\beta \cos[\phi_T(u,v)], \qquad G = \kappa_\beta^2.$$

Similarly, the components of the second fundamental form of the surface $M_T$ are given by:

$$l = \kappa_\alpha \kappa_n^{T_\alpha}, \qquad m = 0, \qquad n = \kappa_\beta \kappa_n^{T_\beta}.$$

The Gaussian curvature $K$ and the mean curvature $H$ of the translation surface $M_T$ are given as follows:

$$K = \frac{\kappa_n^{T_\alpha} \kappa_n^{T_\beta}}{\kappa_\alpha \kappa_\beta \sin^2[\phi_T(u,v)]},$$

$$H = \frac{\kappa_\alpha \kappa_n^{T_\beta} + \kappa_\beta \kappa_n^{T_\alpha}}{2\kappa_\alpha \kappa_\beta \sin^2[\phi_T(u,v)]}.$$

**Theorem 3.2** *The surface $M_T$ is developable if and only if one of the two generating curves is an asymptotic line.*

**Corollary 3.5** *If the surface $M_T$ is developable, then the angle $\theta_{T_\alpha}$ is a function that depends only on $u$ or the angle $\theta_{T_\beta}$ is a function that depends only on $v$.*

This result is obtained using the corollary 3.1.

**Corollary 3.6** *If the curves $\alpha$ and $\beta$ are general helices and the surface $M_T$ is developable, then one of the angles $\theta_{T_\alpha}$ or $\theta_{T_\beta}$ is constant.*

**Proof:** According to Theorems 3.1 and 3.2, $M_T$ is developable if and only if

$$\frac{\tau_\alpha}{\kappa_\alpha} = \cot \theta_{T_\alpha} \quad \texttt{or} \quad \frac{\tau_\beta}{\kappa_\beta} = \cot \theta_{T_\beta}, \tag{3.1}$$

and we conclude with the proposition 2.1. □

**Corollary 3.7** *If the curves $\alpha$ and $\beta$ are general helices and the surface $M_T$ is developable, then the surface $M_T$ is a constant angle surface.*

**Proof:** According to Corollary 3.6, one of the two angles $\theta_{T_\alpha}$ and $\theta_{T_\beta}$ is constant.
Without loss of generality, we assume that $\theta_{T_\alpha} = \theta_0$ is constant. Since $\alpha$ is a general helix, there exists a constant unit direction $d_\alpha$ that makes a constant angle with the tangent vector $T_\alpha$, such that,

$$\langle T_\alpha, d_\alpha \rangle = \cos \delta_0 = \ cste.$$

We can define $d_\alpha$ as follows:

$$d_\alpha = \cos \delta_0 T_\alpha + \sin \delta_0 B_\alpha,$$

then

$$\begin{aligned} \langle U_1, d_\alpha \rangle &= \langle \cos \theta_0 T_\alpha + \sin \theta_0 B_\alpha, \cos \delta_0 T_\alpha + \sin \delta_0 B_\alpha \rangle \\ &= \cos \theta_0 \cos \delta_0 + \sin \theta_0 \sin \delta_0 \\ &= cste, \end{aligned}$$

which completes the proof. □

Considering the expression for the mean curvature $H$, we get:

**Theorem 3.3** *If $u \to T_\alpha(u)$ is an asymptotic line, then the surface $M_T$ is minimal if and only if $v \to T_\beta(v)$ is also an asymptotic line.*

**Corollary 3.8** *If the surface $M_T$ is minimal, then all its points are parabolic or hyperbolic.*

**Proof:** If $M_T$ is a minimal surface then,

$$\kappa_\alpha \kappa_n^{T_\beta} + \kappa_\beta \kappa_n^{T_\alpha} = 0.$$

And therefore the Gaussian curvature $K$ takes the following values:

$$K = -\Big(\frac{\kappa_n^{T_\alpha}}{\kappa_\alpha \sin[\phi_T(u,v)]}\Big)^2 \qquad \Big( or \qquad K = -\Big(\frac{\kappa_n^{T_\beta}}{\kappa_\beta \sin[\phi_T(u,v)]}\Big)^2\Big),$$

hence the desired result. □

**Example 3.1** *Let $\alpha$ and $\beta$ be two coplanar curves defined by:*

$$\alpha(u) = \big(0, 2\cos(\frac{u}{2}), 2\sin(\frac{u}{2})\big),$$

$$\beta(v) = \big(0, v^2, v^3\big),$$

*The tangent indicatrices of the curves $\alpha$ and $\beta$, are as follows:*

$$T_\alpha(u) = \big(0, -\sin(\frac{u}{2}), \cos(\frac{u}{2})\big),$$

$$T_\beta(v) = \big(0, \frac{2}{\sqrt{4+9v^2}}, \frac{3v}{\sqrt{4+9v^2}}\big),$$

*The translation surface generated by the tangent indicatrices of $\alpha$ and $\beta$ is defined by:*

$$M_T : X(u,v) = \big(0, -\sin(\frac{u}{2}) + \frac{2}{\sqrt{4+9v^2}}, \cos(\frac{u}{2}) + \frac{3v}{\sqrt{4+9v^2}}\big).$$

*The normal vector of the translation surface $M_T$ is given by:*

$$\begin{aligned} U(u,v) &= \frac{X_u \wedge X_v}{\| X_u \wedge X_v \|} \\ &= (-1, 0, 0). \end{aligned}$$

*The normal curvature, geodesic curvature and geodesic torsion of the curve $u \to T_\alpha(u)$ and of the curve $v \to T_\beta(v)$ lying on $M_T$, are given as:*

$$\kappa_n^{T_\alpha} = 0, \qquad \kappa_g^{T_\alpha} = -\frac{1}{2}, \qquad \tau_g^{T_\alpha} = 0,$$

$$\kappa_n^{T_\beta} = 0, \qquad \kappa_g^{T_\beta} = -\frac{24 + 54v^2}{(4+9v^2)^2}, \qquad \tau_g^{T_\beta} = 0.$$

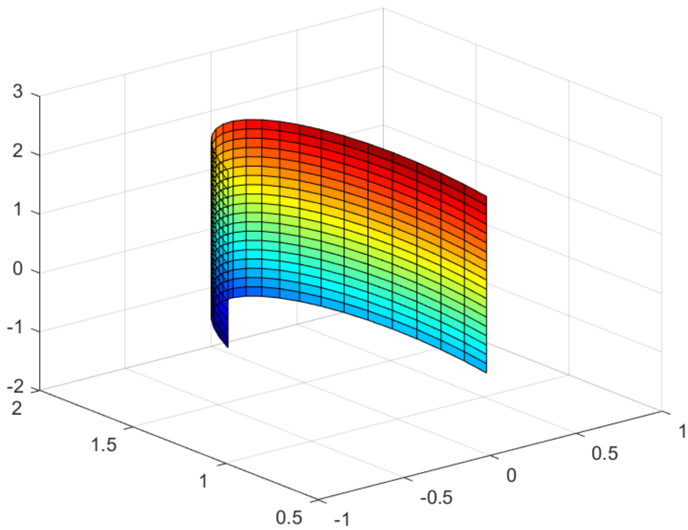


Figure 1: Translation surface $M_T$ generated by tangent indicatrices

## 4. Translation surfaces generated by principal normal indicatrices of $\mathbb{E}^3$ regular curves

Let $u \to \alpha(u)$ and $v \to \beta(v)$ be two non-degenerate curves of class $C^4$ of $\mathbb{E}^3$, and denote by $(N_\alpha, C_\alpha, W_\alpha, f_\alpha, g_\alpha)$ and $(N_\beta, C_\beta, W_\beta, f_\beta, g_\beta)$ the alternative frames of the curves $\alpha$ and $\beta$, respectively. The translation surface generated by the principal normal indicatrices of the curves $\alpha$ and $\beta$ is defined by:

$$M_N: \; X(u,v) \;=\; N_\alpha(u) \;+\; N_\beta(v),$$

and the unit normal vector of $M_N$ is given by:

$$U(u,v) = \frac{C_\alpha \wedge C_\beta}{\sin[\phi_N(u,v)]},$$

where $\phi_N = \phi_N(u,v)$ is the angle between the vectors $C_\alpha$ and $C_\beta$.
As $\langle U, C_\alpha \rangle = \langle U, C_\beta \rangle = 0$, the unit normal vector $U$ of the surface $M_N$ can be expressed respectively in the two frames $\{N_\alpha, C_\alpha, W_\alpha\}$ and $\{N_\beta, C_\beta, W_\beta\}$ as follows:

$$U = U_1 = \cos\theta_{N_\alpha} N_\alpha + \sin\theta_{N_\alpha} W_\alpha,$$
$$U = U_2 = \cos\theta_{N_\beta} N_\beta + \sin\theta_{N_\beta} W_\beta,$$

where $\theta_{N_\alpha}$ and $\theta_{N_\beta}$ are the angles between the vectors $N_\alpha$, $U_1$ and $N_\beta$, $U_2$, respectively.
Likewise, we can describe the expression for the tangent vector of the curve $u \to N_\alpha(u)$ as:

$$\frac{N'_\alpha(u)}{\parallel N'_\alpha(u) \parallel} = C_\alpha(u).$$

The normal curvature, geodesic curvature, and geodesic torsion of the generator $u \to N_\alpha(u)$, denoted respectively by $\kappa_n^{N_\alpha}$, $\kappa_g^{N_\alpha}$, and $\tau_g^{N_\alpha}$, assume the following values:

$$\begin{aligned}
\kappa_n^{N_\alpha} &= -\, f_\alpha \cos\theta_{N_\alpha} + g_\alpha \sin\theta_{N_\alpha}, \\
\kappa_g^{N_\alpha} &= f_\alpha \sin\theta_{N_\alpha} + g_\alpha \cos\theta_{N_\alpha}, \\
\tau_g^{N_\alpha} &= -\, \theta'_{N_\alpha},
\end{aligned}$$

where $\theta'_{N_\alpha}$ represents the derivative of $\theta_{N_\alpha}$ with respect to $u$.

**Theorem 4.1**
- *The curve $u \to N_\alpha(u)$ is an asymptotic line if and only if* $\frac{g_\alpha}{f_\alpha} = \cot\theta_{N_\alpha}$.
- *The curve $u \to N_\alpha(u)$ is a geodesic line if and only if* $\frac{g_\alpha}{f_\alpha} = -\tan\theta_{N_\alpha}$.
- *The curve $u \to N_\alpha(u)$ is a line of curvature if and only if* $\frac{\partial\theta_{N_\alpha}}{\partial u} = 0$.

**Corollary 4.1** *If the generating curve $u \to N_\alpha(u)$ is an asymptotic line (resp. geodesic), then the angle $\theta_{N_\alpha}$ is independent of $v$.*

**Corollary 4.2** *The generating curve $u \to N_\alpha(u)$ is a line of curvature if and only if the angle $\theta_{N_\alpha}$ is independent of $u$.*

**Corollary 4.3** *If the generating curve $u \to N_\alpha(u)$ is an asymptotic line (resp. geodesic), then $u \to N_\alpha(u)$ is a line of curvature if and only if $\alpha$ is a slant helix.*

**Proof:** Assuming that $u \to N_\alpha(u)$ is an asymptotic line, then according to the Theorem 4.1, this is equivalent to

$$\frac{g_\alpha}{f_\alpha} = \cot\theta_{N_\alpha},$$

and therefore $\theta_{N_\alpha}$ does not depend on $v$. It follows that $\frac{g_\alpha}{f_\alpha}$ is constant if and only if $\theta_{N_\alpha}$ does not depend on $u$, and we conclude using Proposition 2.2 and Theorem 4.1.
The reasoning is the same for a geodesic line. □

**Corollary 4.4** *If the generating curve $u \to N_\alpha(u)$ is an asymptotic line (resp. geodesic), then the curve $\alpha$ is a general helix if and only if the angle $\theta_{N_\alpha} = \frac{\pi}{2} + k\pi$ (resp. $\theta_{N_\alpha} = k\pi$).*

**Proof:** Using the Theorem 4.1 and the formula (2.1), $u \to N_\alpha(u)$ is an asymptotic line if and only if

$$\frac{\kappa_\alpha^2}{(\kappa_\alpha^2 + \tau_\alpha^2)^{\frac{3}{2}}}\Big(\frac{\tau_\alpha}{\kappa_\alpha}\Big)' = \cot\theta_{N_\alpha}$$

Therefore $\alpha$ is a general helix, that is, $\frac{\tau_\alpha}{\kappa_\alpha} = cste$ if and only if $\theta_{N_\alpha} = \frac{\pi}{2} + k\pi$.
For a geodesic line, the same reasoning applies. □

The components of the first and second fundamental forms of the surface $M_N$ are given by:

$$E = f_\alpha^2, \qquad F = f_\alpha f_\beta \cos[\phi_N(u,v)], \qquad G = f_\beta^2,$$
$$l = f_\alpha \kappa_n^{N_\alpha}, \qquad m = 0, \qquad n = f_\beta \kappa_n^{N_\beta}.$$

This gives, the Gaussian curvature $K$ and the mean curvature $H$ of the translation surface $M_N$:

$$K = \frac{\kappa_n^{N_\alpha}\kappa_n^{N_\beta}}{f_\alpha f_\beta \sin^2[\phi_N(u,v)]},$$
$$H = \frac{f_\alpha \kappa_n^{N_\beta} + f_\beta \kappa_n^{N_\alpha}}{2 f_\alpha f_\beta \sin^2[\phi_N(u,v)]}.$$

**Theorem 4.2** *The surface $M_N$ is developable if and only if one of the two generating curves is an asymptotic line.*

By using the corollary 4.1, it follows:

**Corollary 4.5** *If the surface $M_N$ is developable, then the angle $\theta_{N_\alpha}$ is a function that depends only on $u$ or the angle $\theta_{N_\beta}$ is a function that depends only on $v$.*

**Corollary 4.6** *If the curves $\alpha$ and $\beta$ are slant helices and the surface $M_N$ is developable, then one of the angles $\theta_{N_\alpha}$ or $\theta_{N_\beta}$ is constant.*

**Proof:** Using respectively the Theorems 4.2 and 4.1, as well as the characterization of the slant helix (Proposition 2.2), we achieve the desired result. □

**Corollary 4.7** *If the curves $\alpha$ and $\beta$ are slant helices and the surface $M_N$ is developable, then the surface $M_N$ is a constant angle surface.*

**Proof:** According to Corollary 4.6, we have one of the angles $\theta_{N_\alpha}$ and $\theta_{N_\beta}$ is constant.
Without loss of generality, we assume that $\theta_{N_\alpha} = \theta_0$ is constant. As $\alpha$ is a slant helix, there is a unit vector $d_\alpha$ which makes a constant angle with the normal vector $N_\alpha$, i.e.

$$\langle N_\alpha, d_\alpha \rangle = \cos\delta_0 = \ cste.$$

Let's assume:

$$d_\alpha = \cos\delta_0 N_\alpha + \sin\delta_0 W_\alpha,$$

hence

$$\begin{aligned}\langle U_1, d_\alpha \rangle &= \langle \cos\theta_0 N_\alpha + \sin\theta_0 W_\alpha, \cos\delta_0 N_\alpha + \sin\delta_0 W_\alpha \rangle \\ &= \cos\theta_0 \cos\delta_0 + \sin\theta_0 \sin\delta_0 \\ &= \ cste.\end{aligned}$$

□

**Theorem 4.3** *If $u \to N_\alpha(u)$ is an asymptotic line, then the surface $M_N$ is minimal if and only if $v \to N_\beta(v)$ is also an asymptotic line.*

**Corollary 4.8** *If the surface $M_N$ is minimal, then all of its points are either parabolic or hyperbolic.*

**Proof:** Indeed, if $M_N$ is minimal, then

$$f_\alpha \kappa_n^{N_\beta} = -f_\beta \kappa_n^{N_\alpha},$$

and substituting this to the expression of $K$, we obtain a negative or zero value. □

**Example 4.1** *Consider $\alpha$ and $\beta$, two curves parametrized by arc length and defined by:*

$$\begin{aligned}\alpha(u) &= \frac{1}{5}\Big(2\sin(2u) - \frac{\sin(8u)}{8}, -2\cos(2u) + \frac{\cos(8u)}{8}, \frac{4\sin(3u)}{3}\Big), \\ \beta(v) &= \Big(\frac{-\cos(4v)}{12} - \frac{\cos(2v)}{3}, -\frac{\sin(4v)}{12} - \frac{\sin(2v)}{3}, -\frac{2\sqrt{2}}{3}\cos(v)\Big).\end{aligned}$$

*By determining the principal normal indicatrices of the curves $\alpha$ and $\beta$, we obtain:*

$$N_\alpha(u) = \frac{1}{5}\big(4\cos(5u), 4\sin(5u), -3\big),$$
$$N_\beta(v) = \frac{1}{3}\big(2\sqrt{2}\cos(3v), 2\sqrt{2}\sin(3v), 1\big).$$

*The translation surface generated by the principal normal indicatrices of the curves $\alpha$ and $\beta$ is given by:*

$$M_N : X(u,v) = \Big(\frac{4}{5}\cos(5u) + \frac{2\sqrt{2}}{3}\cos(3v), \frac{4}{5}\sin(5u) + \frac{2\sqrt{2}}{3}\sin(3v), \frac{-4}{15}\Big).$$

*The normal vector of the translation surface* $M_N$ *is given by*:

$$\begin{aligned} U(u,v) &= \frac{X_u \wedge X_v}{\| X_u \wedge X_v \|} \\ &= (0,0,1). \end{aligned}$$

*The normal curvature, the geodesic curvature and the geodesic torsion of the curves* $u \to N_\alpha(u)$ *and* $v \to N_\beta(v)$, *are given as*:

$$\begin{aligned} \kappa_n^{N_\alpha} &= 0. & \kappa_g^{N_\alpha} &= 5, & \tau_g^{N_\alpha} &= 0, \\ \kappa_n^{N_\beta} &= 0, & \kappa_g^{N_\beta} &= 3, & \tau_g^{N_\beta} &= 0. \end{aligned}$$

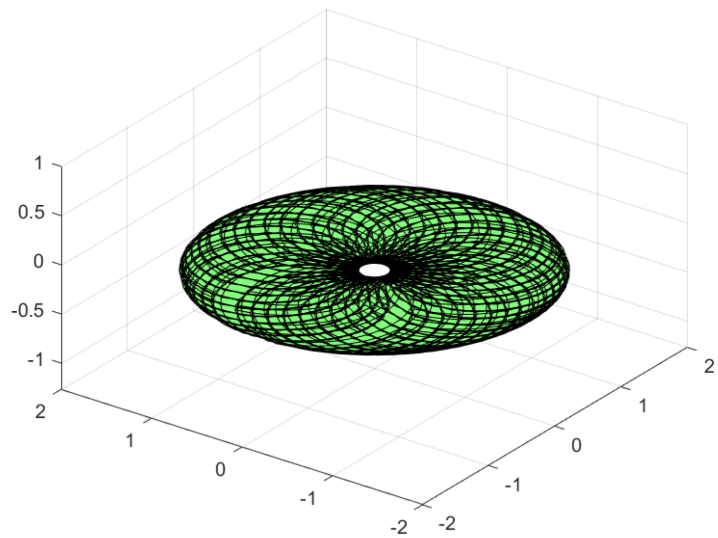


Figure 2: Translation surface $M_N$ generated by the principal normal indicatrices

## 5. Translation surfaces generated by spherical k-indicatrices of regular curves in $\mathbb{E}^3$

In this paragraph, we aim to generalize the findings from the two preceding paragraphs. Let's consider a curve $\alpha$ of class $C^n (n \geq k+2)$, where $k \in \mathbb{N}$. We define:

$$\begin{aligned} C_0(s) &= \alpha(s), \\ C_1(s) &= C_0'(s) = T(s), \\ C_2(s) &= \frac{C_1'(s)}{\| C_1'(s) \|} = N(s), \end{aligned}$$

and in general,

$$C_k(s) = \frac{C_{k-1}'(s)}{\| C_{k-1}'(s) \|}, \quad \text{for} \quad k \in \{1,2,...\}.$$

The frame

$$(C_k, C_{k+1}, W_{k+1}) \qquad \text{where} \qquad W_{k+1} = C_k \wedge C_{k+1},$$

is a direct orthonormal frame, representing the Serret-Frenet frame of the curve $s \to C_{k-1}(s)$, with the following derivative formulas:

$$\begin{pmatrix} C'_k(s) \\ C'_{k+1}(s) \\ W'_{k+1}(s) \end{pmatrix} = \begin{pmatrix} 0 & f_{k-1}(s) & 0 \\ -f_{k-1}(s) & 0 & g_{k-1}(s) \\ 0 & -g_{k-1}(s) & 0 \end{pmatrix} \begin{pmatrix} C_k(s) \\ C_{k+1}(s) \\ W_{k+1}(s) \end{pmatrix},$$

where $f_{k-1}$ and $g_{k-1}$ are the Frenet invariants of $s \to C_{k-1}(s)$. It is obvious that $f_0 = \kappa$, $f_1 = f$, $g_0 = \tau$ and $g_1 = g$.
On the other hand, $(C_{k+1}, C_{k+2}, W_{k+2})$ is the Serret-Frenet frame of $s \to C_k(s) = \frac{C'k-1(s)}{|C'k-1(s)|}$, which represents the unit tangent vector of $s \to C_{k-1}(s)$. So, their respective invariants are related by

$$\sigma_k = \frac{\sigma'_{k-1}}{f_{k-1}\left(1+\sigma^2_{k-1}\right)^{\frac{3}{2}}}, \tag{5.1}$$

where $\sigma_k = \frac{g_k}{f_k}$, $k \geq 1$. Note that $\sigma_0 = \frac{\tau}{\kappa}$ and $\sigma_1 = \sigma$.

**Definition 5.1** *[3] A regular arc of class $C^n$ $(n \geq k+1)$ is called a k-slant helix if the vector $C_{k+1} = \frac{C'_k}{\|C'_k(s)\|}$ makes a constant angle with a fixed direction.*

**Proposition 5.1** *[3] A regular arc of class $C^n$ $(n \geq k+1)$ is a k-slant helix if and only if the function $\sigma_k$ is constant.*

Indeed, if $\alpha$ is a k-slant helix, then the arc $C_k$ of Frenet frame $(C_{k+1}, C_{k+2}, W_{k+2})$ and invariants $f_k$ and $g_k$ is a general helix and therefore $\sigma_k = \frac{g_k}{f_k} = cste$.

Let $u \to \alpha(u)$ and $v \to \beta(v)$ two non-degenerate curves of class $C^n$ $(n \geq k+2)$ of $\mathbb{E}^3$, and denote by $(C_{k_\alpha}, C_{k+1_\alpha}, W_{k+1_\alpha}, f_{k-1_\alpha}, g_{k-1_\alpha})$ and $(C_{k_\beta}, C_{k+1_\beta}, W_{k+1_\beta}, f_{k-1_\beta}, g_{k-1_\beta})$ the Serret-Frenet frames of the curve $u \to C_{k-1_\alpha}(u)$ and the curve $v \to C_{k-1_\beta}(v)$, respectively.
The translation surface generated by the curves $C_{k_\alpha}$ and $C_{k_\beta}$, associated with the curves $\alpha$ and $\beta$, is defined by:

$$M_{C_k}: \; X(u,v) = C_{k_\alpha}(u) + C_{k_\beta}(v).$$

The unit normal vector of the translation surface $M_{C_k}$ is given by:

$$U(u,v) = \frac{C_{k+1_\alpha} \wedge C_{k+1_\beta}}{\sin[\phi_{C_k}(u,v)]},$$

where $\phi_{C_k} = \phi_{C_k}(u,v)$ is the angle between the vectors $C_{k+1_\alpha}$ and $C_{k+1_\beta}$.
The unit normal vector $U$ of surface $M_{C_k}$ can be expressed in the $\{C_{k_\alpha}, C_{k+1_\alpha}, W_{k+1_\alpha}\}$ frame and the $\{C_{k_\beta}, C_{k+1_\beta}, W_{k+1_\beta}\}$ frame as follows:

$$U = U_1 = \cos\theta_{C_{k_\alpha}} C_{k_\alpha} + \sin\theta_{C_{k_\alpha}} W_{k+1_\alpha},$$
$$U = U_2 = \cos\theta_{C_{k_\beta}} C_{k_\beta} + \sin\theta_{C_{k_\beta}} W_{k+1_\beta},$$

where $\theta_{C_{k_\alpha}}$ and $\theta_{C_{k_\beta}}$, are respectively the angles between the vectors $C_{k_\alpha}$, $U_1$ and $C_{k_\beta}$, $U_2$.
The tangent vector of the curve $u \to C_{k_\alpha}(u)$ is given by:

$$\frac{C'_{k_\alpha}(u)}{\| C'_{k_\alpha}(u) \|} = C_{k+1_\alpha}(u).$$

The normal curvature, geodesic curvature and geodesic torsion of the curve $u \to C_{k_\alpha}(u)$, have the following values:

$$\kappa_n^{C_{k_\alpha}} = -f_{k-1_\alpha}\cos\theta_{C_{k_\alpha}} + g_{k-1_\alpha}\sin\theta_{C_{k_\alpha}},$$

$$\kappa_g^{C_{k\alpha}} = f_{k-1_\alpha} \sin\theta_{C_{k_\alpha}} + g_{k-1_\alpha} \cos\theta_{C_{k_\alpha}},$$
$$\tau_g^{C_{k\alpha}} = -\,\theta'_{C_{k_\alpha}},$$

where $\theta'_{C_{k_\alpha}}$ is the derivative of $\theta_{C_{k_\alpha}}$ with respect to $u$.

**Theorem 5.1**
- *The curve $u \to C_{k_\alpha}(u)$ is an asymptotic line if and only if* $\sigma_{k-1_\alpha} = \cot\theta_{C_{k_\alpha}}$.
- *The curve $u \to C_{k_\alpha}(u)$ is a geodesic line if and only if* $\sigma_{k-1_\alpha} = -\tan\theta_{C_{k_\alpha}}$.
- *The curve $u \to C_{k_\alpha}(u)$ is a line of curvature if and only if* $\frac{\partial\theta_{C_{k_\alpha}}}{\partial u} = 0$.

We can give the following results without proof:

**Corollary 5.1** *If the generating curve $u \to C_{k_\alpha}(u)$ is an asymptotic line (resp. geodesic), then the angle $\theta_{C_{k_\alpha}}$ does not depend on $v$.*

**Corollary 5.2** *The generating curve $u \to C_{k_\alpha}(u)$ is a line of curvature if and only if the angle $\theta_{C_{k_\alpha}}$ does not depend on $u$.*

**Corollary 5.3** *If the generating curve $u \to C_{k_\alpha}(u)$ is an asymptotic line (resp. geodesic), then $u \to C_{k_\alpha}(u)$ is a line of curvature if and only if $\alpha$ is a (k-1)-slant helix.*

**Corollary 5.4** *If the generating curve $u \to C_{k_\alpha}(u)$ is an asymptotic line (resp. geodesic), then the curve $\alpha$ is a (k-2)-slant helix if only if the angle $\theta_{C_{k_\alpha}} = \frac{\pi}{2} + k\pi$ (resp. $\theta_{C_{k_\alpha}} = k\pi$).*

The components of the two fundamental forms of the surface $M_{C_k}$ are given by:

$$E = f^2_{k-1_\alpha}, \qquad F = f_{k-1_\alpha} f_{k-1_\beta} \cos\phi_{C_k}, \qquad G = f^2_{k-1_\beta},$$
$$l = f_{k-1_\alpha}\kappa_n^{C_{k_\alpha}}, \qquad m = 0, \qquad n = f_{k-1_\beta}\kappa_n^{C_{k_\beta}}.$$

The Gaussian curvature $K$ and the mean curvature $H$ of the translation surface $M_{C_k}$ are given respectively by:

$$K = \frac{\kappa_n^{C_{k_\alpha}} \kappa_n^{C_{k_\beta}}}{f_{k-1_\alpha} f_{k-1_\beta} \sin^2\phi_{C_k}},$$
$$H = \frac{f_{k-1_\alpha}\kappa_n^{C_{k_\beta}} + f_{k-1_\beta}\kappa_n^{C_{k_\alpha}}}{2 f_{k-1_\alpha} f_{k-1_\beta} \sin^2\phi_{C_k}}.$$

**Theorem 5.2** *The surface $M_{C_k}$ is developable if and only if one of the two generating curves is an asymptotic line.*

**Corollary 5.5** *If the surface $M_{C_k}$ is developable, then either the angle $\theta_{C_{k_\alpha}}$ is a function that depends only on $u$, or the angle $\theta_{C_{k_\beta}}$ is a function that depends only on $v$.*

**Corollary 5.6** *If the curves $\alpha$ and $\beta$ are (k-1)-slant helices and the surface $M_{C_k}$ is developable, then one of the angles $\theta_{C_{k_\alpha}}$ or $\theta_{C_{k_\beta}}$ is constant.*

**Corollary 5.7** *If the curves $\alpha$ and $\beta$ are (k-1)-slant helices and the surface $M_{C_k}$ is developable, then the surface $M_{C_k}$ is a constant angle surface.*

**Theorem 5.3** *If $u \to C_{k_\alpha}(u)$ is an asymptotic line, then the surface $M_{C_k}$ is minimal if and only if $v \to C_{k_\beta}(v)$ is also an asymptotic line.*

**Corollary 5.8** *If the surface $M_{C_k}$ is minimal, then all its points are either parabolic or hyperbolic.*

Indeed, the Gaussian curvature $K$ of the minimal surface $M_{C_k}$ takes the following values:

$$K = -\left(\frac{\kappa_n^{C_{k_\alpha}}}{f_{k-1_\alpha}\sin\phi_{C_k}}\right)^2 \quad \left( or \quad K = -\left(\frac{\kappa_n^{C_{k_\beta}}}{f_{k-1_\beta}\sin\phi_{C_k}}\right)^2 \right).$$

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