Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–7.
ISSN-0037-8712
doi:10.5269/bspm.76197

$\diamond$-hyperconnected spaces

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ABSTRACT: The purpose of this paper is to introduce and study the concept of $\diamond$-hyperconnectedness as a generalization of hyperconnectedness in ideal topological spaces. Several characterizations of it are established and it is shown that hyperconnectedness and $\diamond$-hyperconnectedness coincide in case of trivial and codense ideal.

Key Words: Connectedness, hyperconnectedness, ideal.

Contents

1. Introduction

A topological space X is called hyperconnected [16] if every pair of nonempty open sets of X has a nonempty intersection. Several concepts equivalent to hyperconnectedness were defined and investigated in the literature. Levine [9] introduced the notion of D-space and shown that a topological space X is D-space if and only if it is hyperconnected. Pipitone et al. [13] defined the concept of semi-connectedness and shown the equivalence of it with hyperconnectedness. Maheshwari et al. [10] introduced the notion of s-connectedness and shown the equivalence of s-connectedness and semi-connectedness. Thompson [17] called hyperconnected spaces as irreducible spaces. In [1], Ajmal and Kohli investigated further properties of hyperconnected spaces.

A nonempty collection $\mathcal{I}$ of subsets of X is called an ideal in X if it has the following properties: (i) if $M \in \mathcal{I}$ and $N \subseteq M$, then $N \in \mathcal{I}$ (hereditary) (ii) If $M \in \mathcal{I}$ and $N \in \mathcal{I}$, then $M \cup N \in \mathcal{I}$ (finite additivity). The notion of ideal topological spaces was studied by Kuratowski [7] and Vaidyanathaswamy [19]. Jankovic et al. [5] studied further properties of ideal topological spaces. The notion of connectedness and hyperconnectedness in ideal topological spaces was introduced by Ekici et al. [3]. Subsequently, Tyagi et al. [18] defined the notion of hyperconnectedness modulo an ideal in topological spaces. In recent years, several types of connectedness are studied in different branches of topological spaces. Singh et al. [14,15] introduced the notions of δ-connectedness modulo an ideal and δ^*-connectedness in ideal proximity spaces.

This paper is organised as follows. In Section 2, some basic definitions, notations and results are recalled which will be used in the next section. Section 3 is devoted to the notion of $\diamond$-hyperconnectedness. Relation between different connectednesses are obtained and several characterizations of it are given. It is shown that hyperconnectedness and $\diamond$-hyperconnectedness coincide in case of trivial and codense ideal.

2. Preliminaries

In this section, we recall some basic definitions, notations and results which are going to be used in the subsequent section. Throughout this article, $(X, \mathcal{T})$ (or simply X) will denote a topological space on which no separation axioms are assumed unless explicitly stated. For a subset U of X, $Cl(U)$ and $Int(U)$ will denote the closure and the interior of U, respectively.

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2010 *Mathematics Subject Classification:* 54D05, 03E15.
Submitted March 19, 2025. Published September 22, 2025

Given an ideal $\mathcal{I}$ in a space X and a subset U of X, the collection $I|_U = \{A \cap U : A \in \mathcal{I}\}$ is also an ideal. An ideal $\mathcal{I}$ in a space X is said to be codense if $\mathcal{I} \cap \mathcal{T} = \{\emptyset\}$. If f is a mapping from a set X to a set Y and $\mathcal{I}$ is an ideal in Y, then the collection $f^{-1}(\mathcal{I}) = \{U \subset X : f(U) \in \mathcal{I}\}$ is also an ideal in X.

Definition 2.1 *[8] Let X be a space. Then a subset U of X is said to be semi-open if there exists an open set P of X such that $P \subset U \subset Cl(P)$.*

The family of all semi-open subsets of X will be denoted by $SO(X, \mathcal{T})$ (or simply $SO(X)$).

Definition 2.2 *[12] Let $\mathcal{I}$ be an ideal in a space $(X, \mathcal{T})$. Then a subset U is said to be $\diamond$-connected if for all pair P, Q of open sets of X with $U = (U \cap P) \sqcup (U \cap Q)$, we have that $U \cap P \in \mathcal{I}$ or $U \cap Q \in \mathcal{I}$. The space $(X, \mathcal{T}, \mathcal{I})$ is said to be $\diamond$-connected if X is $\diamond$-connected.*

Proposition 2.1 *[12] Let $\mathcal{I}$ be an ideal in a space X. Then X is $\diamond$-connected if and only if for each clopen subset P of X, we have $P \in \mathcal{I}$ or $X \backslash P \in \mathcal{I}$.*

Proposition 2.2 *[18] Let Y and Z be subspaces of a space X and $\mathcal{I}$ be an ideal in X. Then $\mathcal{I}|_Y \cup \mathcal{I}|_Z \subseteq \mathcal{I}|_{Y \cup Z}$.*

3. $\diamond$-hyperconnectedness

In this section, we define and study $\diamond$-hyperconnectedness and it's relation with hyperconnectedness and $\diamond$-connectedness.

Definition 3.1 *Let $\mathcal{I}$ be an ideal in a topological space $(X, \mathcal{T})$. Then the space X is said to be $\diamond$-hyperconnected if for every pair U, V of nonempty disjoint open sets, we have $U \in \mathcal{I}$ or $V \in \mathcal{I}$.*

Definition 3.2 *Let $\mathcal{I}$ be an ideal in a topological space $(X, \mathcal{T})$ and $P \subset X$. Then $(P, \mathcal{T}|_P, \mathcal{I}|_P)$ is said to be $\diamond$-hyperconnected if it is $\diamond$-hyperconnected in the subspace topology.*

Observe that if $\mathcal{I} = \mathcal{P}(X)$, then $X \in \mathcal{I}$. Therefore X is $\diamond$-hyperconnected. Also if $\mathcal{I} = \{\emptyset\}$, then $\diamond$-hyperconnectedness coincide with hyperconnectedness. So, we assume that $\mathcal{I}$ is a non-trivial ideal throughout this paper unless or otherwise stated.

Proposition 3.1 *Every $\diamond$-hyperconnected space is $\diamond$-connected.*

Proof: Let $\mathcal{I}$ be an ideal in a space X. If X is $\diamond$-hyperconnected, then it is to show that X is $\diamond$-connected. So suppose that X is not $\diamond$-connected. Then there exists a clopen set P such that $P \notin \mathcal{I}$ and $X \backslash P \notin \mathcal{I}$. Therefore there is a pair $P, X \backslash P$ of nonempty disjoint open sets such that $P \notin \mathcal{I}$ and $X \backslash P \notin \mathcal{I}$. Thus, X is not $\diamond$-hyperconnected, a contradiction. $\square$

The converse of above proposition may not be true.

Example 3.1 Let $X = \mathbb{R}$ be the set of real numbers and $\mathcal{I}_c$ be an ideal of all the countable subsets of $\mathbb{R}$. Suppose $\mathcal{T}$ be a topology on $\mathbb{R}$ defined as:

$$\mathcal{T} = \{\emptyset, \mathbb{R}, (-\infty, 1), [1, 2), (-\infty, 2)\}$$

Then $\mathbb{R}$ is $\diamond$-connected as for every clopen set P, we have $P \in \mathcal{I}$ or $X \backslash P \in \mathcal{I}$. Note that $U = (-\infty, 1)$ and $V = [1, 2)$ are disjoint open sets such that $U \notin \mathcal{I}$ and $V \notin \mathcal{I}$. Therefore $\mathbb{R}$ is not $\diamond$-hyperconnected.

Proposition 3.2 *Every hyperconnected space is $\diamond$-hyperconnected.*

Proof: Let X be a hyperconnected space and $\mathcal{I}$ be an ideal in X. If X is not $\diamond$-hyperconnected space, then there exists a pair U, V of nonempty disjoint open sets such that $U \notin \mathcal{I}$ and $V \notin \mathcal{I}$. Therefore X is not hyperconnected, a contradiction. $\square$

Next example shows that converse of above proposition need not be true.

Example 3.2 Let $X = \mathbb{R}$ be the set of real numbers and $\mathcal{I}_f$ be an ideal of all the finite subsets of $\mathbb{R}$. Suppose $\mathcal{T}$ be a topology on $\mathbb{R}$ defined as:

$$\mathcal{T} = \{\emptyset, \mathbb{R}, \{a\}, \mathbb{R}\backslash\{a\}\}$$

Then, $Cl\{a\} = \{a\} \neq \mathbb{R}$. Therefore $\mathbb{R}$ is not hyperconnected. Also for every pair of nonempty disjoint open sets, namely $\{a\}$ and $\mathbb{R}\backslash\{a\}$, we have $\{a\} \in \mathcal{I}_f$. Thus, $\mathbb{R}$ is $\diamond$-hyperconnected.

Thus, we have the following relation among several types of connectedness in topological spaces.

$$\begin{array}{ccc} \diamond\text{-hyperconnected} & \Longleftarrow & \text{Hyperconnected} \\ \Downarrow & & \Downarrow \\ \diamond\text{-connected} & \Longleftarrow & \text{Connected} \end{array}$$

Next theorem gives a characterization for $\diamond$-hyperconnectedness in topological spaces.

Theorem 3.1 *Let X be a space and $\mathcal{I}$ be an ideal in X. Then the following statements are equivalent:*

(i) X is $\diamond$-hyperconnected.

(ii) If some open set P is not dense in X, then $P \in \mathcal{I}$ or $int(X\backslash P) \in \mathcal{I}$.

(iii) If F is a proper closed subset of X with nonempty interior, then $int(F) \in \mathcal{I}$ or $X\backslash F \in \mathcal{I}$.

Proof: $(i) \iff (ii)$. Let X be $\diamond$-hyperconnected and P be an open set in X such that $Cl(P) \neq X$. Then P and $X\backslash Cl(P)$ is a pair of disjoint open sets in X. Therefore, $P \in \mathcal{I}$ or $X\backslash Cl(P) = int(X\backslash P) \in \mathcal{I}$.

Conversely, if X is not $\diamond$-hyperconnected, then there exists a pair of disjoint open sets P, Q such that $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Then $Cl(P) \subset X\backslash Q \subset X$ as P and Q are disjoint. Therefore the open set P is not dense in X with $P \notin \mathcal{I}$ and $int(X\backslash P) \notin \mathcal{I}$ as $Q \subset int(X\backslash P)$, a contradiction to hypothesis.

$(i) \iff (iii)$. Let X be $\diamond$-hyperconnected and F be a proper closed set with nonempty interior. Then $int(F)$ and $X\backslash F$ is a pair of disjoint open sets. Therefore, $int(F) \in \mathcal{I}$ or $X\backslash F \in \mathcal{I}$.

Conversely, if X is not $\diamond$-hyperconnected, then there exists a pair of disjoint open sets P, Q such that $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Therefore $Cl(P)$ is a proper closed subset of X such that $int(Cl(P)) \neq \phi$ as $P \subset int(Cl(P))$. Since $P \notin \mathcal{I}$ and $P \subset int(Cl(P))$, therefore $int(Cl(P)) \notin \mathcal{I}$. Also, $Q \subset X\backslash Cl(P)$. Since $Q \notin \mathcal{I}$, therefore $X\backslash Cl(P) \notin \mathcal{I}$. Thus $Cl(P)$ is a proper closed set with nonempty interior such that $int(Cl(P)) \notin \mathcal{I}$ and $X\backslash Cl(P) \notin \mathcal{I}$, a contradiction to the hypothesis. □

Proposition 3.3 *Let $\mathcal{I}$ be an ideal in a space X and X be $\diamond$-hyperconnected. If X is the union of two proper closed sets, then one of the closed sets are not in $\mathcal{I}$.*

Proof: Let X be $\diamond$-hyperconnected and $X = F \cup G$, where F and G are proper closed sets. Then $X\backslash F$ and $X\backslash G$ make a pair of nonempty disjoint open sets. Therefore, $X\backslash F \in \mathcal{I}$ or $X\backslash G \in \mathcal{I}$. Thus, $F \notin \mathcal{I}$ or $G \notin \mathcal{I}$ as $X \notin \mathcal{I}$. □

Every subset of $\diamond$-hyperconnected space need not be $\diamond$-hyperconnected.

Example 3.3 Let $X = \{a, b, c\}$ with topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{I}$ be codense ideal. Then X is $\diamond$-hyperconnected space. If $U = \{b, c\}$, then $\mathcal{T}|_U = \{\emptyset, \{b\}, \{c\}, U\}$ and $\mathcal{I}|_U = \{\emptyset\}$. Therefore, $(U, \mathcal{T}|_U, \mathcal{I}|_U)$ is not $\diamond$-hyperconnected.

Proposition 3.4 *Every open subset of $\diamond$-hyperconnected space is $\diamond$-hyperconnected.*

Proof: Let U be an open subspace of X and X be $\diamond$-hyperconnected. Then for every pair P, Q of nonempty disjoint open subsets of U, P and Q are also open in X. Since X is $\diamond$-hyperconnected, therefore $P \in \mathcal{I}$ or $Q \in \mathcal{I}$. Thus, $P \in \mathcal{I}|_U$ or $Q \in \mathcal{I}|_U$. Hence, U is $\diamond$-hyperconnected. □

Following propostion shows that continuous image of $\diamond$-hyperconnected space is $\diamond$-hyperconnected.

Proposition 3.5 *Let $\mathcal{I}$ be an ideal in a space Y and f be a continuous surjective map from a space X onto Y. If X is $\diamond$-hyperconnected with respect to ideal $f^{-1}(\mathcal{I})$, then Y is also $\diamond$-hyperconnected.*

Proof: Let P, Q be a pair of disjoint open sets of Y. Then $f^{-1}(P), f^{-1}(Q)$ is a pair of disjoint open sets of X as f is a continuous map. Since X is $\diamond$-hyperconnected, therefore $f^{-1}(P) \in f^{-1}(\mathcal{I})$ or $f^{-1}(Q) \in f^{-1}(\mathcal{I})$. Thus, $P = f(f^{-1}(P)) \in f(f^{-1}(\mathcal{I})) = \mathcal{I}$ or $Q = f(f^{-1}(Q)) \in \mathcal{I}$. Hence, Y is also $\diamond$-hyperconnected. $\square$

Next proposition shows that if Y is $\diamond$-hyperconnected and $f : X \longrightarrow Y$ is an open injection, then X is also $\diamond$-hyperconnected.

Proposition 3.6 *Let $\mathcal{I}$ be an ideal in a space Y and f be an open injective map from a space X to Y. If Y is $\diamond$-hyperconnected, then X is $\diamond$-hyperconnected with respect to ideal $f^{-1}(\mathcal{I})$.*

Proof: Suppose that X is not $\diamond$-hyperconnected with respect to ideal $f^{-1}(\mathcal{I})$. Then there exists a pair P, Q of nonempty disjoint open sets in X such that $P \notin f^{-1}(\mathcal{I})$ and $Q \notin f^{-1}(\mathcal{I})$. Therefore, $f(P)$ and $f(Q)$ make a pair of disjoint nonempty open sets in Y with $f(P) \notin \mathcal{I}$ and $f(Q) \notin \mathcal{I}$. Thus, Y is not $\diamond$-hyperconnected, a contradiction. $\square$

Theorem 3.2 *Let X be a space and $\mathcal{I}$ be an ideal in X such that if an open set P is in $\mathcal{I}$, then $Cl(P) \in \mathcal{I}$. Then X is $\diamond$-hyperconnected if and only if $U \in \mathcal{I}$ or $V \in \mathcal{I}$ for any pair of disjoint semi-open sets U, V of X.*

Proof: Let X be $\diamond$-hyperconnected space and U, V be a pair of disjoint semi-open sets of X. Then there exist open sets P, Q such that $P \subset U \subset Cl(P)$ and $Q \subset V \subset Cl(Q)$. Since U, V are disjoint sets, therefore P, Q are also disjoint. Therefore $P \in \mathcal{I}$ or $Q \in \mathcal{I}$ as X is $\diamond$-hyperconnected. Thus by hypothesis, $Cl(P) \in \mathcal{I}$ or $Cl(Q) \in \mathcal{I}$. Therefore, $U \in \mathcal{I}$ or $V \in \mathcal{I}$.

Conversely, assume that U and V are disjoint open sets of X. Then U and V are also semi-open. Therefore by hypothesis, $U \in \mathcal{I}$ or $V \in \mathcal{I}$. Thus, X is $\diamond$-hyperconnected. $\square$

Proposition 3.7 *Let $\mathcal{I}$ be an ideal in a space X and $(U, \mathcal{T}|_U, \mathcal{I}|_U)$ be a $\diamond$-hyperconnected subspace of X. If $W \in \mathcal{I}$, then $(U \cup W, \mathcal{T}|_{U\cup W}, \mathcal{I}|_{U\cup W})$ is also $\diamond$-hyperconnected.*

Proof: Assume that $U \cup W$ is not $\diamond$-hyperconnected. Then there exists a pair P, Q of disjoint nonempty open sets in X such that $P \cap (U \cup W) \notin \mathcal{I}|_{U\cup W}$ and $Q \cap (U \cup W) \notin \mathcal{I}|_{U\cup W}$. Since $W \in I$, therefore $W \in \mathcal{I}|_W$. Thus, $P \cap U \notin \mathcal{I}|_U$ and $Q \cap U \notin \mathcal{I}|_U$. Therefore, $P \cap U$ and $Q \cap U$ are nonempty disjoint open sets of U such that $P \cap U \notin \mathcal{I}|_U$ and $Q \cap U \notin \mathcal{I}|_U$. Hence, U is not $\diamond$-hyperconnected, a contradiction. $\square$

Corollary 3.1 *Let $\mathcal{I}$ be an ideal in a space X and $(U, \mathcal{T}|_U)$ be hyperconnected subspace of X. If $W \in \mathcal{I}$, then $(U \cup W, \mathcal{T}|_{U\cup W}, \mathcal{I}|_{U\cup W})$ is also $\diamond$-hyperconnected.*

Proof: The proof follows from Proposition 3.7 and the fact that hyperconnectedness of X implies the $\diamond$-hyperconnectedness of X. $\square$

Theorem 3.3 *Let $\mathcal{I}$ be an ideal in a space X. If X is $\diamond$-hyperconnected with respect to $\mathcal{I}$, then there exists a maximal ideal $\mathcal{M}$ (with respect to set-theoretic inclusion $\subseteq$) containing $\mathcal{I}$ such that X is $\diamond$-hyperconnected with respect to $\mathcal{M}$.*

Proof: Let $\mathcal{F}$ be a family of all those ideals $\mathcal{J}$ in X which contain $\mathcal{I}$ and X is $\diamond$-hyperconnected with respect to $\mathcal{J}$. Then $\mathcal{F}$ is a partially ordered set with respect to set-theoretic inclusion $\subseteq$. Let $\mathcal{C}$ be a nonempty chain in $\mathcal{F}$ and $\mathcal{I}^* = \bigcup_{\mathcal{J}\in\mathcal{C}} \mathcal{J}$. Then $\mathcal{I}^*$ is an ideal containing $\mathcal{I}$. It is to show that X is $\diamond$-hyperconnected with respect to $\mathcal{I}^*$. If X is not $\diamond$-hyperconnected with respect to $\mathcal{I}^*$, then there exist a pair P, Q of nonempty disjoint open sets such that $P \notin \mathcal{I}^*$ and $Q \notin \mathcal{I}^*$. Therefore, $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Thus, X is not $\diamond$-hyperconnected with respect to $\mathcal{I}$, a contradiction. Hence, every nonempty chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. □

Next theorem shows that there may exist an ideal other than $\{\emptyset\}$ for which hyperconnectedness and $\diamond$-hyperconnectedness coincide.

Theorem 3.4 *Let $\mathcal{I}$ be a codense ideal in a space X. Then X is $\diamond$-hyperconnected if and only if X is hyperconnected.*

Proof: Let X be $\diamond$-hyperconnected but not hyperconnected. Then there exists a pair P, Q of disjoint open sets. Since $\mathcal{I}$ is a codense ideal, therefore $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Thus, X is not $\diamond$-hyperconnected, a contradiction.

Conversely, assume that X is not $\diamond$-hyperconnected. Then there exists a pair P, Q of disjoint open sets such that $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Therefore X is not hyperconnected. □

Proposition 3.8 *Let I be a codense ideal in a space X. If X is $\diamond$-hyperconnected, then X is pseudocompact.*

Proof: The proof follows from Theorem 3.4 and the fact that hyperconnectedness of X implies the pseudocompactness of X. □

Proposition 3.9 *Let I be a codense ideal in a space X. If X is $\diamond$-hyperconnected, then X is locally connected.*

Proof: The proof follows from Theorem 3.4 and the fact that hyperconnectedness of X implies the local connectedness of X. □

Theorem 3.5 *[4] A space X is hyperconnected if and only if the collection of not dense sets and nowhere dense sets are equal.*

Proposition 3.10 *Let I be a codense ideal in a space X. Then X is $\diamond$-hyperconnected if and only if the collection of not dense sets and nowhere dense sets are equal.*

Proof: The proof follows from Theorem 3.4 and Theorem 3.5. □

Theorem 3.6 *[11] Let X be a space. Then X is hyperconnected if and only if $SO(X, \mathcal{T}) \backslash \{\emptyset\}$ forms a filter on X.*

Proposition 3.11 *Let I be a codense ideal in a space X. Then X is $\diamond$-hyperconnected if and only if $SO(X, \mathcal{T}) \backslash \{\emptyset\}$ forms a filter on X.*

Proof: The proof follows from Theorem 3.4 and Theorem 3.6. □

Definition 3.3 *[2] Let $\mathcal{I}$ be an ideal in a space X. Then X is said to be submaximal if every dense set is open.*

Theorem 3.7 *Let $\mathcal{I}$ be a codense ideal in a submaximal space X. Then X is $\diamond$-hyperconnected if and only if $\mathcal{T} \backslash \{\emptyset\}$ is a filter on X.*

Proof: Let X be $\diamond$-hyperconnected and P be a nonempty open set such that $P \subset Q$. Then $X \backslash Cl(Q) \subset X \backslash P$. Therefore, P and $X \backslash Cl(Q)$ are disjoint open sets. Since X is $\diamond$-hyperconnected, therefore $X \backslash Cl(Q) \in \mathcal{I}$. Since $\mathcal{I}$ is codense, therefore $Cl(Q) = X$. Thus by submaximality of X, Q is open in X. Now let P and Q be two nonempty open sets in X. Then $P \cap Q$ is also nonempty open set in X otherwise X will not be $\diamond$-hyperconnected as $\mathcal{I}$ is codense. Hence, $\mathcal{T} \backslash \{\emptyset\}$ is a filter on X.

Conversely, suppose that $\mathcal{T} \backslash \{\emptyset\}$ is a filter on X. If X is not $\diamond$-hyperconnected, then there exists a pair P, Q of disjoint nonempty open sets such that $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Since $P \cap Q = \emptyset$, therefore $P \cap Q \notin \mathcal{T} \backslash \{\emptyset\}$ which is a contradiction. □

Definition 3.4 *[11] A topological property $\mathcal{P}$ is called contractive (expansive) if $(X, \mathcal{T})$ has the property $\mathcal{P}$ and $\mathcal{T}' \subseteq \mathcal{T}$ ($\mathcal{T} \subseteq \mathcal{T}'$), then $(X, \mathcal{T}')$ also has the property $\mathcal{P}$.*

Proposition 3.12 *$\diamond$-hyperconnectedness is a contractive property.*

Proof: Let $\mathcal{I}$ be an ideal in a space $(X, \mathcal{T})$ and $(X, \mathcal{T})$ be $\diamond$-hyperconnected. If $\mathcal{T}'$ is a topology containing $\mathcal{T}$, then it is to show that $(X, \mathcal{T}')$ is $\diamond$-hyperconnected. If $(X, \mathcal{T}')$ is not $\diamond$-hyperconnected, then there exist a pair P, Q of disjoint open sets in $\mathcal{T}'$ such that $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Since $\mathcal{T}' \subseteq \mathcal{T}$, therefore P and Q are also disjoint open sets in $\mathcal{T}$ with $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Thus, $(X, \mathcal{T})$ is not $\diamond$-hyperconnected, a contradiction. □

Definition 3.5 *[6] Let X be a space. Then X is called a door space if for each subset P of X, either P or $X \backslash P$ is open.*

Theorem 3.8 *Let $(X, \mathcal{T})$ be a space and $\mathcal{I}$ be a codense ideal in X. Then $(X, \mathcal{T})$ is $\diamond$-hyperconnected door space if and only if $\mathcal{T} \backslash \{\emptyset\}$ is an ultrafilter on X.*

Proof: Let $(X, \mathcal{T})$ be a $\diamond$-hyperconnected door space and $P, Q \in \mathcal{T} \backslash \{\emptyset\}$. Then it is to show that $P \cap Q \in \mathcal{T} \backslash \{\emptyset\}$. Suppose that $P \cap Q = \emptyset$. Since $(X, \mathcal{T})$ is $\diamond$-hyperconnected, therefore $P \in \mathcal{I}$ or $Q \in \mathcal{I}$. Thus, $P \in \mathcal{I} \cap (\mathcal{T} \backslash \{\emptyset\})$ or $Q \in \mathcal{I} \cap (\mathcal{T} \backslash \{\emptyset\})$ which is a contradiction as $\mathcal{I}$ is codense. Now let $P \in \mathcal{T} \backslash \{\emptyset\}$ and $P \subset Q$. If $Q = X$, then $Q \in \mathcal{T} \backslash \{\emptyset\}$. Otherwise, $Q \neq X$. So, suppose that $Q \notin \mathcal{T} \backslash \{\emptyset\}$. Then, $X \backslash Q \in \mathcal{T} \backslash \{\emptyset\}$ as X is door space. Since X is $\diamond$-hyperconnected, therefore $P \in \mathcal{I}$ or $X \backslash Q \in \mathcal{I}$. Thus, $P \in \mathcal{I} \cap (\mathcal{T} \backslash \{\emptyset\})$ or $X \backslash Q \in \mathcal{I} \cap (\mathcal{T} \backslash \{\emptyset\})$ which is a contradiction as $\mathcal{I}$ is codense. Assume that P is a nonempty proper subset of X. Then, by definition of door space, either P or $X \backslash P$ is open. Hence, $\mathcal{T} \backslash \{\emptyset\}$ is an ultrafilter on X.

Conversely, assume that $\mathcal{T} \backslash \{\emptyset\}$ is an ultrafilter on X. If X is not $\diamond$-hyperconnected, then there exists a pair P, Q of disjoint nonempty open sets such that $P \notin \mathcal{I}$ and $Q \notin \mathcal{I}$. Since $P \cap Q = \emptyset$, therefore $P \cap Q \notin \mathcal{T} \backslash \{\emptyset\}$ which is a contradiction. Since $\mathcal{T} \backslash \{\emptyset\}$ is an ultrafilter on X, therefore X is a door space. □

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