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Existence and Blow-up Results for the Logarithmic-Viscoelastic Wave Equation with Fractional Kirchhoff-type Nonlinearity and Variable Exponents

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ABSTRACT: In this paper, we investigate the existence of solutions for a viscoelastic wave equation with fractional Kirchhoff type and logarithmic source term and variables exponents. In the initial boundary value problem of this work, we focus on using the Gagliardo seminorm $[.]_{\alpha,2}$, and the fractional Laplace operator $(-\Delta)^{\alpha}$ where $0<\alpha<1$. Firstly, we prove the existence of the weak solutions, concerning this issue under suitable assumptions on the variables exponents of nonlinear source term, we use the Galerkin's approximation method. Subsequently, we establish the long-time behavior of solutions with nonpositive initial energy.

Key Words: Kirchhoff problems, Galerkin approximation, blow-up, a priori estimate, fractional Laplacian, viscoelastic wave equation, weak solution.

Contents

1. Introduction

The fractional viscoelastic wave equation with logarithmic source term, the well-posedness and long time behavior have been studied by many authors, they studied the cases where the exponents are either constants or variables, and they addressed two types of these equations, in [15], R. Aounallah, A. Choucha and S. Boulaaras, employed the Gamma or Beta functions, and they focused on a specific type of fractional viscoelastic wave equation given by

$$w_{tt}-\Delta w+\int_0^t f(t-s)\Delta w(s)ds+\partial_t^{\alpha,\beta}w-\Delta w_t=|w|^{p-2}w\ln|w|,$$

where $\partial_t^{\alpha,\beta}w$ is the fractional derivatives type, they used the energy method combined with the Faedo-Galerkin procedure for establish the existence of global solution and general decay behavior.

On the other hand, some other researchers have used the fractional Laplacian operator. This field, has attracted the attention of many researchers (see [5,6,7,9]). In [7], Xiang, Zhang and Hu investigated the existence of weak solution of the following fractional Kirchhoff type differential wave equation with discontinuous nonlinearity

$$u_{tt}+M\left([u]_{s,p}^p\right)(-\Delta)_p^s u+(-\Delta)^{\alpha}u_t+\mu(x,t)=f, \qquad (x,t)\in Q_T,$$

where $\alpha,s\in(0,1)$, $1<p<N/s$, by using a regularisation method combined with the Galerkin method.

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In [9], the researchers applied the Galerkin method to studie the local and global existence of solutions, then they established that the solution blows-up in finite time of the following initial boundary value problem

$$u_{tt} + M\left([u]_{\alpha,2}^2\right)(-\Delta)^\alpha u + (-\Delta)^s u_t = \int_0^t g(t-\tau)(-\Delta)^\alpha u(\tau)d\tau + \lambda|u|^{q-2}u,$$

where $0 < s \le \alpha < 1$, $1 < q < \infty$, and λ is a positive parameter. In [16] Li Zhang and Yang Liu studied a class of fractional viscelastic Kirchhoff equations with two nonlinear source terms of diferent signs, they used both potential well theory and Galerkin approximations to examine the asymptotic behavior of global solutions for the given initial boundary value problem

$$u_{tt} + \left(a + b[u]_m^{2p-2}\right)(-\Delta)^m u - \int_0^t g(t-\tau)(-\Delta)^m u(\tau)d\tau = |u|^{q-2}u - |u|^{r-2}u,$$

where $0 < m < 1$, $a > 0$, $b \ge 0$ and $p > 1$.

In this overage, we consider the existence and blow-up of weak solution for the following problem

$$\begin{cases} u_{tt} + K\left([u]_{\alpha,2}^2\right)(-\Delta)^\alpha u - \int_0^t f(t-s)(-\Delta)^\alpha u(s)ds + |u_t|^{z(.)-2}u_t \\ = |u|^{m(.)-2}u\ln|u|, \quad (x,t) \in \mathcal{Q}_T, \\ u(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad x \in \Omega. \end{cases} \tag{1.1}$$

Where $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ is a bounded domain, with a smooth boundary $\partial\Omega$, and $\mathcal{Q}_T = \Omega \times (0,T)$, $K : (0,+\infty) \longrightarrow (0,+\infty)$ is a continuous and nondecreacing function, $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is nonincreasing function. We denote by $[u]_{\alpha,2}$ the Gagliardo-seminorm of $u(x,t)$, defined by

$$[u]_{\alpha,2} = \left(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy\right)^{\frac{1}{2}}, \quad \alpha \in (0,1). \tag{1.2}$$

And $(-\Delta)^\alpha$ is the fractional Laplacian defined by

$$(-\Delta)^p g(x) = 2\lim_{\epsilon\to0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{g(x)-g(y)}{|x-y|^{N+2p}}dy \quad x \in \mathbb{R}^N,$$

where $B_\epsilon(x) = \{y \in \mathbb{R}^N : |y-x| < \epsilon\}$, and $g \in C_0^\infty(\mathbb{R}^N)$. For more details and properties of the fractional Laplacian and fractional Sobolev spaces see [13,14].

To illustrate the physical relevance of the model, consider a thin viscoelastic membrane, such us a synthetic rubber sheet, subjected to time-dependent loading. In this context, the fractional derivative captures the memory-dependent behavior of the material, the Kirchhoff-type term accounts for the stiffness variation due to accumulated deformation, and the logarithmic term describes the nonlinear stress response under significant compression. The use of variable exponents allows modeling heterogeneous materials with spatially varying properties.

2. Preliminaries

In this section, we present the following well-known lemmas, definition and propositions, which will be needed later. We will simplify the notation as follows $\|.\|_m$ the norm of $L^{m(.)}(\Omega)$.
To present our main results, we first provide the definition of weak solutions for the problem (1.1).

Definition 2.1 *[9], Let u is the weak solution of the problem (1.1), where*

$$u \in L^\infty(0,T,W^{\alpha,2}(\Omega)) \cap C(0,T,W^{\alpha,2})),$$

if $u_t \in L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,W^{z(x),2}(\Omega))$, *with*

$$\int_\Omega u_t(x,T)g(x,T)dx - \int_{Q\Omega} u_t g_t dxdt + \int_0^T K\left([u]^2_{\alpha,2}\right)(u,g)_{\alpha,2}dt - \int_0^T\int_0^t f(t-s)(u(s),g)_{\alpha,2}dsdt$$
$$+\int_0^T |u_t|^{z(x)-2}u_t g dt = \int_\Omega u_1 g(x,0)dx + \int_{Q\Omega}|u|^{m(x)-2}u\ln|u|gdxdt, \tag{2.1}$$

for all $g \in C^1(0,T,C_0^\infty(\Omega))$. *Here* $(u,g)_{\upsilon,2}$ *is defined as*

$$(u,g)_{\upsilon,2} = \int\int_{\mathbb{R}^{2N}} \frac{(u(x,t)-u(y,t))(g(x,t)-g(x,t))}{|x-y|^{N+2\upsilon}}dxdy, \qquad \upsilon \in (0,1).$$

We can say that u is a global weak solution of problem (1.1), if (3.9) holds for any $0<T<\infty$, *u is a local weak solution, if there exists* $T_0>0$ *such that (3.9) holds for* $0<T<T_0$.

In order to obtain the existence of weak solutions and the blow-up result of the problem (1.1), the following useful results are required.

Proposition 2.1 *[2,9] Let* $\mathcal{V}$ *be a Banach space which is dense and continuously embedded in Hilbert space* $\mathcal{H}$. *We identify* $\mathcal{H}=\mathcal{H}'$, *so that* $\mathcal{V}\hookrightarrow\mathcal{H}=\mathcal{H}'\hookrightarrow\mathcal{V}'$. *Then, the Banach space*

$$W_{z(.)} = \{v : v \in L^{z(.)}(0,T;\mathcal{V}), v_t \in L^{z'(.)}(0,T;\mathcal{V}')\} \subset C(0,T;\mathcal{H}).$$

Further, if $v \in W_{z(.)}$ *then* $\|v(.)\|_{L^2(\Omega)}$ *is absolutely continuous on* $[0,T]$, *we have*

$$\frac{d}{dt}\|v(.)\|^2_{L^2(\Omega)} = 2(v'(.),v(.))_{L^2(\Omega)} \quad a.e. \;\; on \;\; [0,T].$$

and there is a constant $C>0$ *such that*

$$\|v(.)\|_{C(0,T;\mathcal{H})} \le C\|v(.)\|_{W_{z(.)}}, \qquad \forall v \in W_{z(.)}$$

Proposition 2.2 *[2,9] Let* $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{B}_1$ *be Banach space, with* $\mathcal{B}_0\subset\mathcal{B}\subset\mathcal{B}_1$. *Assume that* $\mathcal{B}_0\hookrightarrow\mathcal{B}$ *is compact and* $\mathcal{B}\hookrightarrow\mathcal{B}_1$ *is continuous. Let (2.3) holds, with* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *be reflexive, and define*

$$W = \{v : v \in L^{z(.)}(0,T;\mathcal{B}_0), v_t \in L^{m(.)}(0,T;\mathcal{B}_1)\},$$

then the embedding $W \hookrightarrow L^{z(.)}(0,T;\mathcal{B})$ *is compact.*

Proposition 2.3 *[7,9] Let* Ω *be a bounded domain in* $\mathbb{R}^N$, *and let* $\{a_i\}_{i=1}^\infty$ *be an orthogonal basis in* $L^2(\Omega)$, *Then, for any* $\epsilon>0$, *there exists a constant* $M>0$ *such that*

$$\|u\|_{L^2(\Omega)} \le \left(\sum_{i=1}^M\left(\int_\Omega u a_i dx\right)^2\right)^{\frac{1}{2}} + \epsilon[u]_{\alpha,2},$$

for all $u \in W_0^{\alpha,2}(\Omega)$, *where* $2\alpha<N$.

With regard to the problem (1.1), We establish the following assumptions :

1. Let $\alpha \in]0,1[$, the fractional critical exponent is defined by

$$m^*_\alpha = \begin{cases} \dfrac{2N}{N-2\alpha}; & 2\alpha<N, \\ \infty; & 2\alpha\ge N. \end{cases} \tag{2.2}$$

2. Let Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, and

$$W_0^{\alpha,2}(\Omega)=\{u\in L^2(\Omega):[u]_{\alpha,2}<\infty, u=0 \quad a.e. \quad in \quad \mathbb{R}^N\setminus\Omega\},$$

endowed with the norm

$$[u]_{\alpha,2}=\left(\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy\right)^{\frac{1}{2}},$$

with the inner product given by

$$(u,\varphi)_{\alpha,2}=\int\int_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}}dxdy,$$

for any $u,\varphi\in W_0^{\alpha,2}$, makes $W_0^{\alpha,2}$ a Hilbert space.

3. The exponent functions $z(.)$, $m(.)$ are measrurable functions on Ω satisfying

$$2\le z_-\le z(x)\le z_+<m_-\le m(x)\le m_+<m^*, \tag{2.3}$$

for

$$m^*=\begin{cases}\dfrac{2N}{N-2}; & N\ge 3,\\ \infty; & N=1,2.\end{cases} \tag{2.4}$$

With

$$m^-:=ess\inf_{x\in\Omega}m(x), \qquad m^+:=ess\sup_{x\in\Omega}m(x).$$

4. Denote by $\mathcal{C}_\alpha$ the embedding constant from $W_0^{\alpha,2}(\Omega)$ to $L^2(\Omega)$ (see Lemma 2.1).

5. We assume that $K:(0,+\infty)\longrightarrow(0,+\infty)$ is a continuous and nondecreacing function satisfies
$(K_1): k_0:=K(0)>0$.
$(K_2): \exists\omega\ge 1:\omega\tilde{K}(\sigma)\ge K(\sigma)\sigma$, $\forall\sigma>0$, where $\tilde{K}(\sigma)=\int_0^\sigma K(s)ds$.

Lemma 2.1 *For $0<\alpha<1$, there exists a constant $\mathcal{C}_\alpha>0$ such that*

$$[u]^2_{\alpha,2}\le\mathcal{C}_\alpha\|\nabla u\|_2^2, \qquad u\in W_0^{\alpha,2}(\Omega). \tag{2.5}$$

Proof: Let $u\in W_0^{\alpha,2}(\Omega)$. By (1.2) we have

$$\begin{aligned}
[u]^2_{\alpha,2}&=\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy\\
&\le\int\int_{\{(x,y)\in\mathbb{R}^{2N};|x-y|\le 1\}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy+\int\int_{\{(x,y)\in\mathbb{R}^{2N};|x-y|>1\}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}dxdy\\
&\le 2\int\int_{\{(x,y)\in\mathbb{R}^{2N};|x-y|>1\}}\frac{|u(x)|^2+|u(y)|^2}{|x-y|^{N+2\alpha}}dxdy\\
&\le C_1\int_\Omega|u(x)|^2dx\int_1^\infty\frac{1}{\theta^{1+2\alpha}}d\theta+C_2\int_\Omega|u(y)|^2dx\int_1^\infty\frac{1}{\theta^{1+2\alpha}}d\theta\\
&\le C_3\int_\Omega|u(x)|^2dx\\
&\le C_3\|u\|_2^2\\
&\le\|\nabla u\|_2^2,
\end{aligned}$$

then

$$[u]_{\alpha,2}^2 \leq \mathcal{C}_\alpha \|\nabla u\|_2^2.$$

□

For more details on the properties of the Sobolev space with variable exponents, we refer to [11,13,14].

Lemma 2.2 *[12] For each $m > 0$, $|\tau^m \ln \tau| \leq \frac{1}{em}$, with $0 < \tau < 1$, and $0 \leq \tau^{-m} \ln \tau \leq \frac{1}{em}$ for $\tau \geq 1$.*

Lemma 2.3 *[11] Let $v \in L^\infty(0, T; H_0^1(\Omega) \setminus \{0\})$, then*

$$\int_\Omega |v|^{m(x)-2} v \ln |v| u_t^n dx \leq \int_\Omega u_t^2 dx + \mathcal{C}^\delta \|\nabla v\|_2^\delta < \infty,$$

where

$$\mathcal{C}^\delta = \frac{1}{4}\left[2\frac{|\Omega|}{e^2} + \frac{1}{e^2}C'\left(\frac{2}{\delta + 2 - 2m_+}\right) + \frac{1}{e^2}C'\left(\frac{2}{\delta + 2 - 2m_-}\right)\right].$$

Lemma 2.4 *[11] Let the assumptions (2.3) holds, and let u be the solution of (1.1), then*

$$\int_\Omega |u|^{z(x)} dx \geq \int_{\Omega_2} |u|^{z_-} dx := \|u\|_{z_-,\Omega_2}^{z_-}, \tag{2.6}$$

where $\Omega_2 = \{x \in \Omega : |u| \geq 1\}$.

Remark 2.1 *[11] Without loss of generality, we also state a well-known algebric inequality*

$$a^s \leq a + 1 \leq \left(1 + \frac{1}{b}\right)(a + b), \tag{2.7}$$

for all $a \geq 0$, $0 \leq s \leq 1$, and $b \geq 0$.

Now, we introduce the energy function associated of problem (1.1) as follows :

Lemma 2.5 *Suppose that (2.3), (K_1) and (K_2) hold, then*

$$\begin{aligned} E(t) = &\frac{1}{2}\left(\|u_t\|^2 + \tilde{K}([u]_{\alpha,2}^2) - \int_0^t f(t-s)ds[u]_{\alpha,2}^2 + \int_0^t f(t-s)[u(s) - u(t)]_{\alpha,2}^2 ds\right) \\ &- \int_\Omega \frac{1}{m(x)}|u|^{m(x)} \ln |u| dx + \int_\Omega \frac{1}{m^2(x)}|u|^{m(x)} dx, \end{aligned} \tag{2.8}$$

and

$$E'(t) = -\int_\Omega |u_t|^{z(x)} dx + \int_0^t f'(t-s)[u(s) - u(t)]_{\alpha,2}^2 ds - \frac{1}{2}f(t)[u]_{\alpha,2}^2.$$

Proof: Multiplying the equation in (1.1) by u_t and integrating over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\tilde{K}([u]_{\alpha,2}^2) - \int_\Omega \frac{1}{m(x)}|u|^{m(x)} \ln |u| dx - \int_\Omega \frac{1}{m^2(x)}|u|^{m(x)} dx\right) \\ - &\int_0^t f(t-s)(-\Delta)^\alpha (u(s), u_t(t))_{\alpha,2} ds - \int_\Omega |u_t|^{z(x)} dx, \end{aligned} \tag{2.9}$$

we estimate the following term as

$$\begin{aligned} &\int_0^t f(t-s)(-\Delta)^\alpha (u(s), u_t(t))_{\alpha,2} ds \\ &= \frac{1}{2}\frac{d}{dt}\left(\int_0^t f(t-s)[u(s) - u(t)]_{\alpha,2}^2 ds - \int_0^t f(t-s)ds[u]_{\alpha,2}^2\right) - \frac{1}{2}f(t)[u]_{\alpha,2}^2 \\ &+ \int_0^t f'(t-s)[u(s) - u(t)]_{\alpha,2}^2 ds, \end{aligned}$$

therefore, (2.9) becomes

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\tilde{K}([u]^2_{\alpha,2})-\frac{1}{2}\int_0^t f(t-s)ds[u]^2_{\alpha,2}+\frac{1}{2}\int_0^t f(t-s)[u(s)-u(t)]^2_{\alpha,2}ds\right)\\
-\ &\frac{d}{dt}\left(\int_\Omega \frac{1}{m(x)}|u|^{m(x)}\ln|u|dx-\int_\Omega \frac{1}{m^2(x)}|u|^{m(x)}dx\right)-\frac{1}{2}f(t)[u]^2_{\alpha,2}\\
+\ &\int_0^t f'(t-s)[u(s)-u(t)]^2_{\alpha,2}ds-\int_\Omega |u_t|^{z(x)}dx.
\end{aligned}$$

□

3. Existence of weak solutions

In this section, we will prove the existence of weak solutions of problem (1.1) by applying Galerkin method.

Theorem 3.1 *Let $2\alpha < N$, and $0 < \alpha < 1$. Suppose that K satisfies (K_1), and $u_0 \in W_0^{\alpha,2}(\Omega)$, $u_1 \in L^2(\Omega)$. Let f be a positive C^1 function satisfies*

$$l := \int_0^\infty f(t)dt < k_0, \tag{3.1}$$

and there exists $C_f > 0$ such that

$$f'(s) \leq C_f f(s), \qquad \forall (s \geq 0). \tag{3.2}$$

If $2 < m(x) \leq \frac{m^*_\alpha(z^*-1)+z^*}{z^*}$, then the problem (1.1) admits a local weak solution.

Remark 3.1 *Note that, it is commonly assumed that $f'(s) \leq 0$ for all $s \geq 0$. Definitely, our assumption (3.2) is weaker than in previous studies.*

To apply Galerkin method, it is allowed to choose a sequence

$$\{e_j\}_{j=1}^\infty \subset C_0^\infty(\Omega),$$

such that $C_0^\infty(\Omega) \subset \overline{\cup_{n=1}\infty \mathcal{V}_n}^{C^1(\overline{\Omega})}$, $\|e_j\|_{L^2(\Omega)} = 1$ and $\{e_j\}_{j=i}^\infty$ is a complete orthonormal basis in $L^2(\Omega)$, where $\mathcal{V} = span\{e_1, \ldots, e_n\}$, (see [7,9] for more details).
Moreover, since $C_0^\infty(\Omega) \subset \overline{\cup_{n=1}\infty \mathcal{V}_n}^{C^1(\overline{\Omega})}$, we have the following lemma :

Lemma 3.1 *[7,9] For $u_0 \in W_0^{\alpha,2}(\Omega)$, there existe a sequence $\{\chi_n\}$ with $\chi_n \in \mathcal{V}_n$ such that $\chi_n \to u_0$ in $W_0^{\alpha,2}(\Omega)$ as $n \to \infty$*

Below, we apply the Galerkin method to obtain the weak solutions of problem (1.1). For every $n \in \mathbb{N}$, we will find the sequence of approximate solutions with the following form

$$u^n(x,t) = \sum_{j=1}^n (\zeta_n(t))_j\, e_j(x),$$

where $(\zeta_n(t))_j$ are unkown functions determined by the following second order ordinary differential system

$$\begin{cases} \zeta''(t) + F_n(t,\zeta(t),\zeta'(t)) = L_n, \\ \zeta(0) = (U_{0n}), \qquad \zeta'(0) = U_{1n}. \end{cases} \tag{3.3}$$

Where $(U_{0n})_i = \int_\Omega \chi_n e_i dx$, $(U_{1n})_i = \int_\Omega \psi_n e_i dx$,

$$(L_n)_i = \int_\Omega |\sum_{j=1}^n \zeta_j e_j|^{m(x)-2}(\sum_{j=1}^n \zeta_j e_j)\ln|\sum_{j=1}^n \zeta_j e_j|e_i dx,$$

and $\chi_n \in \mathcal{V}_n$, $\psi_n \in \mathcal{V}_n$, and $\chi_n \longrightarrow u_0$ strongly in $W_0^{\alpha,2}(\Omega)$, $\psi_n \longrightarrow u_1$ stronly in $L^2(\Omega)$. We have $F_n(t,\eta,\zeta)$ is a vector valued function where $F_n(t,\eta,\zeta) : [0,T] \times \mathbb{R}^{2n}$ is defined as

$$(F_n(t,\eta,\zeta))_i = K\left([\sum_{j=1}^{n}\eta_j e_j]^2_{\alpha,2}\right)(\sum_{j=1}^{n}\eta_j e_j, e_i)_{\alpha,2}\int_0^t f(t-s)(\sum_{j=1}^{n}\eta_j e_j, e_i)_{\alpha,2}ds$$
$$+(\sum_{j=1}^{n}\zeta'_j e_j, e_i), \qquad (i=1,\dots,n),$$

here $\eta = (\eta_1,\dots,\eta_n)$, and $\zeta' = (\zeta'_1,\dots,\zeta'_n)$.
In [9], they used this method under a specific assumptions, such as $2 < q \le \frac{2^*_\alpha(2^*_s-1)}{2^*_s}+1$ where 2^*_α is the fractional critical exponent, and they studied two cases of λ, $\lambda > 0$ and $\lambda < 0$.
In our main results, we use the Galerkin method for the following assumptions :

Theorem 3.2 *If $2 < m_- \le m(x) \le m_+ \le \frac{m^*_\alpha(z^*-1)+z^*}{z^*}$, then there existes $T^{**} > 0$ such the problem (3.3) has a solution on $[0,T]$ with $0 < T < T^{**}$.*

Proof: Let $X(t) = (\zeta(t), \zeta'(t))$ and $H_n(t,X) = (\zeta'(t), L_n - F_n(t,\zeta,\zeta'))$, then (3.3) will be as follow

$$\begin{cases} X'(t) = H_n(t, X(t)), \\ \\ X(0) = (U_{0n}, U_{1n}). \end{cases} \tag{3.4}$$

By integrating, we have

$$\int_0^t f(t-s)(u^n(s), u^n_t)_{\alpha,2}ds = \frac{1}{2}\frac{d}{dt}\left(\int_0^t f(t-s)ds[u^n]^2_{\alpha,2} + \int_0^t f(t-s)[u^n(s)-u^n(t)]^2_{\alpha,2}ds\right)$$
$$+\frac{1}{2}\int_0^t f'(t-s)[u^n(s)-u^n(t)]^2_{\alpha,2}ds - \frac{1}{2}f(t)[u^n]^2_{\alpha,2}, \tag{3.5}$$

by (3.5) and (2.6), we have

$$\begin{aligned} F_n(t,\zeta,\zeta')\zeta' &= K\left([u^n]^2_{\alpha,2}\right)(u^n, u^n_t)_{\alpha,2}\int_0^t f(t-s)(u^n(s),u^n_t)_{\alpha,2}ds \\ &+ \int_\Omega |u^n_t|^{z(x)}dx \\ &= \frac{1}{2}\frac{d}{dt}\left(\tilde{K}\left([u^n]^2_{\alpha,2}\right) - \int_0^t f(t-s)ds[u^n]^2_{\alpha,2} - \int_0^t f(t-s)[u^n(s)-u^n(t)]^2_{\alpha,2}\right) \\ &- \frac{1}{2}\int_0^t f'(t-s)[u^n(s)-u^n(t)]^2_{\alpha,2}ds + \frac{1}{2}f(t)[u^n]^2_{\alpha,2} + \|u^n_t\|^{z_-}_{z_-,\Omega_2}, \end{aligned}$$

where $\tilde{K}\left([u^n]^2_{\alpha,2}\right) = \int_0^{[u^n]^2_{\alpha,2}} K(s)ds$.
From the Sobolev embedding, Hölder inequality, and Young's inequality, (3.4), and Lemmas 2.1 and Lemma 2.3, we obtain

$$\begin{aligned} &\int_\Omega u^n_{tt}u^n_t dx + \frac{1}{2}\frac{d}{dt}\left(\tilde{K}\left([u^n]^2_{\alpha,2}\right) - \int_0^t f(t-s)ds[u^n]^2_{\alpha,2} - \int_0^t f(t-s)[u^n(s)-u^n(t)]^2_{\alpha,2}\right) \\ &\le \frac{C_f}{2}\int_0^t f(t-s)[u^n(s)-u^n(t)]^2_{\alpha,2}ds - \|u^n_t\|^{z_-}_{z_-,\Omega_2} + \mathcal{C}^\delta\|\nabla u^n\|^\delta_2 \\ &\le \frac{C_f}{2}\int_0^t f(t-s)[u^n(s)-u^n(t)]^2_{\alpha,2}ds - \|u^n_t\|^{z_-}_{z_-,\Omega_2} + \lambda[u^n]^{2\delta}_{\alpha,2}, \end{aligned} \tag{3.6}$$

where λ is apositive constant associated with $(\delta, m_-, m_+, \mathcal{C}^\delta)$.
Denote

$$E_n(t) = \frac{1}{2}\int_\Omega |u_t^n|^2 dx + \frac{1}{2}\left(\tilde{K}\left([u^n]^2_{\alpha,2}\right) - \int_0^t f(t-s)ds[u^n]^2_{\alpha,2} + \int_0^t f(t-s)[u^n(s) - u^n(t)]^2_{\alpha,2}\right). \quad (3.7)$$

Since K is nondecreasing and $k_0 > l := \int_0^\infty f(t)dt$, we have

$$\tilde{K}\left([u^n]^2_{\alpha,2}\right) - \int_0^t f(t-s)ds[u^n]^2_{\alpha,2} \geq \left(k_0 - \int_0^\infty f(t)dt\right)[u^n]^2_{\alpha,2},$$

then

$$[u^n]^2_{\alpha,2} \leq \mathcal{K}E_n(t),$$

where $\mathcal{K} = \frac{2}{k_0 - l}$. From (3.6), (3.7) and Lemma 2.1 we can get

$$E_n'(t) \leq C_f E_n(t) + \kappa E_n^\delta(t).$$

This together with $\delta \neq 0$ yields that

$$\begin{aligned} E_n(t) &\leq \left(\sup_{n\geq 1} E_n^\delta(t)(0) - \kappa_1(e^{C_f\delta t} - 1)\right)^{-\frac{1}{\delta}} e^{C_f t} \\ &\leq \left(\sup_{n\geq 1} E_n^\delta(t)(0) - \kappa_1(e^{C_f\delta T} - 1)\right)^{-\frac{1}{\delta}} e^{C_f T} \\ &:= C(T), \end{aligned}$$

for each $t \in [0, T]$, where $\kappa_1 = \frac{\kappa}{C_f}$. Then

$$0 < T < T^{**} = \frac{\ln\left(1 + \frac{\sup_{n\geq 1} E_n^\delta(0)}{\kappa_1}\right)}{C_f\delta}.$$

By the definition of $H_n(t, X)$ and the continuity of K, we know that $H_n(t, X)$ is continuous with respect to (t, X). It follows from Peano's Theorem (see [1]) that there exists $\varsigma_n \in (0, T]$ such that (3.4) admits a C^1 solution on $[0, \varsigma_n]$. Then using a similar discussion as that in [7], the solution can be extended to interval $[0, T]$ with $0 < T < T^{**}$. □

Lemma 3.2 *(A priori estimate)*
If $2 < m_- \leq m(x) \leq m_+ \leq \frac{m^*_\alpha(z^*-1)+z^*}{z^*}$ *there exists* $C_T > 0$ *independent of* n*, such that the following estimate*

$$\int_\Omega |u_t^n|^2 dx + [u^n]^2_{\alpha,2} + \|u_t^n\|^{z_-}_{z_-,\Omega_2} \leq C(T), \qquad \forall t \in [0, T],$$

and

$$\int_0^T [u^n]^2_{\alpha,2}dt + \int_0^T \|u_t^n\|^{z_-}_{z_-,\Omega_2}dt \leq C(T),$$

hold, where

$$0 < T < T^{**} = \frac{\ln\left(1 + \frac{\sup_{n\geq 1} E_n^\delta(0)}{\kappa_1}\right)}{C_f\delta}.$$

Proof: By (3.3), for each $1 \le i \le n$, we have

$$\int_\Omega u^n_{tt} e_i dx + K\left([u^n]^2_{\alpha,2}\right)(u^n, e_i)_{\alpha,2} - \int_\Omega f(t-s)(u^n(s), e_i)_{\alpha,2} ds + \int_\Omega |u^n_t|^{z(x)-1} e_i dx$$
$$= \int_\Omega |u^n|^{m(x)-2} u^n \ln |u^n| e_i dx. \tag{3.8}$$

Multiplying (3.8) by $(\zeta'(t))_i$, we obtain

$$\int_\Omega u^n_{tt} u^n_t dx + K\left([u^n]^2_{\alpha,2}\right)(u^n, u^n_t)_{\alpha,2} - \int_\Omega f(t-s)(u^n(s), u^n_t)_{\alpha,2} ds + \int_\Omega |u^n_t|^{z(x)-1} u^n_t dx$$
$$= \int_\Omega |u^n|^{m(x)-2} u^n \ln |u^n| u^n_t dx, \tag{3.9}$$

so (3.9) implies

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |u^n_t|^2 dx + \tilde{K}\left([u^n]^2_{\alpha,2}\right) - \int_0^t f(t-s) ds [u^n]^2_{\alpha,2} + \int_0^t f(t-s)[u^n(s) - u^n(t)]^2_{\alpha,2} ds\right)$$
$$+ \|u^n_t\|^{z_-}_{z_-,\Omega} + \frac{1}{2} f(t)[u^n]^2_{\alpha,2}$$
$$\le \frac{C_f}{2}\int_0^t f(t-s)[u^n(s) - u^n(t)]^2_{\alpha,2} ds + C(\varepsilon)[u^n]^{2\delta}_{\alpha,2} + \varepsilon \|u^n_t\|^{z_-}_{z_-,\Omega}. \tag{3.10}$$

Similarly, we choose $\varepsilon = 1$, it yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |u^n_t|^2 dx + \tilde{K}\left([u^n]^2_{\alpha,2}\right) - \int_0^t f(t-s) ds [u^n]^2_{\alpha,2} + \int_0^t f(t-s)[u^n(s) - u^n(t)]^2_{\alpha,2} ds\right)$$
$$\le \frac{C_f}{2}\int_0^t f(t-s)[u^n(s) - u^n(t)]^2_{\alpha,2} ds + \|u^n_t\|^{z_-}_{z_-,\Omega_2} + \lambda [u^n]^{2\delta}_{\alpha,2}.$$

We use the similar discussion as Theorem 3.2, we obtain

$$E_n(t) \le C(T), \qquad (\forall t \in [0,T]),$$

where

$$0 < T < T^{**} = \frac{\ln\left(1 + \frac{\sup_{n\ge 1} E^\delta_n(0)}{\kappa_1}\right)}{C_f \delta}.$$

Furthermore, integrating (3.10) with respect to t over $(0,T)$ we have

$$\int_0^T [u^n]^2_{\alpha,2} dt + \int_0^T \|u^n_t\|^{z_-}_{z_-,\Omega_2} dt \le C(T),$$

for each $t \in [0,T]$, consequently, we complete the proof. □

Proof of Theorem 3.1 We apply Lemma 3.2, and using Proposition 2.3, and similar procecing as that in [7], one can prove that $u^n \to u$ strongly in $L^2(0,T; W^{\alpha,2}_0(\Omega))$, and $u^n_t \to u_t$ strongly in $L^2(\mathcal{Q}_T)$. Moreover, employing Proposition 2.1 and 2.2, one can obtain that u is a weak solution of problem (1.1).

4. Blow-up

In this section, we investigate the blow-up of solutions for problem (1.1), we assume that the condition (2.3) is satisfied, and $E(0) < 0$ where

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \tilde{K}([u]^2_{\alpha,2}) - \int_0^t f(t-s) ds [u]^2_{\alpha,2} + \int_0^t f(t-s)[u(s) - u(t)]^2_{\alpha,2} ds\right)$$
$$- \int_\Omega \frac{1}{m(x)} |u|^{m(x)} \ln |u| dx + \int_\Omega \frac{1}{m^2(x)} |u|^{m(x)} dx.$$

To proceed, we need to the following lemma indicating the decrease in energy E.

Lemma 4.1 *The energy related to the problem (1.1), given by (2.8) satisfies*

$$E'(t) = -\int_\Omega |u_t|^{z(x)}dx + \int_0^t f'(t-s)[u(s)-u(t)]^2_{\alpha,2}ds - \frac{1}{2}f(t)[u]^2_{\alpha,2} \leq 0, \tag{4.1}$$

and $E(t) \leq E(0)$ *holds.*

Let

$$H(t) = -E(t), \qquad t \geq 0. \tag{4.2}$$

As $E(t)$ is absolutely continuous, $H'(t) \geq 0$, and

$$\begin{aligned} H(t) &= -E(t) \\ &\leq \int_\Omega \frac{1}{m(x)}|u|^{m(x)} \ln |u| dx. \end{aligned}$$

Lemma 4.2 *Based on the assumption of Theorem 3.1 and (2.3) and (2.4), the function* $H(t)$ *introduced the following estimates*

$$0 < H(0) < H(t) \leq \frac{|\Omega|}{m_-} + \frac{C_\rho}{em_-(\rho - m_+)}\|\nabla u\|_2^\rho, \qquad (C_\rho > 0, \quad t \geq 0), \tag{4.3}$$

where C_ρ *is a constant of embedding* $H_0^1(\Omega)$ *in* $L^\rho(\Omega)$ *such that*

$$\|u\|_\rho \leq C_\rho \|\nabla u\|_2, \qquad \forall u \in H_0^1(\Omega), \tag{4.4}$$

and ρ *is chosen small enough, such that*

$$\begin{cases} m_- \leq m_+ \leq \rho \leq \dfrac{2N}{N-2}; & N \geq 3, \\ m_- \leq m_+ \leq \rho < \infty; & N = 1, 2. \end{cases} \tag{4.5}$$

Proof: By Lemma 4.1, we have $H(t)$ is nondecreasing function, thus we have

$$H(t) \geq H(0) = E(0) > 0, \quad (t \geq 0), \tag{4.6}$$

combining (2.8), (2.9) and (4.2) and Lemma 2.2, for any $\delta > 0$ we obtain

$$\begin{aligned} 0 < H(t) < \frac{1}{m_-}\int_\Omega |u|^{m(x)} \ln |u| dx &= \frac{1}{m_-}\int_{\Omega_1} |u|^{m(x)-1} \ln |u| dx + \frac{1}{m_-}\int_{\Omega_2} |u|^{m(x)} \ln |u| dx \\ &\leq \frac{|\Omega|}{m_-} + \frac{1}{\delta e m_-}\int_{\Omega_2} |u|^{m_+ + \delta} dx \\ &\leq \frac{|\Omega|}{m_-} + \frac{1}{\delta e m_-}\|u\|_{m_+ + \delta}^{m_+ + \delta} \\ &\leq \frac{|\Omega|}{m_-} + \frac{C_\rho}{em_-(\rho - m_+)}\|\nabla u\|_2^\rho, \end{aligned} \tag{4.7}$$

where $\Omega_1 = \{x \in \Omega : |u| < 1\}$ and $\Omega_2 = \{x \in \Omega : |u| \geq 1\}$. □

Theorem 4.1 *Suppose that the conditions of Theorem 3.1 are satisfied, furthermore, let (2.3) and (2.4) hold as well as* $E(0) < 0$*, then the solution of problem (1.1) blows up in finite time.*

Proof: For each $t \in [0, T)$, let

$$\Psi(t) = H^{1-\tau}(t) + \epsilon \int_\Omega u u_t dx, \tag{4.8}$$

with ϵ is small positive parameter enough to be chosen later, and τ such that

$$0<\tau\leq min\begin{Bmatrix} m_-(1-\lambda);\left(1+\dfrac{m_-(1-\lambda)}{2}\right);\dfrac{(m_-(1-\lambda)-1)}{2};\left(\dfrac{m_-(1-\lambda)}{2}-\theta\right);\dfrac{m_-(1-\lambda)}{2};\\ \left(1-\dfrac{\epsilon}{4\theta}\right)\displaystyle\int_0^t f(s)ds;\dfrac{m_-(1-\lambda)}{2}\displaystyle\int_0^t f(t-s)ds;\lambda \end{Bmatrix},$$

$$\Psi'(t)=(1-\tau)H'(t)H^{-\tau}+\epsilon\|u_t\|_2^2+\epsilon\int_\Omega uu_{tt}dx, \tag{4.9}$$

multiplying (1.1) by u and integrating over Ω we obtain

$$\begin{aligned}\int_\Omega u(t)u_ttdx=&-\int_\Omega K\left([u]^2_{\alpha,2}\right)(-\Delta)^\alpha uudx+\int_\Omega\int_0^t f(t-s)(-\Delta)^\alpha u(s)u(t)dsdx-\int_\Omega |u_t|^{z(.)-2}u_tu(t)dx\\&+\int_\Omega |u|^{m(.)}\ln|u|dx\\&=\int_\Omega |u|^{m(.)}\ln|u|dx-\frac{1}{2}\tilde{K}\left([u]^2_{\alpha,2}\right)-\int_\Omega |u_t|^{z(.)-2}u_tu(t)dx+\int_0^t f(s)ds[u]^2_{\alpha,2}\\&+\int_0^t\int_\Omega f(t-s)(u(s),u(t))^2_{\alpha,2}dxds,\end{aligned}$$

we apply Cauchy-Schwartz and Young's enquality on the last term, we obtain

$$\int_0^t\int_\Omega f(t-s)(u(s),u(t))_{\alpha,2}dxds\leq\theta\int_0^t f(t-s)[u(s)-u(t)]^2_{\alpha,2}ds+\frac{1}{4\theta}\int_0^t f(s)ds[u]^2_{\alpha,2},\qquad(\theta>0).$$

So (4.9) becomes

$$\begin{aligned}\Psi'(t)\geq&(1-\tau)H'(t)H^{-\tau}+\epsilon\|u_t\|_2^2+\epsilon\int_\Omega|u|^{m(.)}\ln|u|dx-\frac{\epsilon}{2}\tilde{K}\left([u]^2_{\alpha,2}\right)-\epsilon\int_\Omega|u_t|^{z(.)-2}u_tu(t)dx\\&+\epsilon\theta\int_0^t f(t-s)[u(s)-u(t)]^2_{\alpha,2}ds+\frac{\epsilon}{4\theta}\int_0^t f(s)ds[u]^2_{\alpha,2}+\epsilon\int_0^t f(s)ds[u]^2_{\alpha,2}.\end{aligned}\tag{4.10}$$

By subtracting and adding $\epsilon m_-(1-\lambda)H(t)$ (with $0<\lambda<\frac{m_--2}{m_-}$) on the right-hand side of (4.10), we obtain

$$\begin{aligned}\Psi'(t)\geq&(1-\tau)H'(t)H^{-\tau}+\epsilon m_-(1-\lambda)H(t)+\epsilon\left(1+\frac{m_-(1-\lambda)}{2}\right)\|u_t\|_2^2+\epsilon\int_\Omega|u|^{m(x)}\ln|u|dx\\&+\frac{\epsilon}{2}(m_-(1-\lambda)-1)\tilde{K}\left([u]^2_{\alpha,2}\right)+\epsilon\left(\frac{m_-(1-\lambda)}{2}-\theta\right)\int_0^t f(t-s)[u(s)-u(t)]^2_{\alpha,2}ds\\&+\epsilon\left(\left(1-\frac{\epsilon}{4\theta}\right)\int_0^t f(s)ds-\frac{m_-(1-\lambda)}{2}\int_0^t f(t-s)ds\right)[u]^2_{\alpha,2}+\epsilon m_-(1-\lambda)\int_\Omega\frac{1}{m^2(x)}|u|^{m(x)}dx\\&-\epsilon m_-(1-\lambda)\int_\Omega\frac{1}{m(x)}|u|^{m(x)}\ln|u|dx-\epsilon\int_\Omega|u_t|^{z(x)-2}u_tudx,\end{aligned}$$

then, for λ small enough, we obtain

$$
\begin{aligned}
\Psi'(t) \geq & (1-\tau)H'(t)H^{-\tau} + \epsilon m_-(1-\lambda)H(t)\epsilon\left(1+\frac{m_-(1-\lambda)}{2}\right)\|u_t\|_2^2 + \frac{\epsilon}{2}(m_-(1-\lambda)-1)\tilde{K}\left([u]^2_{\alpha,2}\right) \\
& + \epsilon\left(\frac{m_-(1-\lambda)}{2}-\theta\right)\int_0^t f(t-s)[u(s)-u(t)]^2_{\alpha,2}ds + \frac{\epsilon m_-(1-\lambda)}{2}\|u\|^{m_-}_{m_-,\Omega_2} - \epsilon\int_\Omega |u_t|^{z(x)-2}u_t u dx \\
& + \epsilon\left(\left(1-\frac{\epsilon}{4\theta}\right)\int_0^t f(s)ds - \frac{m_-(1-\lambda)}{2}\int_0^t f(t-s)ds\right)[u]^2_{\alpha,2} + \epsilon\lambda\int_\Omega |u|^{m(x)}\ln|u|dx \\
\geq & (1-\tau)H'(t)H^{-\tau} - \epsilon\int_\Omega |u_t|^{z(x)-2}u_t u dx + \epsilon\eta\Big(H(t) + \|u_t\|_2^2 + \tilde{K}\left([u]^2_{\alpha,2}\right) + \|u\|^{m_-}_{m_-,\Omega_2} \\
& + \int_0^t f(t-s)[u(s)-u(t)]^2_{\alpha,2}ds + [u]^2_{\alpha,2} + \int_\Omega |u|^{m(x)}\ln|u|dx\Big),
\end{aligned} \tag{4.11}
$$

where

$$
\eta = min\left\{\begin{array}{l} m_-(1-\lambda);\left(1+\frac{m_-(1-\lambda)}{2}\right);\frac{(m_-(1-\lambda)-1)}{2};\left(\frac{m_-(1-\lambda)}{2}-\theta\right);\frac{m_-(1-\lambda)}{2}; \\ \left(1-\frac{\epsilon}{4\theta}\right)\int_0^t f(s)ds;\frac{m_-(1-\lambda)}{2}\int_0^t f(t-s)ds;\lambda \end{array}\right\}.
$$

We estimate the following term, by using Young's enquality as below

$$
\begin{aligned}
\int_\Omega |u_t|^{z(x)-1}u dx &\leq \int_\Omega \frac{1}{z(x)}\beta^{z(x)}|u|^{z(x)}dx + \int_\Omega \frac{z(x)-1}{z(x)}\beta^{\frac{-z(x)}{z(x)-1}}|u_t|^{z(x)}dx \\
&\leq \frac{1}{z_-}\int_\Omega \beta^{z(x)}|u|^{z(x)}dx + \frac{z_+-1}{z_+}\int_\Omega \beta^{\frac{-z(x)}{z(x)-1}}|u_t|^{z(x)}dx, \qquad \forall\beta>0,
\end{aligned} \tag{4.12}
$$

we choose β such that $\beta^{\frac{-z(x)}{z(x)-1}} = \xi H^{-\tau}(t)$, for $\xi>0$ is large enough determined later, then (4.12) will be

$$
\begin{aligned}
\int_\Omega |u_t|^{z(x)-1}u dx &\leq \frac{1}{z_-}\int_\Omega \xi^{1-z(x)}H^{\tau(z(x)-1)}(t)|u|^{z(x)}dx + \frac{z_+-1}{z_+}\int_\Omega \xi H^{-\tau}(t)|u_t|^{z(x)}dx \\
&\leq \frac{1}{z_-}\xi^{(1-z_-)}H^{\tau(z_+-1)}(t)\int_\Omega |u|^{z(x)}dx + \frac{z_+-1}{z_+}H^{-\tau}(t)H'(t),
\end{aligned} \tag{4.13}
$$

we substitute (4.13) into (4.11) we obtain

$$
\begin{aligned}
\Psi'(t) \geq & \left((1-\tau)-\epsilon\frac{z_+-1}{z_+}\right)H'(t)H^{-\tau} - \epsilon\frac{1}{z_-}\xi^{(1-z_-)}H^{\tau(z_+-1)}(t)\int_\Omega |u|^{z(x)}dx + \epsilon\eta\Big[H(t)+\|u_t\|_2^2 \\
& + \tilde{K}\left([u]^2_{\alpha,2}\right) + \|u\|^{m_-}_{m_-,\Omega_2} + [u]^2_{\alpha,2} + \int_0^t f(t-s)[u(s)-u(t)]^2_{\alpha,2}ds + \int_\Omega |u|^{m(x)}\ln|u|dx\Big],
\end{aligned} \tag{4.14}
$$

using lemma 4.2 on the second term of the right-hand side of (4.14), we obtain

$$
\begin{aligned}
H^{\tau(z_+-1)}(t)\int_\Omega |u|^{z(x)}dx \leq & C\left(2^{\tau(z_+-1)-1}\left(\frac{|\Omega|}{em_-}\right)^{\tau(z^+-1)} + 2^{\tau(z_+-1)-1}\frac{1}{em_-(\rho-m_+)}\|\nabla u\|_2^{\rho\tau(z_+-1)}\right) \\
& \times\left(\|u\|^{z_-}_{m_-,\Omega_2} + \|u\|^{z_+}_{m_-,\Omega_2}\right) \\
\leq & 2^{\tau(z_+-1)-1}C\left(\frac{|\Omega|}{em_-}\right)^{\tau(z^+-1)}\left(\left(\|u\|^{m_-}_{m_-,\Omega_2}\right)^{\frac{z_-}{m_-}} + \left(\|u\|^{m_-}_{m_-,\Omega_2}\right)^{\frac{z_+}{m_-}}\right) \\
& + 2^{\tau(z_+-1)-1}\frac{C}{em_-(\rho-m_+)}\|\nabla u\|_2^{\rho\tau(z_+-1)}\left(\|u\|^{z_-}_{m_-,\Omega_2} + \|u\|^{z_+}_{m_-,\Omega_2}\right),
\end{aligned} \tag{4.15}
$$

we use Young's inequality on the last term of the right-hand side of (4.15), we obtain

$$\begin{aligned}\|\nabla u\|_2^{\rho\tau(z_+-1)}\|u\|_{m_-,\Omega_2}^{z_-} &\leq \frac{z_-}{m_-}\|u\|_{m_-,\Omega_2}^{m_-}+C\frac{m_--z_-}{m_-}\|\nabla u\|_2^{\frac{\rho\tau(z^+-1)m_-}{m_--z_-}}\\ &= \frac{z_-}{m_-}\|u\|_{m_-,\Omega_2}^{m_-}+C\frac{m_--z_-}{m_-}\left(\|\nabla u\|_2^2\right)^{\frac{\rho\tau(z^+-1)m_-}{2(m_--z_-)}},\end{aligned} \tag{4.16}$$

and

$$\|\nabla u\|_2^{\rho\tau(z_+-1)}\|u\|_{m_-,\Omega_2}^{z_+} \leq \frac{z_+}{m_-}\|u\|_{m_-,\Omega_2}^{m_-}+C\frac{m_--z_+}{m_-}\left(\|\nabla u\|_2^2\right)^{\frac{\rho\tau(z^+-1)m_-}{2(m_--z_+)}},$$

by using the algebric inequality (2.7) and (2.3) and (2.4) holds, then we obtain

$$\begin{aligned}\left(\|u\|_{m_-,\Omega_2}^{m_-}\right)^{\frac{z_-}{m_-}} &\leq \left(1+\frac{1}{H(0)}\right)\left(\|u\|_{m_-,\Omega_2}^{z_-}+H(0)\right) \leq \left(1+\frac{1}{H(0)}\right)\left(\|u\|_{m_-,\Omega_2}^{z_-}+H(t)\right),\\ \left(\|u\|_{m_-,\Omega_2}^{m_-}\right)^{\frac{z_+}{m_-}} &\leq \left(1+\frac{1}{H(0)}\right)\left(\|u\|_{m_-,\Omega_2}^{z_-}+H(0)\right) \leq \left(1+\frac{1}{H(0)}\right)\left(\|u\|_{m_-,\Omega_2}^{z_-}+H(t)\right),\\ \left(\|\nabla u\|_2^2\right)^{\frac{\rho\tau(z^+-1)m_-}{2(m_--z_+)}} &\leq \left(1+\frac{1}{H(0)}\right)\left(\|\nabla u\|_2^2+H(0)\right) \leq \left(\|\nabla u\|_2^2+H(t)\right), \qquad (C>0),\end{aligned}$$

we substitute these inequality into (4.15) and using Lemma 2.1 we get

$$H^{\tau(z_+-1)}(t)\int_\Omega |u|^{z(x)}dx \leq \left(\|u\|_{m_-,\Omega_2}^{z_-}+[u]_{\alpha,2}^2+H(t)\right), \qquad \forall t\in[0,T], \tag{4.17}$$

then, (4.14) becomes

$$\begin{aligned}\Psi'(t) &\geq \left((1-\tau)-\epsilon\frac{z_+-1}{z_+}\right)H'(t)H^{-\tau}-\epsilon\frac{C\xi^{(1-z_-)}}{z_-}\left(\|u\|_{m_-,\Omega_2}^{z_-}+[u]_{\alpha,2}^2+H(t)\right)+\epsilon\eta\Big[H(t)+\|u_t\|_2^2\\ &+\tilde{K}\left([u]_{\alpha,2}^2\right)+\|u\|_{m_-,\Omega_2}^{m_-}+[u]_{\alpha,2}^2+\int_0^t f(t-s)[u(s)-u(t)]_{\alpha,2}^2ds+\int_\Omega |u|^{m(x)}\ln|u|dx\Big]\\ &\geq \left((1-\tau)-\epsilon\frac{z_+-1}{z_+}\right)H'(t)H^{-\tau}+\epsilon K_1\Big[H(t)+\|u_t\|_2^2+\tilde{K}\left([u]_{\alpha,2}^2\right)+\|u\|_{m_-,\Omega_2}^{m_-}+[u]_{\alpha,2}^2\\ &+\int_0^t f(t-s)[u(s)-u(t)]_{\alpha,2}^2ds+\int_\Omega |u|^{m(x)}\ln|u|dx\Big],\end{aligned}$$

wich $K_1=\eta-\frac{C\xi^{(1-z_-)}}{z_-}>0$, for ξ being fixed we choose ϵ small such that $(1-\tau)-\epsilon\frac{z_+-1}{z_+}>0$ and

$$\Psi(0)=H^{1-\tau}(0)+\epsilon\int_\Omega u_0u_1dx>0, \qquad (\forall t\geq 0).$$

Then, we have

$$\begin{aligned}\Psi'(t) &\geq \epsilon K_1\Big[H(t)+\|u_t\|_2^2+\tilde{K}\left([u]_{\alpha,2}^2\right)+\|u\|_{m_-,\Omega_2}^{m_-}+[u]_{\alpha,2}^2+\int_0^t f(t-s)[u(s)-u(t)]_{\alpha,2}^2ds\\ &+\int_\Omega |u|^{m(x)}\ln|u|dx\Big].\end{aligned} \tag{4.18}$$

Therefore, we have

$$\Psi(t)\geq\Psi(0)\geq 0, \forall t\geq 0.$$

On the other hand of (4.8)

$$\begin{aligned}\Psi^{\frac{1}{1-\tau}}(t) &= \left(H^{1-\tau}(t) + \epsilon \int_\Omega u u_t dx\right)^{\frac{1}{1-\tau}} \\ &\leq 2^{\frac{\tau}{1-\tau}} \left(H(t) + \epsilon^{\frac{1}{1-\tau}} \left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\tau}}\right), \end{aligned} \tag{4.19}$$

by using Hölder inequality and Young's inequality on the second term of the right-hand side of (4.19), we get

$$\begin{aligned}\left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\tau}} &\leq C_1 \left(\|u\|_{m_-}^{\frac{1}{1-\tau}} \|u_t\|^{\frac{1}{1-\tau}}\right) \\ &\leq C_1 \left(\|u\|_{m_-}^{\frac{\nu_1}{1-\tau}} + \|u_t\|_2^{\frac{\nu_2}{1-\tau}}\right),\end{aligned}$$

for $\frac{1}{\nu_1} + \frac{1}{\nu_2} = 1$, we take $\nu_2 = 2(1-\tau)$, then $\nu_1 = \frac{2(1-\tau)}{1-2\tau}$, and $\frac{\nu_1}{1-\tau} = \frac{2}{1-2\tau} \leq m_-$. We use the algebric inequality (2.7), we obtain

$$\begin{aligned}\left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\tau}} &\leq C_1 \left(\|u\|_{m_-}^{\frac{2}{1-2\tau}} + \|u_t\|_2^2\right) \\ &\leq C_1 \left(\|u\|_{m_-,\Omega_2}^{m_-} + \|u_t\|_2^2 + H(t)\right),\end{aligned} \tag{4.20}$$

we substitute (4.20) into (4.19) we get

$$\begin{aligned}\Psi^{\frac{1}{1-\tau}}(t) &\leq 2^{\frac{\tau}{1-\tau}} \left(H(t) + \epsilon^{\frac{1}{1-\tau}} \left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\tau}}\right) \\ &\leq C\left(H(t) + \|u\|_{m_-,\Omega_2}^{m_-} + \|u_t\|_2^2\right) \\ &\leq C\Big(H(t) + \|u_t\|_2^2 + \tilde{K}\left([u]_{\alpha,2}^2\right) + \|u\|_{m_-,\Omega_2}^{m_-} + [u]_{\alpha,2}^2 + \int_0^t f(t-s)[u(s)-u(t)]_{\alpha,2}^2 ds \\ &+ \int_\Omega |u|^{m(x)} \ln|u| dx\Big),\end{aligned} \tag{4.21}$$

in view (4.21) and (4.18) we have

$$\Psi'(t) \geq \epsilon K_1 \left(\frac{1}{C}\Psi^{\frac{1}{1-\tau}}(t)\right) = K_3 \Psi^{\frac{1}{1-\tau}}(t), \tag{4.22}$$

where K_3 is a positive constant depending on (K_1, ϵ, C). We integrate (4.22) over $(0,t)$ we obtain

$$\Psi^{\frac{\tau}{1-\tau}}(t) \geq \frac{1}{\frac{\tau}{1-\tau}K_3 t + \Psi^{-\frac{\tau}{1-\tau}}(0)}. \tag{4.23}$$

Consequently, the solution of (1.1) blow-up in a finit time T^{**}, where

$$T^{**} \leq \frac{1-\tau}{\Psi^{\frac{\tau}{1-\tau}}(0) K_3}.$$

□

5. Conclusion

In this paper, we used the Galerkin's approximation to obtain the existence of solution of a fractional Kirchhoff-type problem (1.1), also we studied the long time behavior of solution for problem (1.1).

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