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Solution to Transport Equation with Interactive Fuzzy Data via Drastic T-norm

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ABSTRACT: In this paper we study the transport equation with uncertain parameter and initial condition modeled by interactive fuzzy number and fuzzy-number-valued function respectively. The fuzzy solution to the problem is obtained by extending the classical one using the extension principle via drastic t-norm (the weakest t-norm T_D). The interactivity considered is the one associated to T_D. Some results of T_D-based addition and product are given, and three illustrative examples are presented.

Key Words: Interactive fuzzy number, extension principle, drastic t-norm, transport equation.

Contents

1. Introduction

The transport equation represents a fundamental concept in the fields of fluid dynamics and other areas of physics and engineering. It is employed to model the transport of mass, energy or other scalar quantities in a fluid flow. The general form of a homogeneous linear transport equation is given by

$$\begin{cases} U_t(t,x) = \eta_0 U_x(t,x) \\ U(0,x) = h(x) \end{cases}$$

where $\eta_0 \in \mathbb{R}$, $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a function, U_t and U_x are the partial derivatives with respect to time t and space x respectively. It is often challenging to determine the precise value of the parameter η_0. Consequently, it is possible to consider this parameter to be uncertain and to model it using a fuzzy number [1]. Similarly, determining the exact value of $h(x)$ at any given position x can be a complex task. In such case, one can treat this value as uncertain and model it using a fuzzy number-valued function [1,14]. Knowing that the process depends on the parameter and the initial condition, the fuzzy relation of interactivity is considered. This relation of interactivity is defined via the concept of joint possibility distribution (JPD) [5]. Authors in [13,14] have employed this concept to solve fuzzy differential equations, taking interactivity into account. In this case, the fuzzy solution is the extension of the solution to the associated deterministic equation, using the sup-J extension principle. JPD can be defined via a t-norm and then solving fuzzy differential equation is done using the extension principle defined via t-norms [3]. Two important t-norms are the minimum t-norm T_M (the strongest one) and the drastic t-norm T_D (the weakest one) : they frame any other t-norm (see proposition 2.1). The T_D-based operations are studied in [6,7,8,10] and it is well known that T_D-based addition and product preserve the shape of L-R fuzzy numbers [6,7].

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In this paper we will study the transport equation where the parameter and initial condition are modeled by interactive fuzzy numbers via drastic t-norm. The paper is organized as follows : Section 2 provides preliminary results that will be needed in the paper. Section 3 gives some results about T_D-based addition and product of two fuzzy numbers. Section 4 presents a solution to the transport equation with fuzzy data under interactivity via drastic t-norm, illustrated by three examples. Finally, in section 5, some concluding remarks are given.

2. Preliminaries

We denote by $\mathcal{K}$ the collection of all nonempty compact convex sets of real numbers. For $A, B \in \mathcal{K}$ and $\lambda \in \mathbb{R}$ the addition and the scalar multiplication are defined by

$$A + B = \{a + b : \ a \in A, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a : \ a \in A\}. \tag{2.1}$$

A fuzzy subset η of $\mathbb{R}$ is defined as a mapping $\eta : \mathbb{R} \longrightarrow [0,1]$ where $\eta(x)$ is the membership grade of x to the fuzzy set η [2]. The α-level sets of η are the crisp sets defined by

$$\begin{cases} [\eta]_\alpha = \{x \in \mathbb{R} : \ \eta(x) \geq \alpha\} & \text{for } \ 0 < \alpha \leq 1 \\ [\eta]_0 = cl\,\{x \in \mathbb{R} : \ \eta(x) > 0\} \end{cases}$$

where $cl(A)$ denotes the closure of A. We denote E the space of all fuzzy subsets $\eta : \mathbb{R} \longrightarrow [0,1]$ which are normal (i.e $\exists x_0 \in \mathbb{R}$, $\eta(x_0) = 1$), fuzzy convex (i.e $\eta(tx + (1-t)y) \geq \min(\eta(x), \eta(y))$, $\forall t \in [0,1]$, $\forall x, y \in \mathbb{R}$), upper semi-continuous and compactly supported. We have $\eta \in E$ if and only if $[\eta]_\alpha \in \mathcal{K}$, $\forall \alpha \in [0,1]$. Each $\eta \in E$ is called a fuzzy number [2].

Definition 2.1 *(L-R fuzzy number [2]) Let $L, R : [0,1] \longrightarrow [0,1]$ be two continuous increasing functions fulfilling $L(0) = R(0) = 0$ and $L(1) = R(1) = 1$. Let $\eta_1 \leq \eta_2 \leq \eta_3 \leq \eta_4$ be real numbers. A fuzzy number η is said to be an L-R fuzzy number and we write $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)_{L,R}$ if*

$$\eta(x) = \begin{cases} L\left(\frac{x-\eta_1}{\eta_2-\eta_1}\right) & \text{if } \eta_1 \leq x < \eta_2 \\ 1 & \text{if } \eta_2 \leq x < \eta_3 \\ R\left(\frac{\eta_4-x}{\eta_4-\eta_3}\right) & \text{if } \eta_3 \leq x < \eta_4 \\ 0 & \text{otherwise} \end{cases}.$$

If $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)_{L,R}$, then for any $\alpha \in [0,1]$:

$$[\eta]_\alpha = \left[\eta_1 + L^{-1}(\alpha)(\eta_2 - \eta_1), \eta_4 - R^{-1}(\alpha)(\eta_4 - \eta_3)\right]. \tag{2.2}$$

Example 2.1 For $L(x) = R(x) = x$, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)_{L,R} = (\eta_1, \eta_2, \eta_3, \eta_4)$ is called a trapezoidal fuzzy number and we have

$$[\eta]_\alpha = [\eta_1 + \alpha(\eta_2 - \eta_1), \eta_4 - \alpha(\eta_4 - \eta_3)], \ \forall \alpha \in [0,1]. \tag{2.3}$$

If in addition we have $\eta_2 = \eta_3$ then η is called triangular fuzzy number and a triplet $(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ is sufficient to represent it and we have

$$[\eta]_\alpha = [\eta_1 + \alpha(\eta_2 - \eta_1), \eta_3 - \alpha(\eta_3 - \eta_2)] \ , \forall \alpha \in [0,1]. \tag{2.4}$$

Definition 2.2 *([2,9]) An operator $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a triangular norm (or simply a t-norm) if for $x, y, z, w \in [0,1]$ we have:*
i) $T(1, x) = x$;
ii) $T(x, T(y, z)) = T(T(x, y), z)$;
iii) $T(x, y) = T(y, x)$;
iv) If $x \leq z$ and $y \leq w$, then $T(x, y) \leq T(z, w)$.

The minimum t-norm and the drastic t-norm are defined respectively by $T_M(x, y) = \min(x, y)$ and $T_D(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}.$

Proposition 2.1 *Let T be a t-norm. Then we have:*

$$T_D(x,y) \leq T(x,y) \leq T_M(x,y),\ \forall (x,y) \in [0,1]\times[0,1].$$

For the proof of proposition 2.1 and more information about t-norms see [9].

Definition 2.3 *(Zadeh's extension principle [16]) Let $\mathcal{F}(\mathbb{R})$ be the space of all fuzzy sets on $\mathbb{R}$ and $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. The Zadeh's extension of Φ is the function $\tilde{\Phi} : \mathcal{F}(\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R})$ defined for each $\eta \in \mathcal{F}(\mathbb{R})$ by*

$$\tilde{\Phi}(\eta)(y) = \begin{cases} \sup\limits_{x\in\mathbb{R},\Phi(x)=y} \eta(x) & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all y in $\mathbb{R}$.

If $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then $\left[\tilde{\Phi}(\eta)\right]_\alpha = \Phi\left([\eta]_\alpha\right), \forall \eta \in E, \alpha \in [0,1]$ (see [2,11]). The following statement is a more general definition that has the Zadeh's extension principle for a function with two variables as a particular case when the t-norm used is T_M.

Definition 2.4 *([3,4]) Let T be a t-norm and $\Phi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a function. The extension of Φ via T is the function $\tilde{\Phi}_T : \mathcal{F}(\mathbb{R})\times\mathcal{F}(\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R})$ defined for each $\eta, \xi \in \mathcal{F}(\mathbb{R})$ by*

$$\tilde{\Phi}_T(\eta,\xi)(z) = \begin{cases} \sup\limits_{(x,y)\in\mathbb{R}^2,\Phi(x,y)=z} T(\eta(x),\xi(y)) & \text{if } \Phi^{-1}(z) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all z in $\mathbb{R}$.

Theorem 2.1 *Let $\Phi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, T an upper semi-continuous t-norm and $\eta, \xi \in E$. Then*

$$\left[\tilde{\Phi}_T(\eta,\xi)\right]_\alpha = \bigcup_{T(\beta,\gamma)\geq\alpha} \Phi\left([\eta]_\beta \times [\xi]_\gamma\right),\ \forall \alpha \in (0,1].$$

For the proof of the theorem 2.1 see [4]. For any $\eta, \xi \in E$ and $\alpha \in [0,1]$ we have

$$\left[\tilde{\Phi}_{T_M}(\eta,\xi)\right]_\alpha = \Phi\left([\eta]_\alpha \times [\xi]_\alpha\right) \tag{2.5}$$

$$\left[\tilde{\Phi}_{T_D}(\eta,\xi)\right]_\alpha = \Phi\left([\eta]_1 \times [\xi]_\alpha\right) \cup \Phi\left([\eta]_\alpha \times [\xi]_1\right). \tag{2.6}$$

The operations obtained using the extension principle via t-norm T are said to be interactive when $T \neq T_M$ and non-interactive when $T = T_M$ [3]. The scalar multiplication and the non-interactive addition and product are defined level-wise [2] for all $\eta, \xi \in E$ and $\lambda \in \mathbb{R}$ by

$$[\lambda\eta]_\alpha = \lambda\,[\eta]_\alpha \tag{2.7}$$

$$[\eta \oplus_M \xi]_\alpha = [\eta]_\alpha + [\xi]_\alpha \tag{2.8}$$

$$[\eta \otimes_M \xi]_\alpha = \left[\min_{(x,y)\in[\eta]_\alpha\times[\xi]_\alpha} xy,\ \max_{(x,y)\in[\eta]_\alpha\times[\xi]_\alpha} xy\right]. \tag{2.9}$$

Theorem 2.2 *Let $\Phi : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, T_1 and T_2 be upper semi-continuous t-norms with $T_2(v,w) \geq T_1(v,w)$ for all $v, w \in (0,1]$ and $\eta, \xi \in E$. Then*

$$\tilde{\Phi}_{T_1}(\eta,\xi) \subseteq \tilde{\Phi}_{T_2}(\eta,\xi) \text{ i.e. } \left[\tilde{\Phi}_{T_1}(\eta,\xi)\right]_\alpha \subseteq \left[\tilde{\Phi}_{T_2}(\eta,\xi)\right]_\alpha,\ \forall \alpha \in (0,1].$$

For the proof of the theorem 2.2 see [3]. Using this theorem and according to the proposition 2.1, we get the following result

Corollary 2.1 *Let $\Phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Then for any upper semi-continuous t-norm T we have*

$$\tilde{\Phi}_{T_D}(\eta,\xi) \subseteq \tilde{\Phi}_T(\eta,\xi) \subseteq \tilde{\Phi}_{T_M}(\eta,\xi), \ \forall \eta,\xi \in E.$$

By ii)-iii) in the definition 2.2 we have for all $a, b, c \in \mathbb{R}$

$$T(T(a,b),c) = T(a,T(b,c)) = T(T(a,c),b).$$

We denote $T(a,b,c) := T(T(a,b),c)$.

Definition 2.5 *Let T be a t-norm and $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function. The extension of Φ via T is the function $\tilde{\Phi}_T : \mathcal{F}(\mathbb{R})^3 \longrightarrow \mathcal{F}(\mathbb{R})$ defined for each $\eta,\xi,\theta \in \mathcal{F}(\mathbb{R})$ by*

$$\tilde{\Phi}_T(\eta,\xi,\theta)(w) = \begin{cases} \sup\limits_{(x,y,z)\in\Phi^{-1}(w)} T(\eta(x),\xi(y),\theta(z)) & \text{if } \Phi^{-1}(w) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all w in $\mathbb{R}$.

Now, we present and prove the following result that we will need in section 4.

Theorem 2.3 *Let T be an upper semi-continuous t-norm and $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function defined by $\Phi(x,y,z) = F(x,y) + G(z)$ where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $G : \mathbb{R} \longrightarrow \mathbb{R}$ are two continuous functions. Then for all $\eta,\xi,\theta \in E$:*

$$\tilde{\Phi}_T(\eta,\xi,\theta) = \tilde{F}_T(\eta,\xi) \oplus_T \tilde{G}(\theta)$$

where $\tilde{\Phi}_T$ and $\tilde{F}_T$ are the extensions of Φ and F via T respectively, $\tilde{G}$ is the Zadeh's extension of G and $\oplus_T$ denotes the T-based addition.

Proof: Let $\eta,\xi,\theta \in E$ and $w \in \mathbb{R}$. If $\Phi^{-1}(w) = \emptyset$, then $\tilde{\Phi}_T(\eta,\xi,\theta)(w) = 0$ and

$$F(x,y) + G(z) \neq w, \forall x,y,z \in \mathbb{R}. \tag{2.10}$$

It is sufficient to show that $\left(\tilde{F}_T(\eta,\xi) \oplus_T \tilde{G}(\theta)\right)(w) = 0$. Let $u, v \in \mathbb{R}$ such that $u + v = w$. (2.10) implies that $F^{-1}(u) = \emptyset$ or $G^{-1}(v) = \emptyset$. Then $\tilde{F}_T(\eta,\xi)(u) = 0$ or $\tilde{G}(\theta)(v) = 0$. Hence, $T\left(\tilde{F}_T(\eta,\xi)(u), \tilde{G}(\theta)(v)\right)$ $= 0$. So $\left(\tilde{F}_T(\eta,\xi) \oplus_T \tilde{G}(\theta)\right)(w) = 0$.

Now assume that $\Phi^{-1}(w) \neq \emptyset$. Then

$$\begin{aligned} \tilde{\Phi}_T(\eta,\xi,\theta)(w) &= \sup_{(x,y,z)\in\mathbb{R}^3,\, F(x,y)+G(z)=w} T\left(T\left(\eta(x),\xi(y)\right),\theta(z)\right) \\ &= \sup_{(u,v)\in\mathbb{R}^2,\, u+v=w} \ \sup_{(x,y)\in F^{-1}(u)} \ \sup_{z\in G^{-1}(v)} T\left(T(\eta(x),\xi(y)),\theta(z)\right). \end{aligned} \tag{2.11}$$

Let $v \in \mathbb{R}$. Obviously,

$$\sup_{z\in G^{-1}(v)} \theta(z) = \sup_{z\in G^{-1}(v)\cap[\theta]_0} \theta(z)$$

since $\theta(z) = 0$ for all $z \notin [\theta]_0$. However, $[\theta]_0$ is compact and $G^{-1}(v)$ is closed by continuity of G; hence, $G^{-1}(v) \cap [\theta]_0$ is compact. Since θ is upper semi-continuous, it attains its maximum in $G^{-1}(v)$. Since T is non-decreasing with respect to the second argument, then

$$\sup_{z\in G^{-1}(v)} T\left(T\left(\eta(x),\xi(y)\right),\theta(z)\right) = T\left(T\left(\eta(x),\xi(y)\right), \sup_{z\in G^{-1}(v)} \theta(z)\right). \tag{2.12}$$

Let $u \in \mathbb{R}$. We have

$$\sup_{(x,y)\in F^{-1}(u)} T\left(\eta(x),\xi(y)\right) = \sup_{(x,y)\in F^{-1}(u)\cap\left([\eta]_0\times[\xi]_0\right)} T\left(\eta(x),\xi(y)\right)$$

since $T(\eta(x), \xi(y)) = 0$ for all $(x,y) \notin [\eta]_0 \times [\xi]_0$. However, $[\eta]_0 \times [\xi]_0$ is compact and $F^{-1}(u)$ is closed by continuity of F; hence, $F^{-1}(u) \cap ([\eta]_0 \times [\xi]_0)$ is compact. Since η and ξ are upper semi-continuous, and T is non-decreasing and upper semi-continuous, then the function $(x,y) \longrightarrow T(\eta(x), \xi(y))$ is upper semi-continuous and then attains its maximum in $F^{-1}(u)$. Since T is non-decreasing with respect to the first argument, then from (2.12), we get

$$\sup_{(x,y)\in F^{-1}(u)} \sup_{z\in G^{-1}(v)} T\left(T(\eta(x), \xi(y)), \theta(z)\right) = T\left(\sup_{(x,y)\in F^{-1}(u)} T(\eta(x), \xi(y)), \sup_{z\in G^{-1}(v)} \theta(z)\right). \tag{2.13}$$

Combining equations (2.11) and (2.13), we obtain

$$\begin{aligned}
\tilde{\Phi}_T(\eta,\xi,\theta)(w) &= \sup_{(u,v)\in\mathbb{R}^2,\, u+v=w} T\left(\sup_{(x,y)\in F^{-1}(u)} T(\eta(x), \xi(y)), \sup_{z\in G^{-1}(v)} \theta(z)\right)\\
&= \sup_{(u,v)\in\mathbb{R}^2,\, u+v=w} T\left(\tilde{F}_T(\eta,\xi)(u), \tilde{G}(\theta)(v)\right)\\
&= \left(\tilde{F}_T(\eta,\xi) \oplus_T \tilde{G}(\theta)\right)(w).
\end{aligned}$$

□

3. T_D-based addition and product of fuzzy numbers

In this section, we denote $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$ for all $\alpha \in [0,1]$ and $\eta \in E$.

3.1. T_D-based Addition

Denote $\oplus_D$ the addition based on the extension principle via drastic t-norm.

Proposition 3.1 *Let $\eta, \xi \in E$. Then for any $\alpha \in [0,1]$, we have*
i) $[\eta \oplus_D \xi]_\alpha = ([\eta]_1 + [\xi]_\alpha) \cup ([\eta]_\alpha + [\xi]_1)$;
ii) $[\eta \oplus_D \xi]_\alpha = \left[\min(\eta_1^- + \xi_\alpha^-, \eta_\alpha^- + \xi_1^-), \max(\eta_1^+ + \xi_\alpha^+, \eta_\alpha^+ + \xi_1^+)\right]$.

Proof: The function $\Phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\Phi(x,y) = x + y$ is continuous on $\mathbb{R}^2$. Then, by the theorem 2.1 and the equation (2.6), we have

$$\begin{aligned}
[\eta \oplus_D \xi]_\alpha = \left[\tilde{\Phi}_{T_D}(\eta,\xi)\right]_\alpha &= \Phi([\eta]_1 \times [\xi]_\alpha) \cup \Phi([\eta]_\alpha \times [\xi]_1)\\
&= ([\eta]_1 + [\xi]_\alpha) \cup ([\eta]_\alpha + [\xi]_1)\\
&= \left([\eta_1^-, \eta_1^+] + [\xi_\alpha^-, \xi_\alpha^+]\right) \cup \left([\eta_\alpha^-, \eta_\alpha^+] + [\xi_1^-, \xi_1^+]\right)\\
&= [\eta_1^- + \xi_\alpha^-, \eta_1^+ + \xi_\alpha^+] \cup [\eta_\alpha^- + \xi_1^-, \eta_\alpha^+ + \xi_1^+]\\
&= \left[\min(\eta_1^- + \xi_\alpha^-, \eta_\alpha^- + \xi_1^-), \max(\eta_1^+ + \xi_\alpha^+, \eta_\alpha^+ + \xi_1^+)\right]
\end{aligned}$$

where $+$ in the second and the third equalities is as defined in equation (2.1). □

Remark 3.1 *For any $\eta \in E$, we have $[\eta \oplus_D \eta]_\alpha = \left[\eta_1^- + \eta_\alpha^-, \eta_1^+ + \eta_\alpha^+\right]$. In general $\eta \oplus_D \eta \neq 2\eta$ where 2η is the scalar multiplication of η by 2 defined in equation (2.7).*

Remark 3.2 *For any $\eta \in E$, we have $[\eta \ominus_D \eta]_\alpha = ([\eta]_1 - [\eta]_\alpha) \cup ([\eta]_\alpha - [\eta]_1)$. So $\eta \ominus_D \eta$ is symmetric and in general $\eta \ominus_D \eta \neq \chi_{\{0\}}$.*

Proposition 3.2 *Let $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)_{L,R}$ and $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)_{L,R}$ be two L-R fuzzy numbers. Then*

$$\eta \oplus_D \xi = (\eta_2 + \xi_2 - l, \eta_2 + \xi_2, \eta_3 + \xi_3, \eta_3 + \xi_3 + r)_{L,R}$$

where $l = \max(\eta_2 - \eta_1, \xi_2 - \xi_1)$ and $r = \max(\eta_4 - \eta_3, \xi_4 - \xi_3)$.

Proof: Let $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)_{L,R}$, $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)_{L,R}$ and $\alpha \in [0, 1]$. Then

$$\begin{cases} [\eta]_\alpha = \left[\eta_1 + L^{-1}(\alpha)(\eta_2 - \eta_1)\ ,\ \eta_4 - R^{-1}(\alpha)(\eta_4 - \eta_3)\right] \\ [\xi]_\alpha = \left[\xi_1 + L^{-1}(\alpha)(\xi_2 - \xi_1)\ ,\ \xi_4 - R^{-1}(\alpha)(\xi_4 - \xi_3)\right] \end{cases}.$$

So

$$\begin{cases} \eta_1^- + \xi_\alpha^- = \eta_2 + \xi_1 + L^{-1}(\alpha)(\xi_2 - \xi_1) = \eta_2 + \xi_2 + \left(L^{-1}(\alpha) - 1\right)(\xi_2 - \xi_1) \\ \eta_\alpha^- + \xi_1^- = \eta_1 + \xi_2 + L^{-1}(\alpha)(\eta_2 - \eta_1) = \eta_2 + \xi_2 + \left(L^{-1}(\alpha) - 1\right)(\eta_2 - \eta_1). \end{cases}$$

Since $L^{-1}(\alpha) - 1 \leq 0$, then

$$\min(\eta_1^- + \xi_\alpha^-, \eta_\alpha^- + \xi_1^-) = \eta_2 + \xi_2 + \left(L^{-1}(\alpha) - 1\right)\max(\eta_2 - \eta_1, \xi_2 - \xi_1).$$

By the same procedure, we prove that

$$\max(\eta_1^+ + \xi_\alpha^+, \eta_\alpha^+ + \xi_1^+) = \eta_3 + \xi_3 + \left(1 - R^{-1}(\alpha)\right)\max(\eta_4 - \eta_3, \xi_4 - \xi_3).$$

By the proposition 3.1, we have

$$\begin{aligned} [\eta \oplus_D \xi]_\alpha &= \left[\eta_2 + \xi_2 + \left(L^{-1}(\alpha) - 1\right) l, \eta_3 + \xi_3 + \left(1 - R^{-1}(\alpha)\right) r\right] \\ &= \left[\eta_2 + \xi_2 - l + L^{-1}(\alpha) l, \eta_3 + \xi_3 + r - R^{-1}(\alpha) r\right] \end{aligned}$$

where $l = \max(\eta_2 - \eta_1, \xi_2 - \xi_1)$ and $r = \max(\eta_4 - \eta_3, \xi_4 - \xi_3)$. □

The proposition 3.2 was differently proved in [10]. When $\eta, \xi \in E$ are triangular [6,7] and in particular symmetric with respect to real numbers η_0 and ξ_0 respectively, we have the following results

Corollary 3.1 *If $\eta = (\eta_1, \eta_2, \eta_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ then*

$$\eta \oplus_D \xi = (\eta_2 + \xi_2 - l, \eta_2 + \xi_2, \eta_2 + \xi_2 + r)$$

where $l = \max(\eta_2 - \eta_1, \xi_2 - \xi_1)$ and $r = \max(\eta_3 - \eta_2, \xi_3 - \xi_2)$.

Corollary 3.2 *If $\eta = (\eta_0 - l_1, \eta_0, \eta_0 + l_1)$ and $\xi = (\xi_0 - l_2, \xi_0, \xi_0 + l_2)$ then*

$$\eta \oplus_D \xi = (\eta_0 + \xi_0 - l, \eta_0 + \xi_0, \eta_0 + \xi_0 + l)$$

where $l = \max(l_1, l_2)$.

3.2. T_D-based Product

Denote $\otimes_D$ the product based on the extension principle via drastic t-norm.

Proposition 3.3 *Let $\eta, \xi \in E$. Then for any $0 \leq \alpha \leq 1$:*

$$[\eta \otimes_D \xi]_\alpha = [\min \Omega, \max \Omega]$$

where $\Omega = \left\{\eta_1^- \xi_\alpha^-, \eta_1^- \xi_\alpha^+, \eta_1^+ \xi_\alpha^-, \eta_1^+ \xi_\alpha^+, \eta_\alpha^- \xi_1^-, \eta_\alpha^- \xi_1^+, \eta_\alpha^+ \xi_1^-, \eta_\alpha^+ \xi_1^+\right\}$.

Proof: The function $\Phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\Phi(x, y) = xy$ is continuous on $\mathbb{R}^2$. Then by the theorem 2.1 and the equation (2.6), we have

$$\begin{aligned} [\eta \otimes_D \xi]_\alpha &= \left[\tilde{\Phi}_{T_D}(\eta, \xi)\right]_\alpha = \Phi\left([\eta]_1 \times [\xi]_\alpha\right) \cup \Phi\left([\eta]_\alpha \times [\xi]_1\right) \\ &= \left[\inf_{(x,y)\in\Lambda} \Phi(x, y), \sup_{(x,y)\in\Lambda} \Phi(x, y)\right] \end{aligned}$$

where $\Lambda = ([\eta]_1 \times [\xi]_\alpha) \cup ([\eta]_\alpha \times [\xi]_1)$. Since $[\eta]_\alpha$ and $[\xi]_\alpha$ are compact, Λ is compact. Since Φ is continuous and increasing, it attains its extrema at the corners of Λ. □

Proposition 3.4 *Let $\eta, \xi \in E$ such that $[\eta]_1 = \{\eta_1\}$ and $[\xi]_1 = \{\xi_1\}$. Then for any $0 \le \alpha \le 1$:*

$$[\eta \otimes_D \xi]_\alpha = \begin{cases} [\min(\eta_1\xi_\alpha^-, \eta_\alpha^-\xi_1), \max(\eta_1\xi_\alpha^+, \eta_\alpha^+\xi_1)] & \text{if } \eta_1 \ge 0, \xi_1 \ge 0 \\ [\min(\eta_1\xi_\alpha^+, \eta_\alpha^-\xi_1), \max(\eta_1\xi_\alpha^-, \eta_\alpha^+\xi_1)] & \text{if } \eta_1 < 0, \xi_1 \ge 0 \\ [\min(\eta_1\xi_\alpha^-, \eta_\alpha^+\xi_1), \max(\eta_1\xi_\alpha^+, \eta_\alpha^-\xi_1)] & \text{if } \eta_1 \ge 0, \xi_1 < 0 \\ [\min(\eta_1\xi_\alpha^+, \eta_\alpha^+\xi_1), \max(\eta_1\xi_\alpha^-, \eta_\alpha^-\xi_1)] & \text{if } \eta_1 < 0, \xi_1 < 0 \end{cases}.$$

Proof: Let $\eta, \xi \in E$ such that $[\eta]_1 = \{\eta_1\}$ and $[\xi]_1 = \{\xi_1\}$. From the proposition 3.3, for any $0 \le \alpha \le 1$ we have $[\eta \otimes_D \xi]_\alpha = [\min\Omega, \max\Omega]$ where $\Omega = \{\eta_1^-\xi_\alpha^-, \eta_1^-\xi_\alpha^+, \eta_1^+\xi_\alpha^-, \eta_1^+\xi_\alpha^+, \eta_\alpha^-\xi_1^-, \eta_\alpha^-\xi_1^+, \eta_\alpha^+\xi_1^-, \eta_\alpha^+\xi_1^+\}$. Since $\eta_1^- = \eta_1^+ = \eta_1$ and $\xi_1^- = \xi_1^+ = \xi_1$, then $\Omega = \{\eta_1\xi_\alpha^-, \eta_1\xi_\alpha^+, \eta_\alpha^-\xi_1, \eta_\alpha^+\xi_1\}$. If $\eta_1 \ge 0$ and $\xi_1 \ge 0$, then $\eta_1\xi_\alpha^- \le \eta_1\xi_\alpha^+$ and $\eta_\alpha^-\xi_1 \le \eta_\alpha^+\xi_1$. So $\min(\Omega) = \min(\eta_1\xi_\alpha^-, \eta_\alpha^-\xi_1)$ and $\max(\Omega) = \max(\eta_1\xi_\alpha^+, \eta_\alpha^+\xi_1)$. This proves the first case. The proof of the other cases is similar. □

Remark 3.3 If $[\eta]_1 = \{0\}$ and $[\xi]_1 = \{0\}$, then $[\eta \otimes_D \xi]_\alpha = \{0\}$, $\forall \alpha \in [0,1] : \eta \otimes_D \xi = \chi_{\{0\}}$.

4. Solution to transport equation with interactive fuzzy data

Denote $\mathcal{C}^1(\mathbb{R})$ the space of continuously differentiable functions on $\mathbb{R}$. In this section we will study the following homogeneous transport equation

$$\begin{cases} U_t(t,x) = \eta U_x(t,x) \\ U(0,x) = g(x)\xi \oplus_D \theta \end{cases} \tag{4.1}$$

where $g \in \mathcal{C}^1(\mathbb{R})$ and η, ξ and θ are interactive fuzzy numbers via T_D. We will study the equation (4.1) from the point of view of fuzzification of the deterministic solution to the equation

$$\begin{cases} U_t(t,x) = \eta_0 U_x(t,x) \\ U(0,x) = \xi_0 g(x) + \theta_0 \end{cases} \tag{4.2}$$

where $\eta_0 \in [\eta]_0$, $\xi_0 \in [\xi]_0$ and $\theta_0 \in [\theta]_0$. We will say that $U : \mathbb{R}^+ \times \mathbb{R} \longrightarrow E$ is a fuzzy solution to (4.1) if it is the extension via T_D of a solution to (4.2). The following proposition gives the fuzzy solution to (4.1) when η is assumed to be real.

Proposition 4.1 *Let $\eta \in \mathbb{R}$. Then the equation (4.1) has a unique fuzzy solution $U : \mathbb{R}^+ \times \mathbb{R} \longrightarrow E$ defined by*

$$U(t,x) = (g(x+\eta t)\xi) \oplus_D \theta, \ \forall t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{4.3}$$

Proof: Let $\eta \in \mathbb{R}$. Since $g \in \mathcal{C}^1(\mathbb{R})$, it is well known (see [12]) that the equation (4.2) has the unique solution $(t,x) \longrightarrow g(x+\eta t)\xi_0 + \theta_0$ on $\mathbb{R}^+ \times \mathbb{R}$. For each (t,x) fixed, consider the operator

$$\begin{aligned} S_{t,x} : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (\xi_0, \theta_0) &\longrightarrow g(x+\eta t)\xi_0 + \theta_0. \end{aligned}$$

Since $S_{t,x}$ is continuous on $\mathbb{R}^2$, then by the theorem 2.1 and the equation (2.6), the extension via T_D of $S_{t,x}$ is such that

$$\begin{aligned} \left[\tilde{S}_{t,x}(\xi,\theta)\right]_\alpha &= S_{t,x}([\xi]_1 \times [\theta]_\alpha) \cup S_{t,x}([\xi]_\alpha \times [\theta]_1) \\ &= \{g(x+\eta t)\xi_0 + \theta_0 : (\xi_0,\theta_0) \in [\xi]_1 \times [\theta]_\alpha\} \\ &\quad \cup \{g(x+\eta t)\xi_0 + \theta_0 : (\xi_0,\theta_0) \in [\xi]_\alpha \times [\theta]_1\} \\ &= (g(x+\eta t)[\xi]_1 + [\theta]_\alpha) \cup (g(x+\eta t)[\xi]_\alpha + [\theta]_1) \\ &= ([g(x+\eta t)\xi]_1 + [\theta]_\alpha) \cup ([g(x+\eta t)\xi]_\alpha + [\theta]_1) \\ &= [(g(x+\eta t)\xi) \oplus_D \theta]_\alpha. \end{aligned}$$

So the equation (4.1) has the unique fuzzy solution given by equation (4.3). □

If, in addition to $\eta \in \mathbb{R}$, we have $\theta = 0$, then the unique fuzzy solution to the equation (4.1) is given by $U(t,x) = g\,(x+\eta t)\,\xi$, $\forall t \in \mathbb{R}^+, x \in \mathbb{R}$. It is the same solution obtained using the Zadeh's extension principle for a real valued function of one variable. Denote by $\tilde{g}$ the Zadeh extension of g and $\tilde{\lambda} = \chi_{\{\lambda\}}$ for any $\lambda \in \mathbb{R}$.

Proposition 4.2 *Let $\theta = 0$. Then the equation (4.1) has a unique fuzzy solution $U : \mathbb{R}^+ \times \mathbb{R} \longrightarrow E$ defined by*

$$U(t,x) = \tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi, \ \forall t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{4.4}$$

Proof: Let $\theta = \theta_0 = 0$. The equation (4.2) has the unique solution $(t,x) \longrightarrow g(x+t\eta_0)\xi_0$ on $\mathbb{R}^+ \times \mathbb{R}$. For each (t,x) fixed, consider the operator

$$\begin{aligned} P_{t,x} : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (\eta_0, \xi_0) &\longrightarrow g(x+t\eta_0)\xi_0. \end{aligned}$$

Since g is continuous on $\mathbb{R}$, then $P_{t,x}$ is continuous on $\mathbb{R}^2$. By the theorem 2.1 and the equation (2.6), the extension via T_D of $P_{t,x}$ is such that

$$\begin{aligned}
\left[\tilde{P}_{t,x}(\eta,\xi)\right]_\alpha &= P_{t,x}\left([\eta]_1 \times [\xi]_\alpha\right) \cup P_{t,x}\left([\eta]_\alpha \times [\xi]_1\right) \\
&= \{g(x+t\eta_0)\xi_0 : (\eta_0,\xi_0) \in [\eta]_1 \times [\xi]_\alpha\} \\
&\quad \cup \{g(x+t\eta_0)\xi_0 : (\eta_0,\xi_0) \in [\eta]_\alpha \times [\xi]_1\} \\
&= \{g(\bar{\eta}_0)\xi_0 : (\bar{\eta}_0,\xi_0) \in (\{x\} + t[\eta]_1) \times [\xi]_\alpha\} \\
&\quad \cup \{g(\bar{\eta}_0)\xi_0 : (\bar{\eta}_0,\xi_0) \in (\{x\} + t[\eta]_\alpha) \times [\xi]_1\} \\
&= \{g(\bar{\eta}_0)\xi_0 : (\bar{\eta}_0,\xi_0) \in [\tilde{x} \oplus_M t\eta]_1 \times [\xi]_\alpha\} \\
&\quad \cup \{g(\bar{\eta}_0)\xi_0 : (\bar{\eta}_0,\xi_0) \in [\tilde{x} \oplus_M t\eta]_\alpha \times [\xi]_1\} \\
&= \{\zeta_0\xi_0 : (\zeta_0,\xi_0) \in g\left([\tilde{x} \oplus_M t\eta]_1\right) \times [\xi]_\alpha\} \\
&\quad \cup \{\zeta_0\xi_0 : (\zeta_0,\xi_0) \in g\left([\tilde{x} \oplus_M t\eta]_\alpha\right) \times [\xi]_1\} \\
&= \{\zeta_0\xi_0 : (\zeta_0,\xi_0) \in [\tilde{g}\left(\tilde{x} \oplus_M t\eta\right)]_1 \times [\xi]_\alpha\} \\
&\quad \cup \{\zeta_0\xi_0 : (\zeta_0,\xi_0) \in [\tilde{g}\left(\tilde{x} \oplus_M t\eta\right)]_\alpha \times [\xi]_1\} \\
&= [\tilde{g}\left(\tilde{x} \oplus_M t\eta\right) \otimes_D \xi]_\alpha \ .
\end{aligned}$$

So the equation (4.1) has the unique fuzzy solution given by equation (4.4). □

The following theorem gives the unique solution when $\eta, \xi, \theta \in E$.

Theorem 4.1 *Let $\eta, \xi, \theta \in E$. Then the equation (4.1) has a unique fuzzy solution $U : \mathbb{R}^+ \times \mathbb{R} \longrightarrow E$ defined by*

$$U(t,x) = (\tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi) \oplus_D \theta, \ \forall t \in \mathbb{R}^+, x \in \mathbb{R}.$$

Proof: Let $\eta, \xi, \theta \in E$. The unique solution to the equation (4.2) is the function $(t,x) \longrightarrow g(x+t\eta_0)\xi_0+\theta_0$ defined on $\mathbb{R}^+ \times \mathbb{R}$. For each (t,x) fixed, consider the operator

$$\begin{aligned} L_{t,x} : \quad \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (\eta_0, \xi_0, \theta_0) &\longrightarrow P_{t,x}(\eta_0,\xi_0) + W(\theta_0) \end{aligned}$$

where $P_{t,x}(\eta_0,\xi_0) = g(x+t\eta_0)\xi_0$ and $W(\theta_0) = \theta_0$. Since g is continuous on $\mathbb{R}$, then $L_{t,x}$ is continuous on $\mathbb{R}^3$. By theorem 2.3, we have $\tilde{L}_{t,x}(\eta,\xi,\theta) = \tilde{P}_{t,x}(\eta,\xi) \oplus_D \tilde{W}(\theta)$ where $\tilde{P}_{t,x}$ and $\tilde{W}$ are the extension via T_D and the Zadeh's extension of $P_{t,x}$ and W respectively. Since $\tilde{W}(\theta) = \theta$ and $\tilde{P}_{t,x}(\eta,\xi) = \tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi$ (see the proof of the proposition 4.2), then

$$\tilde{L}_{t,x}(\eta,\xi,\theta) = (\tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi) \oplus_D \theta.$$

$\square$

For illustration, we consider the following examples.

Example 4.1 Consider the equation

$$\begin{cases} U_t(t,x) = 0.5U_x(t,x) \\ U(0,x) = (0,0.5,1)\, e^{-(x-2)^2} \oplus_D (0,0.25,0.5) \end{cases}. \tag{4.5}$$

By the proposition 4.1, the unique fuzzy solution (T_D-solution) to the equation (4.5) is given by

$$\begin{aligned} U(t,x) &= g(x+0.5t)\,(0,0.5,1) \oplus_D (0,0.25,0.5) \\ &= \left(0, 0.5e^{-(x+0.5t-2)^2}, e^{-(x+0.5t-2)^2}\right) \oplus_D (0,0.25,0.5) \end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}$. From the corollary 3.2, we have $U(t,x) = (\sigma - l, \sigma, \sigma + l)$ where $\sigma = 0.5e^{-(x+0.5t-2)^2} + 0.25$ and $l = \max\left(0.5e^{-(x+0.5t-2)^2}, 0.25\right)$. It is easy to check that

$$0.5e^{-(x+0.5t-2)^2} \geq 0.25 \Leftrightarrow 2 - 0.5t - \sqrt{\ln(2)} \leq x \leq 2 - 0.5t + \sqrt{\ln(2)}.$$

Thus, for any $t \geq 0$ and $x \in \mathbb{R}$

$$\begin{aligned} U(t,x) &= \begin{cases} \widetilde{0.25} \oplus_M \left(0, 0.5e^{-(x+0.5t-2)^2}, e^{-(x+0.5t-2)^2}\right) & \text{if } x \in I_t \\ 0.5e^{-(x+0.5t-2)^2} \oplus_M (0,0.25,0.5) & \text{if } x \notin I_t \end{cases} \\ &= \begin{cases} \widetilde{0.25} \oplus_M e^{-(x+0.5t-2)^2}\,(0,0.5,1) & \text{if } x \in I_t \\ 0.5e^{-(x+0.5t-2)^2} \oplus_M (0,0.25,0.5) & \text{if } x \notin I_t \end{cases} \end{aligned}$$

where $I_t = \left[2 - 0.5t - \sqrt{\ln(2)}, 2 - 0.5t + \sqrt{\ln(2)}\right]$.

In the example 4.1, if $(0,0.5,1)$ and $(0,0.25,0.5)$ are considered non-interactive, the unique fuzzy solution (T_M-solution) to the equation (4.5) is given by

$$\bar{U}(t,x) = (0,0.5,1)\, e^{-(x+0.5t-2)^2} \oplus_M (0,0.25,0.5)$$

for all $t \geq 0$, $x \in \mathbb{R}$. It is clear that this solution is more fuzzy than the T_D-solution (i.e $[U(t,x)]_\alpha \subset [\bar{U}(t,x)]_\alpha$). Figure 1 illustrates this result by showing the fuzzy solutions via T_M and T_D for three different fixed time values. The use of interactivity results in a reduction in uncertainty when compared to the non-interactive case, as it is shown in [14] using Sup-J extension principle. This result can be seen from the corollary 2.1 for any upper semi-continuous t-norm T.

In the example 4.1, the parameter η is crisp and the initial condition is considered as a function of the form $U(0,x) = \xi g(x) \oplus_D \theta$ where $\xi, \theta \in E$ are interactive via T_D and $g \in \mathcal{C}^1(\mathbb{R})$. The next example will consider the function of the form $U(0,x) = \xi g(x)$ with $\xi, \eta \in E$ are interactive via T_D.

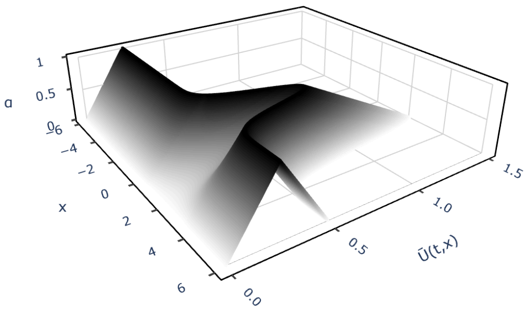
Example 4.2 Consider the equation

$$\begin{cases} U_t(t,x) = (-3,-2,-1)\, U_x(t,x) \\ U(0,x) = (1,2,3)\, e^{-x} \end{cases}. \tag{4.6}$$

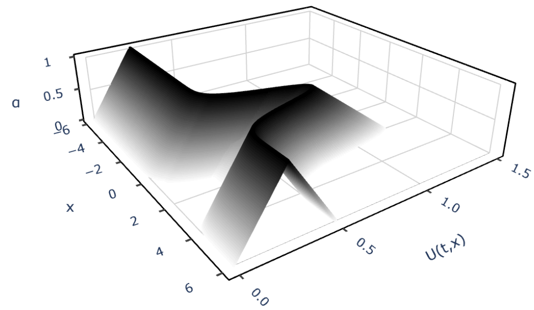
By proposition 4.2, the solution to the equation (4.6) is given by

$$U(t,x) = \tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi, \; \forall t \geq 0, \; x \in \mathbb{R}$$

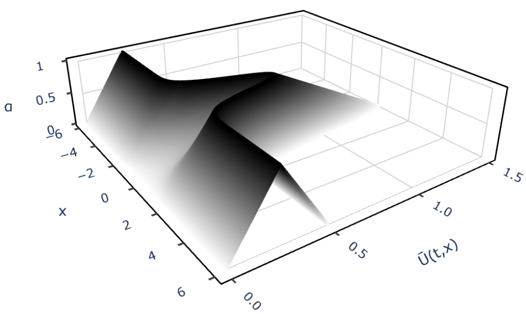
where $\eta = (-3,-2,-1)$, $\xi = (1,2,3)$ and $g : x \longrightarrow e^{-x}$. Let $\alpha \in [0,1]$. By equation (2.4), we have $[\eta]_\alpha = [-3+\alpha, -1-\alpha]$ and $[\xi]_\alpha = [1+\alpha, 3-\alpha]$. Since g is decreasing, then $[\tilde{g}(\tilde{x} \oplus_M t\eta)]_\alpha =$



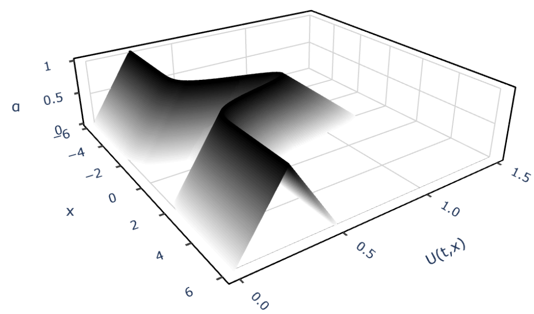
(a) view of the T_M-solution $t = 0$



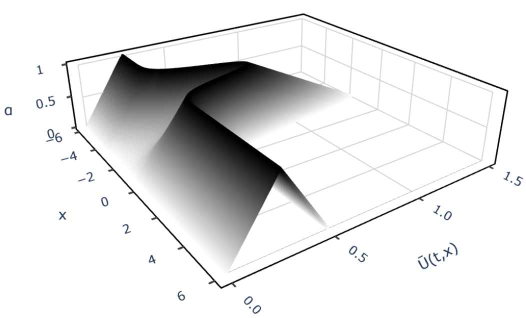
(b) view of the T_D-solution at $t = 0$



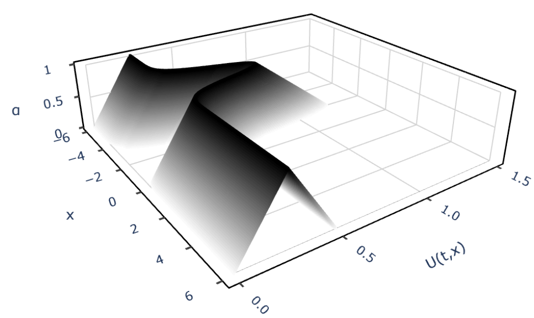
(c) view of the T_M-solution at $t = 4$



(d) view of the T_D-solution at $t = 4$



(e) view of the T_M-solution at $t = 8$



(f) view of the T_D-solution at $t = 8$

Figure 1: Three-dimensional views of the fuzzy solutions via T_M and T_D to the equation (4.5) for three fixed time values. The α-level sets are illustrated using gray lines, where the endpoints correspond to varying α levels, progressing from white to black as α increases from 0 to 1.

$\left[e^{-x+(1+\alpha)t}, e^{-x+(3-\alpha)t}\right]$. Since $[\tilde{g}(\tilde{x} \oplus_M t\eta)]_1 = \left\{e^{-x+2t}\right\}$ and $[\xi]_1 = \{2\}$, then by proposition 3.4, we have

$$[U(t,x)]_\alpha = e^{-x+2t}\left[\min(1+\alpha, 2e^{(\alpha-1)t}), \max(3-\alpha, 2e^{(1-\alpha)t})\right].$$

It is easy to check that

$$(1-\alpha)t \leq \ln\left(\frac{2}{1+\alpha}\right) \Leftrightarrow 2e^{(\alpha-1)t} \geq 1+\alpha$$

and

$$(1-\alpha)t \leq \ln\left(\frac{3-\alpha}{2}\right) \Leftrightarrow 2e^{(1-\alpha)t} \leq 3-\alpha.$$

So

$$[U(t,x)]_\alpha = \begin{cases} e^{-x+2t}\left[1+\alpha, 3-\alpha\right] & \text{if } (1-\alpha)t \leq \mu(\alpha) \\ e^{-x+2t}\left[1+\alpha, 2e^{(1-\alpha)t}\right] & \text{if } \mu(\alpha) \leq (1-\alpha)t \leq \nu(\alpha) \\ e^{-x+2t}\left[2e^{(\alpha-1)t}, 2e^{(1-\alpha)t}\right] & \text{if } \nu(\alpha) \leq (1-\alpha)t \end{cases}$$

where $\mu(\alpha) = \ln\left(\frac{3-\alpha}{2}\right)$ and $\nu(\alpha) = \ln\left(\frac{2}{1+\alpha}\right)$. Therefore, the diameter of $[U(t,x)]_\alpha$ increases with respect to t and decreases with respect to x for any $\alpha \in [0,1]$. For $\alpha = 0$, we obtain

$$diam\left([U(t,x)]_0\right) = \begin{cases} 2e^{-x+2t} & \text{if } t \leq \ln\left(\frac{3}{2}\right) \\ (2e^t - 1)e^{-x+2t} & \text{if } \ln\left(\frac{3}{2}\right) \leq t \leq \ln(2) \\ 2(e^{2t}-1)e^{-x+t} & \text{if } \ln(2) \leq t \end{cases}.$$

It is clear that the obtained solution is precise for a limited time but at infinity the results are non-precise.

In the example 4.2, if η and ξ are considered non-interactive, the unique T_M-solution to the equation (4.6) is given, from equations (2.5) and (2.9), level-wise by

$$\begin{aligned} \left[\bar{U}(t,x)\right]_\alpha &= \{g(x+t\eta_0)\xi_0 : (\eta_0, \xi_0) \in [\eta]_\alpha \times [\xi]_\alpha\} \\ &= \left[(1+\alpha)e^{-x+(1+\alpha)t}, (3-\alpha)e^{-x+(3-\alpha)t}\right], \end{aligned}$$

for all $t \geq 0$, $x \in \mathbb{R}$ and $\alpha \in [0,1]$. The diameter of this solution also increases with respect to t and decreases with respect to x for any $\alpha \in [0,1]$. For $\alpha = 0$, it is given by

$$diam\left(\left[\bar{U}(t,x)\right]_0\right) = 3e^{-x+3t} - e^{-x+t}.$$

The solution $\bar{U}(t,x)$ is more fuzzy than the solution $U(t,x)$ given in the example 4.2 (see figures 2 and 3). Note that $\bar{U}(t,x)$ is also the solution obtained when solving the equation (4.6) using gH-derivative (which is interactive [15]) as we can see in [1].

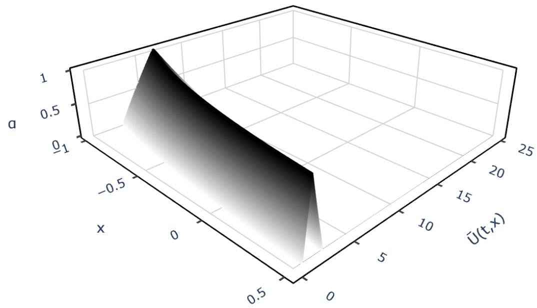
Example 4.3 Consider the equation

$$\begin{cases} U_t(t,x) = (1,2,3)\, U_x(t,x) \\ U(0,x) = (0,1,2)\, x \oplus_D (-4,-3,-1) \end{cases}. \tag{4.7}$$

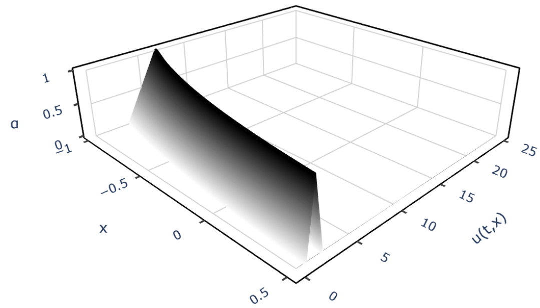
By theorem 4.1, the unique fuzzy solution to the equation (4.7) is given by

$$U(t,x) = (\tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi) \oplus_D \theta,\ \forall t \geq 0,\ x \in \mathbb{R}$$

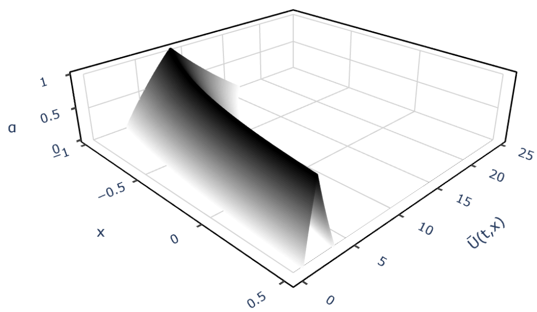
where $\eta = (1,2,3)$, $\xi = (0,1,2)$, $\theta = (-4,-3,-1)$ and $g : x \longrightarrow x$. We have $\tilde{g}(\tilde{x} \oplus_M t\eta) = \tilde{x} \oplus_M t\eta = (x+t, x+2t, x+3t)$.



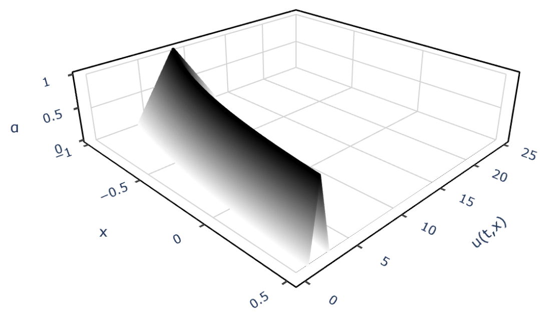
(a) view of the T_M-solution at $t = 0.12$



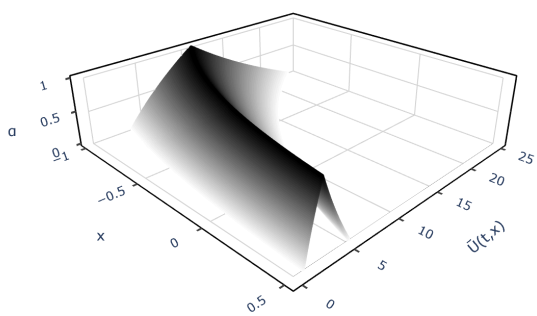
(b) view of the T_D-solution at $t = 0.12$



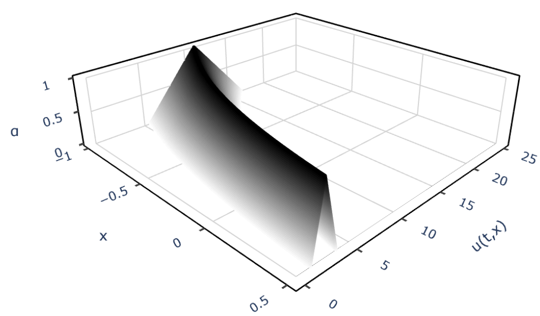
(c) view of the T_M-solution at $t = 0.24$



(d) view of the T_D-solution at $t = 0.24$

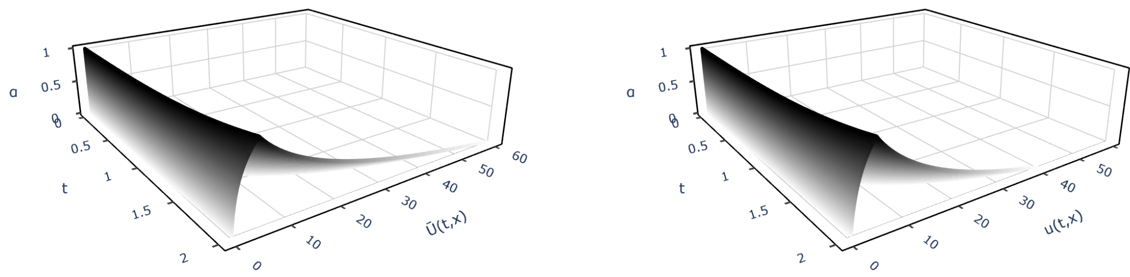


(e) view of the T_M-solution at $t = 0.36$



(f) view of the T_D-solution at $t = 0.36$

Figure 2: Three-dimensional views of the fuzzy solutions via T_M and T_D to the equation (4.6) for three fixed time values. The α-level sets are illustrated using gray lines, where the endpoints correspond to varying α levels, progressing from white to black as α increases from 0 to 1.



(a) view of the T_M-solution (b) view of the T_D-solution

Figure 3: Three-dimensional views of the fuzzy solutions via T_M and T_D to the equation (4.6) for spatial point $x = 3$. The α-level sets are illustrated using gray lines, where the endpoints correspond to varying α levels, progressing from white to black as α increases from 0 to 1.

For any $\alpha \in [0,1]$, by equation (2.4), we have $[\tilde{g}(\tilde{x} \oplus_M t\eta)]_\alpha = [x + t(1+\alpha), x + t(3-\alpha)]$ and $[\xi]_\alpha = [\alpha, 2-\alpha]$. Since $[\tilde{g}(\tilde{x} \oplus_M t\eta)]_1 = \{x+2t\}$ and $[\xi]_1 = \{1\}$, then by proposition 3.4, we have

$$[\tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi]_\alpha =$$
$$\begin{cases} [\min((x+2t)\alpha, x+t(1+\alpha)), \max((x+2t)(2-\alpha), x+t(3-\alpha))] & \text{if } x+2t \geq 0 \\ [\min((x+2t)\alpha, x+t(1+\alpha)), \max((x+2t)(2-\alpha), x+t(3-\alpha))] & \text{if } x+2t \leq 0 \end{cases}.$$

Since

$$(x+2t)\alpha \leq x + t(1+\alpha) \Leftrightarrow x \geq -t \text{ or } \alpha = 1$$
$$\text{and} \quad (x+2t)(2-\alpha) \leq x + t(3-\alpha) \Leftrightarrow x \leq -t \text{ or } \alpha = 1,$$

then

$$[\tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi]_\alpha = \begin{cases} [(x+2t)\alpha, (x+2t)(2-\alpha)] & \text{if } x \geq -t \\ [x+t(\alpha+1), x+t(3-\alpha)] & \text{if } -3t \leq x \leq -t \\ [(x+2t)(2-\alpha), (x+2t)\alpha] & \text{if } x \leq -3t \end{cases}.$$

Thus, by equation (2.4), we have

$$\tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_D \xi = \begin{cases} (x+2t)(0,1,2) & \text{if } x \geq -t \\ \tilde{x} \oplus_M t(1,2,3) & \text{if } -3t \leq x \leq -t \\ (x+2t)(0,1,2) & \text{if } x \leq -3t \end{cases}.$$

Since $\theta = (-4,-3,-1)$, then by corollary 3.1 we obtain

$$U(t,x) =$$
$$\begin{cases} \widetilde{(x+2t-3)} \oplus_M (-\max(x+2t,1), 0, \max(x+2t,2)) & \text{if } x \geq -t \\ \widetilde{(x+2t-3)} \oplus_M (-\max(t,1), 0, \max(t,2)) & \text{if } -3t \leq x \leq -t \\ \widetilde{(x+2t-3)} \oplus_M (\min(x+2t,-1), 0, -\min(x+2t,-2)) & \text{if } x \leq -3t \end{cases}.$$

In the example 4.3, if η, ξ and θ are considered non-interactive, the unique fuzzy solution to the equation (4.7) is given by

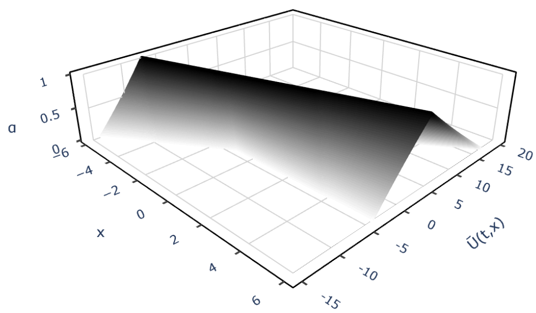
$$\begin{aligned} \bar{U}(t,x) &= (\tilde{g}(\tilde{x} \oplus_M t\eta) \otimes_M \xi) \oplus_M \theta \\ &= ((\tilde{x} \oplus_M t\eta) \otimes_M \xi) \oplus_M \theta \end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}$. By equations (2.5) and (2.9), we have

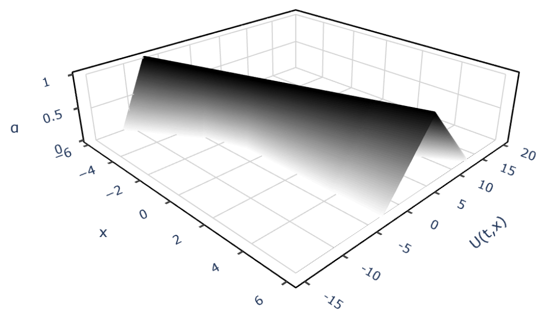
$$\begin{aligned}
[(\tilde{x} \oplus_M t\eta) \otimes_M \xi]_\alpha &= [\tilde{x} \oplus_M t\eta]_\alpha \times [\xi]_\alpha \\
&= [\tilde{x} \oplus_M t\,(1,2,3)]_\alpha \times [(0,1,2)]_\alpha \\
&= [x + t(1+\alpha), x + t(3-\alpha)] \times [\alpha, 2-\alpha] \\
&= \begin{cases} [(x+t(1+\alpha))\,\alpha, (x+t(3-\alpha))\,(2-\alpha)] & \text{if } x \geq -(1+\alpha)t \\ (2-\alpha)\,[x+t(1+\alpha), x+t(3-\alpha)] & \text{if } (\alpha-3)t \leq x \leq -(1+\alpha)t \\ [(x+t(1+\alpha))\,(2-\alpha)\,, (x+t(3-\alpha))\,\alpha] & \text{if } x \leq (\alpha-3)t \end{cases} .
\end{aligned}$$

So by equation (2.8) we get

$$\begin{aligned}
&\left[\bar{U}(t,x)\right]_\alpha = [(\tilde{x} \oplus_M t\eta) \otimes_M \xi]_\alpha + [\theta]_\alpha \\
&= \begin{cases} [(x+t(1+\alpha)+1)\,\alpha - 4, (x+t(3-\alpha)+2)\,(2-\alpha) - 5] & \text{if } x \geq -(1+\alpha)t \\ [(2-\alpha)\,(x+t(1+\alpha)-1) - 2, (2-\alpha)\,(x+t(3-\alpha)+2) - 5] & \text{if } (\alpha-3)t \leq x \leq -(1+\alpha)t \\ [(2-\alpha)\,(x+t(1+\alpha)-1) - 2, (x+t(3-\alpha)-2)\,\alpha - 1] & \text{if } x \leq (\alpha-3)t \end{cases} .
\end{aligned}$$



(a) View of the T_M-solution



(b) View of the T_D-solution

Figure 4: Three-dimensional views of the fuzzy solutions via T_M and T_D to the equation (4.7) at time $t = 1$. The α-level sets are illustrated using gray lines, where the endpoints correspond to varying α levels, progressing from white to black as α increases from 0 to 1.

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5. Concluding remarks

This paper studied the transport equation when the parameter and the initial condition are given by fuzzy number and fuzzy-number-valued function respectively. The interactivity taken into account is the one associated to the drastic t-norm T_D and the fuzzy solution obtained is the T_D-extension of the classical solution. We note that the solutions obtained via T_D and T_M frame any other possible solution when we use the same approach via an upper semi-continuous t-norm : the diameter of any possible solution satisfies the inclusion property in the corollary 2.1. The solution with the biggest diameter is obtained via Zadeh's extension principle and the solution with the smallest diameter is the one obtained by T_D-extension principle. The comparison is only done with the Zadeh's extension and any T-extension solution when T is upper semi-continuous. Other methods, such as those using fuzzy derivatives, approach the dynamical system from different angles. It is noteworthy that the methodology used in this paper is applicable whenever a classical solution can be found.

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