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Some Tauberian Conditions for the Logarithmic Summability Method of Integrals

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ABSTRACT: Let f be a real valued continuous function on $[1,\infty)$ and $s(x)=\int_1^x f(t)dt$. The logarithmic mean of $s(x)$ is defined by $\ell(x)=\frac{1}{\log x}\int_1^x \frac{s(t)}{t}dt$. If the limit $\lim\limits_{x\to\infty}\ell(x)=\alpha$ exists, then we say that the improper integral $\int_1^\infty f(t)dt$ is summable by logarithmic summability method to a finite number α. It is known that if the improper integral $\int_1^\infty f(t)dt$ is summable, then it is also summable by logarithmic summability method to same limit. However the converse implication is not always true. In this paper, we give the concept of slow oscillation with respect to logarithmic summability method and general logarithmic control modulo. Our goal is to obtain some Tauberian theorems for the logarithmic summability method of integrals by using these concepts.

Key Words: Tauberian theorem, slowly oscillating function, logarithmic summability method of integrals, general logarithmic control modulo.

Contents

1. Introduction

Let f be a real valued continuous function on $[1,\infty)$ and $s(x)=\int_1^x f(t)dt$. The logarithmic mean of $s(x)$ is defined by

$$\ell(x)=\frac{1}{\log x}\int_1^x \frac{s(t)}{t}dt.$$

If the limit

$$\lim_{x\to\infty}\ell(x)=\alpha \tag{1.1}$$

exists, then we say that the improper integral $\int_1^\infty f(t)dt$ is summable by logarithmic summability method (in other words; logarithmic summable or ℓ summable) to a finite number α [4]. Additionally if the $\lim\limits_{x\to\infty}s(x)=\alpha$ exists, then the limit (1.1) also exists. This indicates that the logarithmic summability method for integrals is a regular method.

The $(C,1)$ mean of a given integral $s(x)=\int_0^x f(t)dt$ is defined as

$$\sigma(x)=\frac{1}{x}\int_0^x s(t)dt$$

and if the limit

$$\lim_{x\to\infty}\sigma(x)=\alpha \tag{1.2}$$

exists, then the improper integral $\int_0^\infty f(t)dt$ is said to be summable by the $(C,1)$ method. Also we know that if $s(x)$ is $(C,1)$ summable, then it is also logarithmic summable to the same value [4].

At this point, we want to present two examples to better illustrate the inclusion relationship among ordinary convergence, $(C,1)$ summability and logarithmic summability methods for integrals.

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Firstly we present an integral that is not convergent but is $(C,1)$ summable: $s(x) = \int_0^x \sin 2t dt$. Although this integral is not convergent in the ordinary sense, it converges to $\frac{1}{2}$ according to the $(C,1)$ method.

Now we give an integral that is not $(C,1)$ summable but is summable according to the logarithmic method: $s(x) = \int_0^x f(t)dt = x\sin(x-1)$. While this integral does not converge under the $(C,1)$ method, it converges to 0 according to the logarithmic method.

It is necessary to add some conditions to make the converse implications true. Some of these conditions are boundedness, convergence and slow oscillation with respect to logarithmic summability.

The symbol of $s(x) = O(1)$ means that the function $s(x)$ is bounded for large enough x and throughout this paper, R denotes a positive constant and possibly different at each occurence for boundedness. The symbol of $s(x) = o(1)$ means that limit of $s(x)$ is zero as $x \to \infty$.

For a function $s(x)$, we have the identity

$$s(x) - \ell(x) = v_\ell(x) \tag{1.3}$$

where $v_\ell(x) = \frac{1}{\log x}\int_1^x f(t)\log t dt$. It is called Kronecker identity in the sense of logarithmic summability [10].

The concept of classical logarithmic control modulo and general logarithmic control modulo are presented in [5], respectively as follows:

$$\omega_\ell^{(0)}(x) = x\log x f(x) \tag{1.4}$$

and

$$\omega_\ell^{(m)}(x) = \omega_\ell^{(m-1)}(x) - \ell\left(\omega_\ell^{(m-1)}(x)\right). \tag{1.5}$$

A function s is said to be slowly oscillating with respect to logarithmic summability method if

$$\lim_{\lambda\to 1^+}\limsup_{x\to\infty}\sup_{x<t\le x^\lambda}|s(t)-s(x)| = 0 \tag{1.6}$$

holds.

The logarithmic summability of sequences and integrals have been studied by some authors for a long time. Ishiguro [2,1] presented some Tauberian theorems for the logarithmic summability of sequences. On the other hand Móricz [4] presented some classical type Tauberian theorems for the logarithmic summability of integrals. Totur and Okur [10] used the conditions of boundedness to obtain Tauberian theorems. Okur and Totur [5] introduced the classical logarithmic control modulo and general logarithmic control modulo and proved some Tauberian theorems by using these concepts. In addition to the Cesàro and logarithmic methods we have mentioned for integrals, we can also refer to the weighted mean method. The weighted mean method has also been studied by many authors. We will not include definitions or examples for this method. Results similar to those obtained for the logarithmic method, and even more extensive findings, can be found in [9,11,3,6,7,8].

The purpose of this study is to obtain Tauberian theorems by benefit general logarithmic control modulo and slow oscillation with respect to logarithmic summability method of integrals.

2. Auxilary Results

The first lemma presents the relationship between the function $s(x)$ and its logarithmic mean $\ell(x)$ for $\lambda > 1$ and $0 < \lambda < 1$ respectively.

Lemma 2.1 [10]

(i) For $\lambda > 1$,

$$s(x) - \ell(x) = \frac{\lambda}{\lambda-1}\left(\ell(x^\lambda) - \ell(x)\right) - \frac{1}{(\lambda-1)\log x}\int_x^{x^\lambda}\frac{s(t)-s(x)}{t}dt.$$

(ii) For $0<\lambda<1$,

$$s(x)-\ell(x)=\frac{\lambda}{1-\lambda}\left(\ell(x)-\ell(x^{\lambda})\right)+\frac{1}{(1-\lambda)\log x}\int_{x^{\lambda}}^{x}\frac{s(x)-s(t)}{t}dt.$$

The next lemma shows us how to get the slow oscillation with respect to logarithmic summablity method of $s(x)$ from the slow oscillation with respect to logarithmic summablity method of v_ℓ and vice versa.

Lemma 2.2 [5] *The function $s(x)$ is slowly oscillating with respect to logarithmic summability method if and only if $v_\ell(x)$ is slowly oscillating with respect to logarithmic summability method and bounded.*

Lemma 2.3 *If the function $s(x)$ is ℓ summable to α and slowly oscillating with respect to logarithmic summability method, then $s(x)$ converges to α.*

Proof: From Lemma 2.1 (i), we have

$$\begin{aligned}|s(x)-\ell(x)| &\leq \left|\frac{\lambda}{\lambda-1}\left(\ell(x^{\lambda})-\ell(x)\right)\right|+\left|\frac{1}{(\lambda-1)\log x}\int_{x}^{x^{\lambda}}\frac{s(t)-s(x)}{t}dt\right| \\ &\leq \frac{\lambda}{\lambda-1}\left|\ell(x^{\lambda})-\ell(x)\right|+\frac{1}{(\lambda-1)}\sup_{x<t\leq x^{\lambda}}|s(t)-s(x)|\left|\frac{1}{\log x}\int_{x}^{x^{\lambda}}\frac{1}{t}dt\right| \\ &= \frac{\lambda}{\lambda-1}\left|\ell(x^{\lambda})-\ell(x)\right|+\sup_{x<t\leq x^{\lambda}}|s(t)-s(x)|.\end{aligned}$$

Taking lim sup of both sides as $x\to\infty$, we obtain

$$\limsup_{x\to\infty}|s(x)-\ell(x)|\leq\frac{\lambda}{\lambda-1}\limsup_{x\to\infty}\left|\ell(x^{\lambda})-\ell(x)\right|+\limsup_{x\to\infty}\sup_{x<t\leq x^{\lambda}}|s(t)-s(x)|.$$

Since $s(x)$ is ℓ summable to α, the first term on the right-hand side of inequality above vanishes and then, we get

$$\limsup_{x\to\infty}|s(x)-\ell(x)|\leq\limsup_{x\to\infty}\sup_{x<t\leq x^{\lambda}}|s(t)-s(x)|.$$

After taking the limit of both sides as $\lambda\to1^{+}$, we obtain

$$\limsup_{x\to\infty}|s(x)-\ell(x)|\leq\lim_{\lambda\to1^{+}}\limsup_{x\to\infty}\sup_{x<t\leq x^{\lambda}}|s(t)-s(x)|.$$

Because of $s(x)$ is slowly oscillating with respect to logarithmic summability method, we obtain $\limsup_{x\to\infty}|s(x)-\ell(x)|=0$. This result requires $s(x)$ is summable to α.

□

3. Main Results

Theorem 3.1 *Let the function $s(x)$ be ℓ summable to α, $p(x)$ be nondecreasing function and*

$$\frac{d}{dx}\left(s(x)\right)=O\left(\frac{p'(x)}{p(x)}\right) \tag{3.1}$$

as $x\to\infty$, where

$$\lim_{\lambda\to1^{+}}\limsup_{x\to\infty}\frac{p(x^{\lambda})}{p(x)}=1, \tag{3.2}$$

then $s(x)$ converges to α.

Proof: By the condition (3.1), we obtain

$$|s(t)-s(x)| \le R\log\left(\frac{p(t)}{p(x)}\right).$$

Taking supremum of both sides for $x<t\le x^{\lambda}$ and lim sup of both sides as $x\to\infty$ respectively, we have

$$\limsup_{x\to\infty}\sup_{x<t\le x^{\lambda}}|s(t)-s(x)| \le R\log\limsup_{x\to\infty}\left(\frac{p(x^{\lambda})}{p(x)}\right).$$

Now, taking the limit of both sides as $\lambda\to 1^{+}$ in the last inequality, we obtain

$$\lim_{\lambda\to 1^{+}}\limsup_{x\to\infty}\sup_{x<t\le x^{\lambda}}|s(t)-s(x)| = 0,$$

by (3.2). This result means that $s(x)$ is slowly oscillating with respect to logarithmic summability method. Hence the conditions in Lemma 2.3 hold and this result completes the proof. □

Corollary 3.1 *If the function $s(x)$ is ℓ summable to α and*

$$x\log x\frac{d}{dx}(s(x)) = O(1), \tag{3.3}$$

then $s(x)$ converges to α.

Proof:
The proof is completed by using (3.3) and taking $p(x)=\log x$ in Theorem 3.1.

□

Corollary 3.2 *If the function $s(x)$ is ℓ summable to α and*

$$x\log x\frac{d}{dx}(s(x)) = o(1), \tag{3.4}$$

then $s(x)$ converges to α.

Proof:
From (3.4), we have (3.3). So the proof is completed via Corollary 3.1. □

Theorem 3.2 *Let the function $s(x)$ be ℓ summable to α. If $\ell\left(\omega^{(m)}(x)\right)$ is slowly oscillating with respect to logarithmic summability method, then $s(x)$ converges to α.*

Proof:
From ℓ summability of the function $s(x)$, we obtain

$$\ell\left(\omega^{(0)}(x)\right)\to 0(\ell) \tag{3.5}$$

by the identity (1.3). After taking $m=1$ in (1.5) and taking logarithmic mean of both sides of this identity, we obtain

$$\ell\left(\omega^{(0)}(x)\right)-\ell^{(2)}\left(\omega^{(0)}(x)\right)=\ell\left(\omega^{(1)}(x)\right). \tag{3.6}$$

Using the result of (3.5) in (3.6), we get $\ell\left(\omega^{(1)}(x)\right)\to 0(\ell)$. After repeating same steps by using the identity (1.5), we conclude

$$\ell\left(\omega^{(m)}(x)\right)\to 0(\ell). \tag{3.7}$$

Applying Lemma 2.1 to $\ell\left(\omega^{(m)}(x)\right)$ we have following equality:

$$\ell\left(\omega^{(m)}(x)\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)=\frac{\lambda}{\lambda-1}\left(\ell^{(2)}\left(\omega^{(m)}(x^{\lambda})\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)\right)-\frac{1}{(\lambda-1)\log x}\int_{x}^{x^{\lambda}}\frac{\ell\left(\omega^{(m)}(t)\right)-\ell\left(\omega^{(m)}(x)\right)}{t}dt$$

Taking absolute value of both sides of last equality, we obtain

$$\begin{aligned}\left|\ell\left(\omega^{(m)}(x)\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)\right| &\leq \frac{\lambda}{\lambda-1}\left|\ell^{(2)}\left(\omega^{(m)}(x^{\lambda})\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)\right| \\ &\quad+\left|\frac{1}{(\lambda-1)\log x}\int_{x}^{x^{\lambda}}\frac{\ell\left(\omega^{(m)}(t)\right)-\ell\left(\omega^{(m)}(x)\right)}{t}dt\right| \\ &\leq \frac{\lambda}{\lambda-1}\left|\ell^{(2)}\left(\omega^{(m)}(x^{\lambda})\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)\right| \\ &\quad+\sup_{x<t\leq x^{\lambda}}\left|\ell\left(\omega^{(m)}(t)\right)-\ell\left(\omega^{(m)}(x)\right)\right|.\end{aligned}$$

Taking $\limsup$ of both sides of the inequality above as $x\to\infty$, we conclude

$$\limsup_{x\to\infty}\left|\ell\left(\omega^{(m)}(x)\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)\right|\leq\limsup_{x\to\infty}\sup_{x<t\leq x^{\lambda}}\left|\ell\left(\omega^{(m)}(t)\right)-\ell\left(\omega^{(m)}(x)\right)\right|$$

by using (3.7). Now, taking limit of both sides of last inequality as $\lambda\to 1^{+}$, we obtain

$$\limsup_{x\to\infty}\left|\ell\left(\omega^{(m)}(x)\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)\right|\leq\lim_{\lambda\to1^{+}}\limsup_{x\to\infty}\sup_{x<t\leq x^{\lambda}}\left|\ell\left(\omega^{(m)}(t)\right)-\ell\left(\omega^{(m)}(x)\right)\right|.$$

Consequently, it is easy to obtain

$$\limsup_{x\to\infty}\left|\ell\left(\omega^{(m)}(x)\right)-\ell^{(2)}\left(\omega^{(m)}(x)\right)\right|=0$$

by using slow oscillation with respect to logarithmic summability method of $\ell\left(\omega^{(m)}(x)\right)$. And the last result means that

$$\ell\left(\omega^{(m)}(x)\right)\to 0 \tag{3.8}$$

by (3.7). Now, using (3.5), (3.6) and the identity (1.5), we get

$$\ell\left(\omega^{(m-1)}(x)\right)\to 0(\ell). \tag{3.9}$$

Therefore, we obtain

$$\ell\left(\omega^{(m-1)}(x)\right)\to 0$$

by combining (3.8), (3.9) and the identity (1.5). Continuing in this way, we obtain

$$\ell\left(\omega^{(1)}(x)\right)\to 0. \tag{3.10}$$

Then, using (3.5), (3.10) and the identity (1.5), we obtain

$$\ell\left(\omega^{(0)}(x)\right)\to 0 \tag{3.11}$$

this time. It follows from (3.11) and the identity (1.4) that

$$v_\ell(x) \to 0 \tag{3.12}$$

Hence (3.12) implies $s(x) \to \alpha$ by the logarithmic summability of $s(x)$ and the Kronecker identity in the sense of logarithmic summability.

□

Corollary 3.3 *Let the function $s(x)$ be ℓ summable to α. If $v_\ell(x)$ is slowly oscillating with respect to logarithmic summability method, then $s(x)$ converges to α.*

Proof:
The proof completed by taking $m = 0$ in Theorem 3.2.

□

Corollary 3.4 *If the function $s(x)$ is ℓ summable to α and*

$$x \log x \frac{d}{dx} v_\ell(x) = O(1), \tag{3.13}$$

then $s(x)$ converges to α.

Proof:
From (3.13), we have

$$|v_\ell(t) - v_\ell(x)| \leq R \log\left(\frac{\log t}{\log x}\right).$$

It follows from the last inequality above that

$$\sup_{x<t\leq x^\lambda} |v_\ell(t) - v_\ell(x)| \leq R \log\left(\frac{\log t}{\log x^\lambda}\right).$$

Taking lim sup of both sides as $x \to \infty$ and limit of both sides as $\lambda \to 1^+$ respectively, we obtain

$$\lim_{\lambda\to 1^+} \limsup_{x\to\infty} \sup_{x<t\leq x^\lambda} |v_\ell(t) - v_\ell(x)| = 0$$

which means $v_\ell(x)$ is slowly oscillating with respect to logarithmic summability method. Thus the proof of Corollary 3.4 follows from Corollary 3.3.

□

Corollary 3.5 *Let the function $s(x)$ be ℓ summable to α, $p(x)$ be nondecreasing function and*

$$\frac{d}{dx}\left(\ell\left(\omega^{(m)}(x)\right)\right) = O\left(\frac{p'(x)}{p(x)}\right) \tag{3.14}$$

as $x \to \infty$, where (3.2) is satisfied, then $s(x)$ converges to α.

Proof: By the condition (3.14), we obtain

$$\left|\ell\left(\omega^{(m)}(t)\right) - \ell\left(\omega^{(m)}(x)\right)\right| \leq R \log\left(\frac{p(t)}{p(x)}\right).$$

Taking supremum of both sides for $x < t \leq x^\lambda$ and lim sup of both sides as $x \to \infty$ respectively, we have

$$\limsup_{x\to\infty} \sup_{x<t\leq x^\lambda} \left|\ell\left(\omega^{(m)}(t)\right) - \ell\left(\omega^{(m)}(x)\right)\right| \leq R \log \limsup_{x\to\infty} \left(\frac{p(x^\lambda)}{p(x)}\right).$$

By taking the limit of both sides as $\lambda \to 1^+$ in the last inequality, we conclude

$$\lim_{\lambda\to 1^+} \limsup_{x\to\infty} \sup_{x<t\leq x^\lambda} \left|\ell\left(\omega^{(m)}(t)\right) - \ell\left(\omega^{(m)}(x)\right)\right| = 0,$$

by (3.2). So $\ell\left(\omega^{(m)}(x)\right)$ is slowly oscillating with respect to logarithmic summability method. Therefore the conditions in Theorem 3.2 hold and this result completes the proof. □

Corollary 3.6 *If the function $s(x)$ is ℓ summable to α and*

$$x \log x \frac{d}{dx}\left(\ell(\omega^{(m)}(x))\right) = O(1), \tag{3.15}$$

then $s(x)$ converges to α.

Proof: By taking $p(x) = \log x$ in Corollary 3.5, we get (3.15). Hence the proof of Corollary 3.6 follows from Corollary 3.5. □

Corollary 3.7 *Let the function $s(x)$ be ℓ summable to α. If $\omega^{(m)}(x)$ is slowly oscillating with respect to logarithmic summability method, then $s(x)$ converges to α*

Proof: By Lemma 2.2 and the slow oscillation with respect to logarithmic summability method of $\omega^{(m)}(x)$, $\ell\left(\omega^{(m)}(x)\right)$ is slowly oscillating with respect to logarithmic summabity method. Then the proof is completed by Theroem 3.2. □

4. Conclusion

In this paper, we give general logarithmic control modulo of functions and slow oscillation with respect to logarithmic summability method. With the help of these notions, we proved new Tauberian theorems, which extend the results given for Cesàro summability method of integrals. As a continuation of this paper, we plan to generalize some of the results here.

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