Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.76244

Applying Disconnected Domination on Well-known Graphs and Complement Graphs

Ashraf L. Dahham and Mohammed A. Abdlhusein*

ABSTRACT: In this paper, a new domination concept in graph theory, referred to as doubly disconnected domination, is introduced. Let $G = (V, E)$ be an undirected, nontrivial, finite and simple graph. A subset $D \subseteq V$ is called a doubly disconnected dominating set if it is a dominating set, which is meaning that any vertex in $V - D$ has at least one neighbor in the set D, and both the induced subgraphs $G[D]$ and $G[V - D]$ are disconnected. The least number of elements of such a set among all possible doubly disconnected dominating sets in G is called the parameter $\gamma_{dn}(G)$ known as the doubly disconnected domination number. This paper aims to establish several relations for $\gamma_{dn}(G)$, as well as to analyze its behavior in various graph structures. Additionally, we explore the relationship between $\gamma_{dn}(G)$ and some complement graphs, deriving specific results that determine how this domination parameter transforms under the complement of graphs. Furthermore, explicit evaluations of $\gamma_{dn}(G)$ are provided for well-known graphs, and certain classes of graphs are identified that do not admit such a domination structure. The results contribute to a deeper understanding of domination properties in graph theory and open new avenues for further exploration in structural and combinatorial graph analysis.

Key Words: dominating set, doubly disconnected domination, induced subgraph.

Contents

1. Introduction

Let $G = (V, E)$ be a graph, where V denotes the set of vertices and E represents the set of edges, where the number of edges is denoted as $m = |E|$ and the number of vertices as $n = |V|$. The degree of any vertex v, denoted as $deg(v)$, is the number of edges incident to v. A vertex with no incident edges is referred to as an isolated vertex, while a vertex with exactly one incident edge is referred to as a pendant vertex or a leaf. The minimum and maximum degrees of the graph are represented by $\delta(G)$ and $\Delta(G)$, respectively. The subgraph induced by a set of vertices T, which consists of the vertices in T along with the edges between them, is denoted as $G[T]$. For additional definitions and details in graph theory see [25]. The research of domination in graphs is one of the most rapidly expanding fields in graph theory. A comprehensive analysis of fundamental domination concepts can be found in [18]. A subset $D \subseteq V$ is called a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. A minimal dominating set is a dominating set that does not contain any proper subset that is also a dominating set. The domination number, represented by $\gamma(G)$ denotes the number of elements in the smallest dominating set in G. Due to the significance of domination in various applications, numerous variations of domination models have been developed. Some domination parameters impose conditions on the dominating set itself, as discussed in [5,7,9,11,13,15,16,17,23,24]. Other parameters focus on conditions imposed on the complement set $V - D$, as explored in [1,3,8,14]. Additionally, certain domination definitions introduced conditions on both the dominating set D and its complement $V - D$, as in studies such as [2,4,6]. Domination in graphs can be studied either in terms of vertices or edges, where both the dominating set D and its complement $V - D$ may consist of vertices or edges [26,27]. Some studies investigate the relationship between domination in graphs and other branches of

* Corresponding author
Submitted April 17, 2025. Published July 11, 2025
2010 *Mathematics Subject Classification:* 05C69.

mathematics, such as constructing graphs from specific algebraic modules or transforming special types of topological spaces into graphs, known as topological graphs, and applying domination results on these new structures, see [10,12,19,20,21,22,28].

In this work, the idea of doubly disconnected domination is introduced as an innovative type of disconnected domination in graphs. This domination parameter imposes specific conditions on the set D and its complement $V - D$ within the graph. The study explores the constraints on the doubly disconnected domination number and examines its relationship with various structural properties of graphs. Additionally, this parameter is determined for certain well-known graphs, and its behavior is analyzed in graphs formed through operations such as complement and other structural transformations. Furthermore, significant theoretical results are presented to highlight the impact of this parameter on different graph properties, supported by practical examples and comparative studies that reinforce the proposed concepts.

2. Doubly Disconnected Domination in Certain Graphs

A model of doubly disconnected domination is examined for certain types of graphs, including: cycle, complete graph, big helm graph, helm and barbell graphs.

Definition 2.1 *Suppose that $G = (V, E)$ be an undirected graph, finite, simple, and nontrivial. A subset $D \subseteq V$ is a doubly disconnected dominating set (DD-set) of G if the set is dominating and satisfies both conditions $G[D]$ and $G[V - D]$ are disconnected subgraphs. It is said minimum DD-set if it has the least order among all minimal DD-sets in G denoted by γ_{dn}-set and its order is the doubly disconnected domination number in G denoted as $\gamma_{dn}(G)$.*

Example 2.2 *The following figures show the different between the γ-set and γ_{dn}-set.*

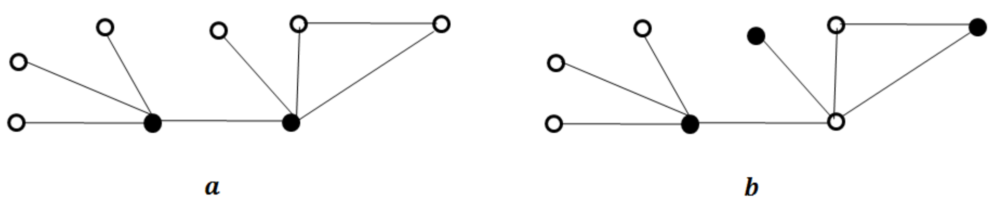

Figure 1: The γ-set and γ_{dn}-set.

Proposition 2.3 *The cycle graph C_3 has no doubly disconnected domination.*

Proof: Let $n = 3$. Then, D has one vertex and $V - D$ has two vertices. Hence, $G[D]$ is connected subgraph (a single vertex is connected). Then, C_3 has no doubly disconnected domination. For example, see Fig 2. □

Theorem 2.4 *Given a cycle graph C_n, then:*

$$\gamma_{dn}(C_n) = \left\lceil \frac{n}{3} \right\rceil \quad \textit{for } n \geq 4.$$

Proof: To demonstrate the existence of a doubly disconnected dominating set of a cycle graph. Let $v_1, v_2, \ldots, v_n$ be the vertices of a cycle graph of order n and let $D \subseteq V(C_n)$, such that

$$D = \begin{cases} \{v_{3i-2} : i = 1, 2, \ldots, \frac{n}{3}\} & \text{if } n \equiv 0 \pmod 3 \\ \{v_{3i-2} : i = 1, 2, \ldots, \frac{(n-1)}{3}\} \cup \{v_{n-1}\} & \text{if } n \equiv 1 \pmod 3 \\ \{v_{3i-2} : i = 1, 2, \ldots, \frac{(n+1)}{3}\} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

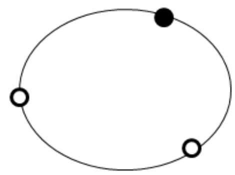

Figure 2: C_3 has no doubly disconnected domination.

Accordingly, the discussion is divided into the following three cases:

Case1. If $n \equiv 0 \pmod 3$. Let $D = \{v_{3i-2} : i = 1, 2, \ldots, \frac{n}{3}\}$. We divide the vertices of C_n in to $n/3$ sets such that each set contains three vertices. Then, we choose the second vertex of each set to be in D which dominates two vertices of $V - D$, where we started counting from the last vertex. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs. Thus, C_n has doubly disconnected dominating set and $\gamma_{dn}(C_n) = \frac{n}{3}$.

Case2. If $n \equiv 1 \pmod 3$. Let $D = \{v_{3i-2} : i = 1, 2, \ldots, \frac{(n-1)}{3}\} \cup \{v_{n-1}\}$. We divide the vertices of C_n in to $\lceil n/3 \rceil$ sets such that each set contains three vertices unless the last set contains one vertex. Then, we choose the second vertex of each set to be in D which dominates two vertices in $V - D$. And the last set contains one vertex belongs to D which dominates only one vertex. Note that we started counting from the last vertex. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs. Thus, C_n has doubly disconnected dominating set and $\gamma_{dn}(C_n) = \lceil \frac{n}{3} \rceil$.

Case3. If $n \equiv 2 \pmod 3$. Let $D = \{v_{3i-2} : i = 1, 2, \ldots, \frac{(n+1)}{3}\}$. We are divided the vertices of C_n in to $\lceil n/3 \rceil$ sets such that each set contains three vertices unless the last set contains two vertices. Then, we choose the second vertex of each set to be in D which dominates two vertices in $V - D$. And the last set the second vertex dominates one vertex in $V - D$. Note that we started counting from the last vertex. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs and C_n has doubly disconnected dominating set and $\gamma_{dn}(C_n) = \lceil \frac{n}{3} \rceil$.

Now, to demonstrate that D is the smallest doubly disconnected The dominating set in all of the aforementioned cases. Let D' is a doubly disconnected dominating set in G, such that $|D'| < |D|$. In that case, there is at least one vertex, or possibly more, that of $V - D$ don't dominated by any vertex of D' and $G[D']$ is connected graph. This contraction in relation to the concept of the doubly disconnected dominating set. Hence, D' is not doubly disconnected dominating set and D is the smallest doubly disconnected dominating set. For example, see Fig.3 □

Proposition 2.5 *The complete graph K_n has no doubly disconnected dominating set.*

Proof: Each vertex in K_n is adjacent with all other vertices, so that each vertex in any dominating set D dominates all vertices of $V - D$. But $G[D]$ and $G[V - D]$ is connected subgraphs. Then, K_n has no doubly disconnected dominating set. For example, see Fig.4 □

Proposition 2.6 *A helm graph H_n and big helm graph $\mathcal{H}_n$ $(n \geq 3)$ have doubly disconnected dominating set, such that $\gamma_{dn}(H_n) = n$ and $\gamma_{dn}(\mathcal{H}_n) = n + 1$.*

Proof: To prove that H_n and $\mathcal{H}_n$ have doubly disconnected dominating set. Since the helm graph H_n A graph obtained by connecting each vertex of the cycle to an additional vertex through a single edge C_n

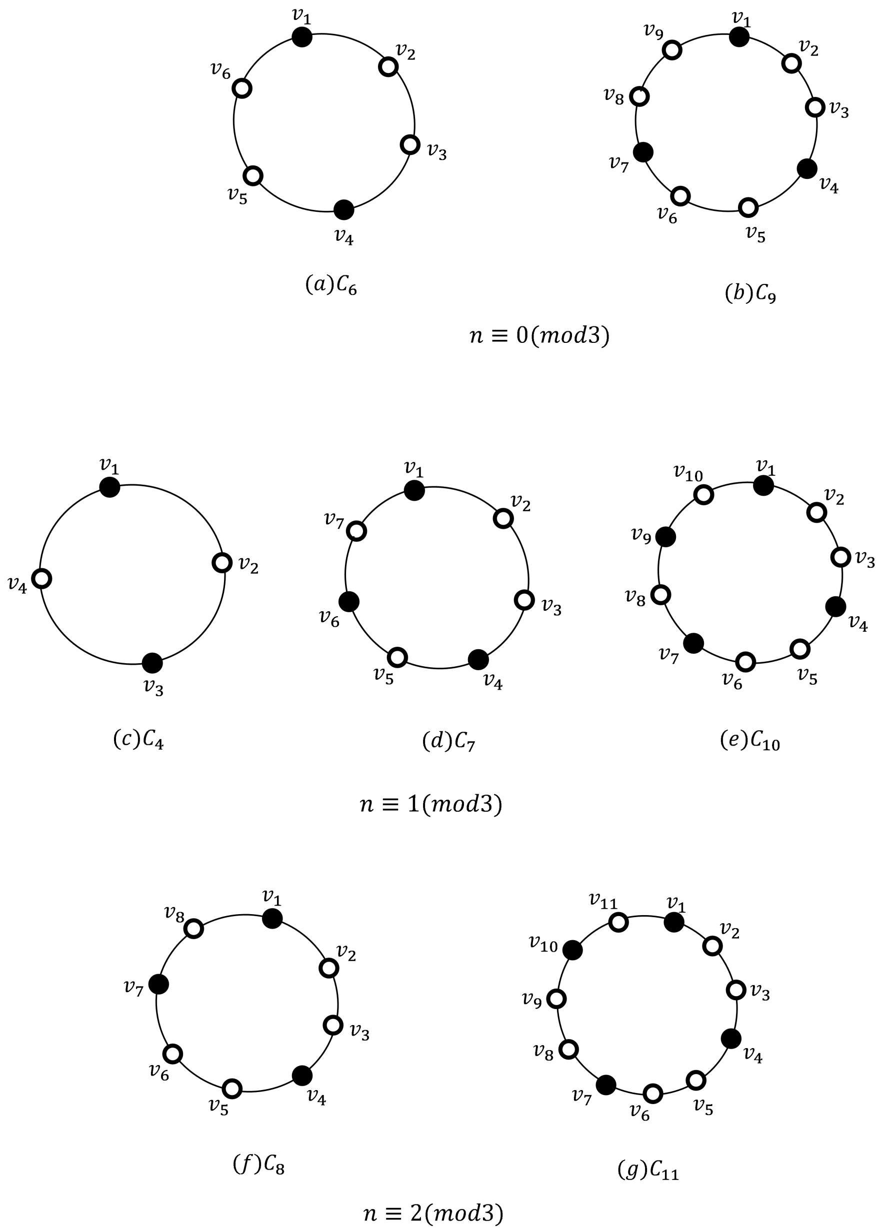

Figure 3: C_n has doubly disconnected dominating set.

in the wheel W_n. Then, every vertex of C_n is adjacent with one end vertex. Since C_n is connected graph. Let D contains $n-1$ vertices from C_n and one vertex from the end vertices that adjacent with the vertex C_n of the cycle. Then, $G[D]$ and $G[V-D]$ are disconnected subgraphs. Such that $G[D] \cong P_{n-1} \cup K_1$ and $G[V-D] \cong N_{n-1} \cup P_2$. Since the extended helm graph $\mathcal{H}_n$ is constructed by adding an individual vertex and an edge to any vertex of the wheel graph W_n. Then, every vertex of W_n are adjacent with one end vertex. Since W_n is connected. Then, let D contains n vertices from W_n and the terminal vertex that is connected to the central vertex. Then, $G[D]$ and $G[V-D]$ are disconnected subgraphs. Where $G[D] \cong C_n \cup K_1$ and $G[V-D] \cong N_{n+1}$. Then, H_n and $\mathcal{H}_n$ have doubly disconnected dominating set.
The set D is the smallest doubly disconnected dominating set in this case and the proof of it similar to

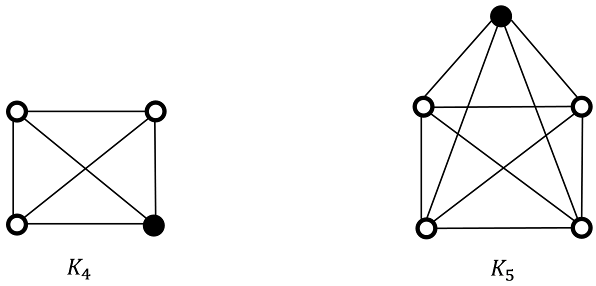

Figure 4: A complete graph has no doubly disconnected domination.

the proof of Theorem 2.4. For example, see Fig 5. □

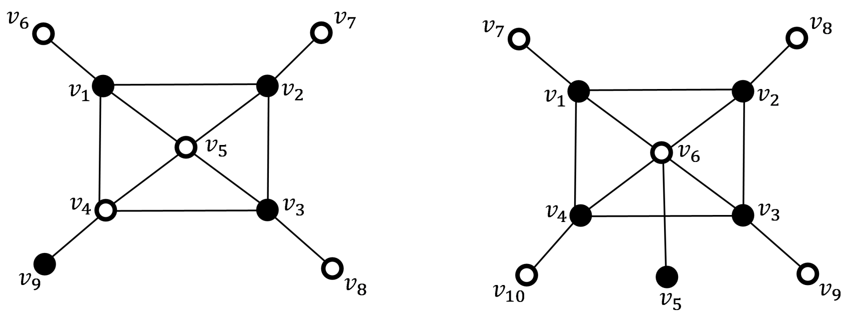

Figure 5: H_4 and $\mathcal{H}_4$ have doubly disconnected domination.

Proposition 2.7 *The barbell graph $B_{n,n}$ $(n \geq 3)$, has doubly disconnected domination and $\gamma_{dn}(B_{n,n}) = 2$.*

Proof: Since $B_{n,n}$ has two copies of complete graph K_n are joined by a bridge, and given that K_n does not possess a doubly disconnected domination, as stated in Proposition 2.5. Then, $G[D]$ and $G[V-D]$ must be disconnected subgraphs in $B_{n,n}$. Therefore, in each instance of the complete graph, there exists a vertex from D that dominates the $n-1$ vertices of the complete graph K_n, with the bridge necessarily lying on a vertex belongs to D and vertex belongs to $V-D$. Then, $B_{n,n}$ has doubly disconnected dominating set.
The set D is the smallest doubly disconnected dominating set in this case and the proof of it similar to the proof of Theorem 2.4. For example, see Fig 6. □

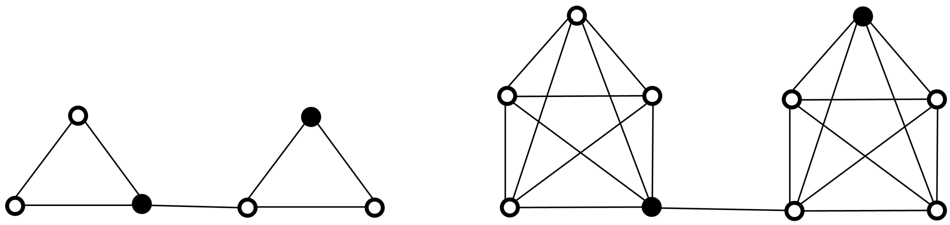

Figure 6: A minimum doubly disconnected dominating set of $B_{n,n}$.

3. Doubly Disconnected Domination for Complement Graphs

This section discuss the doubly disconnected domination for specific complement graphs, such as complement of star graph, path graph, cycle graph, wheel graph, fan graph, corresponding barbell graph, complement of complete graph and complete bipartite graph.

Proposition 3.1 *A complement star graph $\overline{S_n}$, has no doubly disconnected dominating set.*

Proof: As defined by of a star graph is a bipartite graph of the from $K_{1,n}$. Then, $\overline{S_n}$ are union of isolated vertex and complete subgraph . Hence, let the isolated vertex and one vertex from complete subgraph belongs to D. Then, $G[D]$ is disconnected subgraph but $G[V-D]$ connected subgraph. Thus, $\overline{S_n}$ has no doubly disconnected dominating set. For example, see Fig.7. □

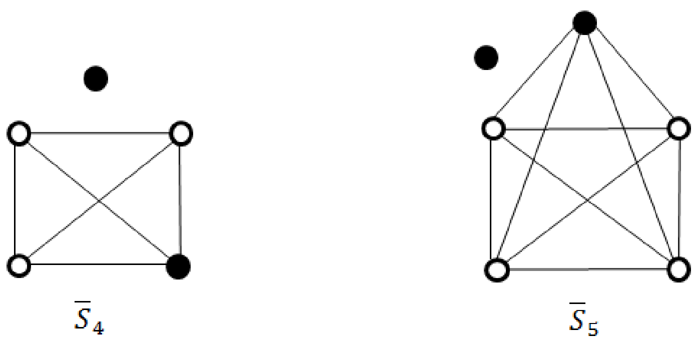

Figure 7: $\overline{S_n}$ has no doubly disconnected domination.

Theorem 3.2 *Suppose that P_n be a path graph, then $\overline{P_n}$ has doubly disconnected domination if and only if $n = \{4, 5, 6\}$. Such that:*

$$\gamma_{dn}(\overline{P_n}) = \begin{cases} 2 & \text{if } n = 4, 5 \\ 3 & \text{if } n = 6 \end{cases}$$

Proof: Let D be a γ_{dn}-set of a graph $\overline{P_n}$, then:
Case1: If $n = 4$, let $D = \{v_1, v_2\}$ and $V - D = \{v_3, v_4\}$. Then, the vertex v_1 dominates two vertices $\{v_3, v_4\}$ and the vertex v_2 dominates only one vertex v_4. Then, $G[D]$ and $G[V-D]$ are disconnected subgraphs. If $n = 5$, let $D = \{v_1, v_2\}$ and $V - D = \{v_3, v_4, v_5\}$. Hence, the vertex v_1 dominates three vertices $\{v_3, v_4, v_5\}$, and the vertex v_2 dominates two vertices $\{v_4, v_5\}$. Therefore, $G[D]$ and $G[V-D]$ are disconnected subgraphs. If $n = 6$, let $D = \{v_1, v_2, v_3\}$ and $V - D = \{v_4, v_5, v_6\}$. Then, vertex v_1 dominates on three vertices $\{v_4, v_5, v_6\}$ and the vertex v_2 dominates three vertices $\{v_4, v_5, v_6\}$ and the vertex v_3 dominates two vertices $\{v_5, v_6\}$. Hence, $G[D]$ and $G[V-D]$ are disconnected subgraphs. Then,

$\overline{P_n}$ has doubly disconnected dominating set. For example, see Fig 8.
Case2: If $n \geq 7$, then $G[D]$ or $G[V-D]$ are connected subgraphs. Therefore, $\overline{P_n}$ has no doubly disconnected domination. For example, see Fig 9.
The set D is the smallest doubly disconnected dominating set in two cases and the proof of it similar to the proof of Theorem 2.4. □

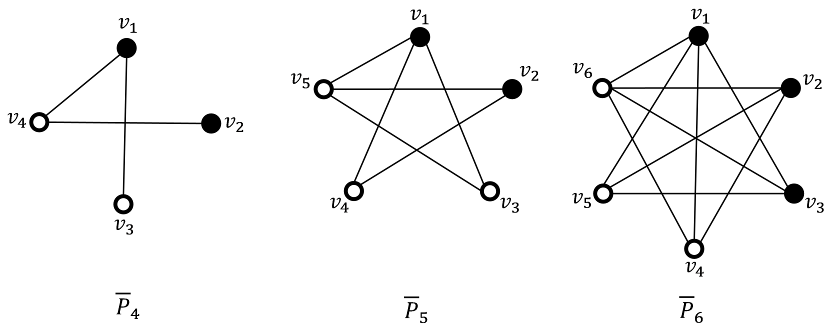

Figure 8: A minimum doubly disconnected dominating sets of $\overline{P_n}$.

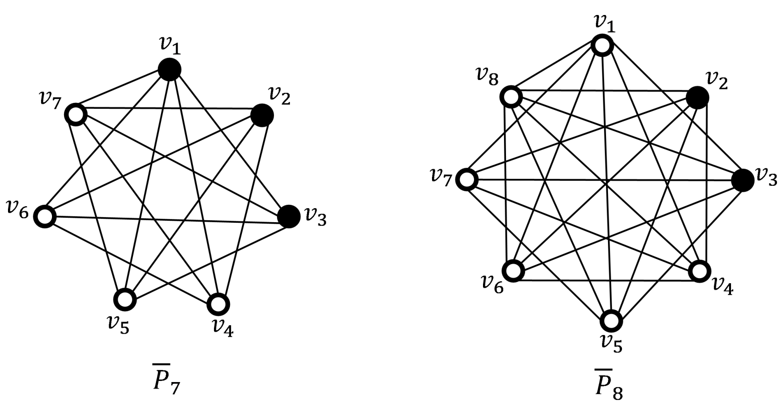

Figure 9: $\overline{P_n}$ $(n \geq 7)$ has no doubly disconnected dominating set.

Proposition 3.3 *Let G_n be a cycle graph, then $\overline{C_n}$ has doubly disconnected domination if and only if $n = \{4, 5, 6\}$. Such that:*

$$\gamma_{dn}(\overline{C_n}) = \begin{cases} 2 & \text{if } n = 4, 5 \\ 3 & \text{if } n = 6 \end{cases}$$

Proof: The argument proceeds as a similar approach to that of Theorem 3.2. For example, see Fig.10, and Fig.11. □

Theorem 3.4 *Let $G = \overline{F_n}$ or $G = \overline{W_n}$, then G has doubly disconnected dominating set, such that*

$$\gamma_{dn}(G) = \begin{cases} 3 & \text{if } n = 4 \\ (n-3)+1 & \text{if } n \geq 5 \end{cases}$$

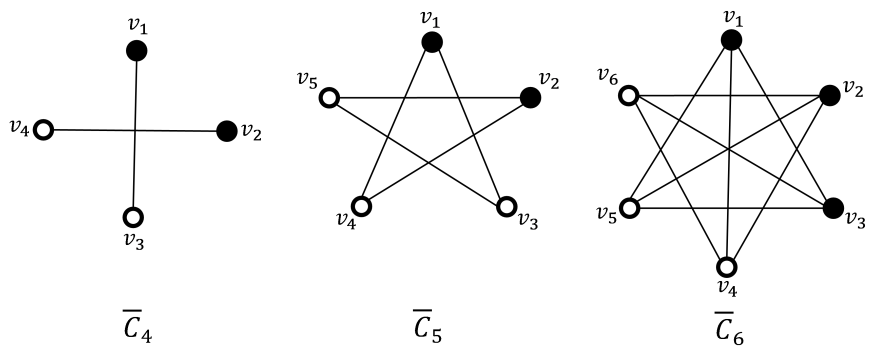

Figure 10: A minimum doubly disconnected dominating set of $\overline{C_n}$.

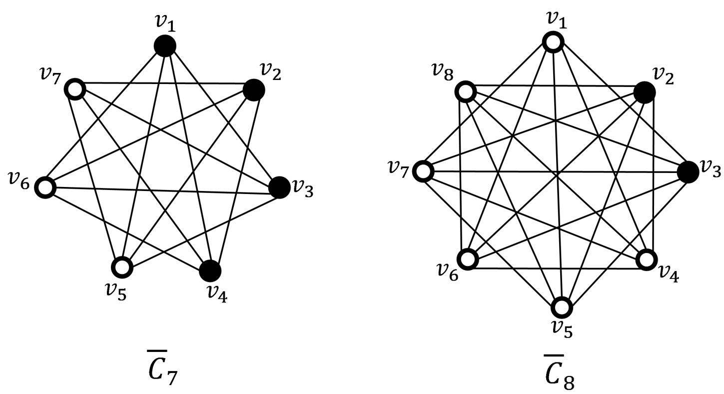

Figure 11: $\overline{C_n}(n \geq 7)$ has no doubly disconnected dominating set.

Proof: Let D be a γ_{dn}-set of G, then:
Case1. Since $F_n = P_n + K_1$. Then, $\overline{F_n}$ are union of isolated vertex and $\overline{P_n}$ subgraph. If $n = 4$, let D contains the isolated vertex and two vertices from $\overline{P_4}$ and $V - D$ contains two vertices from $\overline{P_4}$. Hence, $G[D]$ and $G[V - D]$ are disconnected subgraphs. If $n \geq 5$. Let the isolated vertex and $(n - 3)$ vertices from $\overline{P_n}$ belongs to D and $V - D$ contains three consecutive vertices from $\overline{P_n}$. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs. **Case2.** Since $W_n = C_n + K_1$ has no doubly disconnected dominating set. Then, $\overline{W_n}$ are union of isolated vertex and $\overline{C_n}$ subgraph. If $n = 4$, let D contains the isolated vertex and two vertices from $\overline{C_4}$ and $V - D$ contains two vertices from $\overline{C_4}$. Hence, $G[D]$ and $G[V - D]$ are disconnected subgraphs. If $n \geq 5$. Therefore, let the isolated vertex and $(n - 3)$ vertices from $\overline{C_n}$ belongs to D and $V - D$ contains three consecutive vertices from $\overline{C_n}$. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs. Hence, G has doubly disconnected dominating set. The set D is the smallest doubly disconnected dominating set in two cases and the proof of it similar to the proof of Theorem 2.4. For example, see Fig.12. □

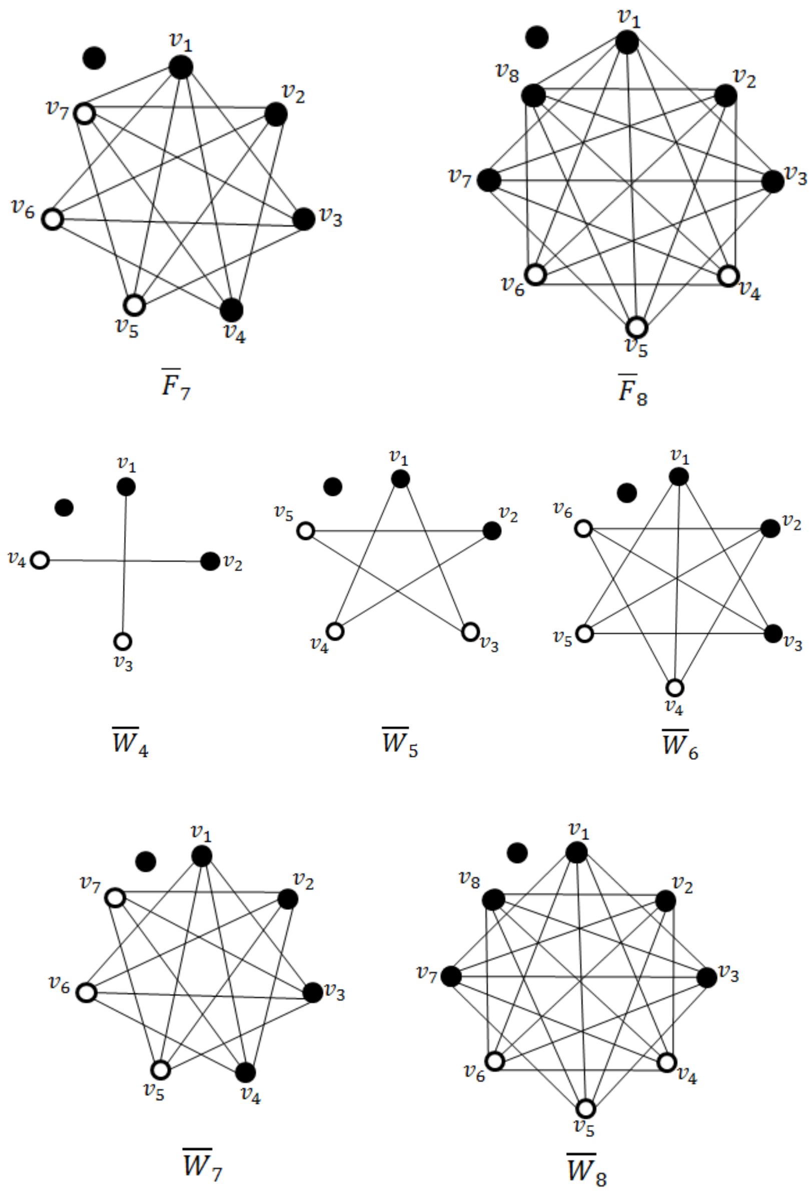

Figure 12: $\overline{F_n}$ and $\overline{W_n}$ have doubly disconnected domination.

Proposition 3.5 *The corresponding barbell graph $B^c_{n,n}$ ($n \geq 3$), has doubly disconnected domination and $\gamma_{dn}(B^c_{n,n}) = n$.*

Proof: Since $B^c_{n,n}$ holds two copies of the complete graph K_n, which are connected through a connection between each pair of corresponding vertices, and given that K_n does not exhibit doubly disconnected domination, as stated in Proposition 2.5. Let $D = \{v_1, v_2, \ldots, v_{\lceil \frac{n}{2} \rceil}\} \cup \{u_{\lceil \frac{n}{2} \rceil+1}, u_{\lceil \frac{n}{2} \rceil+2}, \ldots, u_n\}$. To explain the chosen vertices of D we choose the first $\lceil \frac{n}{2} \rceil$ of the vertices of first copy of complete graph and we choose $n - \lceil \frac{n}{2} \rceil$ of the vertices of second copy of complete graph. Since $\{v_1, v_2, \ldots, v_{\lceil \frac{n}{2} \rceil}\}$ are adjacent with $\{u_1, u_2, \ldots, u_{\lceil \frac{n}{2} \rceil}\}$ respectively and $\{u_{\lceil \frac{n}{2} \rceil+1}, \ldots, u_n\}$ are adjacent with $\{v_{\lceil \frac{n}{2} \rceil+1}, \ldots, v_n\}$ respectively, then D is dominating set and $|D| = n$. Hence, $G[D]$ and $G[V - D]$ are disconnected subgraphs. Then, $\gamma_{dn}(B^c_{n,n}) = n$. The set D is the smallest doubly disconnected dominating set in this case and the proof of it similar to the proof of Theorem 2.4. For example, see Fig.13. □

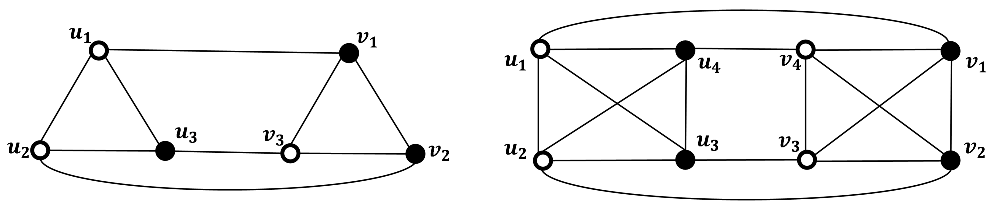

Figure 13: $B^c_{n,n}$ has doubly disconnected dominating set.

Theorem 3.6 *A graph $\overline{K_n}$ has no doubly disconnected dominating set.*

Proof: Considering that each vertex in K_n is adjacent accompanied by every other vertex in the graph. Hence, $\overline{K_n}$ is null graph such that $D = V(G)$. Then, $V - D = \emptyset$. Therefore, $G[V - D]$ is connected subgraph. Then, this contradiction with the definition of doubly disconnected dominating set. Hence, $\overline{K_n}$ has no doubly disconnected dominating set. □

Theorem 3.7 *A graph $K_{n,m}$ be a complete bipartite graph, then $\overline{K}_{n,m}$ has doubly disconnected domination if and only if $n \geq 2$ and $m \geq 2$, such that: $\gamma_{dn}(\overline{K}_{n,m}) = 2$.*

Proof: Let $n \geq 2$ and $m \geq 2$, then $\overline{K}_{n,m}$ consists of two disjoint subgraphs K_n and K_m. Since $\overline{K}_{n,m}$ is disconnected graph. Therefore, there exists a vertex in K_n that dominates all other vertices of K_n, and similarly, A vertex is present in K_n that dominates all other vertices of K_n. Thus, D contains two vertices of $\overline{K}_{n,m}$ such that $G[D]$ and $G[V - D]$ are disconnected subgraphs. Hence, $\gamma_{dn}(\overline{K}_{n,m}) = 2$. The set D is the smallest doubly disconnected dominating set in this case and the proof of it similar to the proof of Theorem 2.4. □

4. Conclusions

This study introduces a new type of domination in graph theory, known as doubly disconnected domination, emphasizing. The connection between doubly disconnected domination in the complement graph and in certain graphs. Additionally, we explored the construction of various conventional graphs and their modified versions, allowing for an accurate calculation of the domination number. Through this research, we have demonstrated the significance of doubly disconnected domination in enhancing our understanding of domination properties in complex graphs, paving the way for future studies to further investigate domination patterns in different types of graphs. In conclusion, these findings contribute to a deeper theoretical understanding of graph structures and open new avenues for analyzing domination in complex graphs, with potential applications in areas such as networks and algorithmic analysis.

References

1. M. A. Abdlhusein, *Applying the (1,2)-pitchfork domination and its inverse on some special graphs*, Bol. Soc. Paran. Mat., 41, (2023).
2. M. A. Abdlhusein, *Doubly connected bi-domination in graphs*, Discr. Math. Algorithm. Appl., 13(2), 2150009, (2021).
3. M. A. Abdlhusein, *Stability of inverse pitchfork domination*, Int. J. Nonlinear Anal. Appl., 12, 1009-1016, (2021).
4. M. A. Abdlhusein and M. N. Al-Harere, *Some modified types of pitchfork domination and its inverse*, Bol. Soc. Paran. Mat., 40, 1-9, (2022).
5. M. A. Abdlhusein and M. N. Al-Harere,*Total pitchfork domination and its inverse in graphs*, Discr. Math. Algorithm. Appl., 13(4), 2150038, (2021).
6. M. A. Abdlhusein and M. N. Al-Harere,*Doubly connected pitchfork domination and it's inverse in graphs*, TWMS J. App. Eng. Math., 12(1), 82-91, (2022).

7. Z. H. Abdulhasan and M. A. Abdlhusein, *Triple effect domination in graphs*, AIP Conf. Proc., 2386, 060013, (2022).

8. Z. H. Abdulhasan and M. A. Abdlhusein, *An inverse triple effect domination in graphs*, Int. J. Nonlinear Anal. Appl., 12(2), 913-919, (2021).

9. K. S. Al'Dzhabri *Enumeration of connected components of acyclic digraphs*, J. Discrete Math. Sci. Cryptogr., 24 (7), 2047-2058, (2021).

10. K. S. Al'Dzhabri and M. F. Almurshidy, *On certain types of topological spaces associated with digraphs*. J. of Physics: Conference Series IOP Publishing, 1591 (1), 012055, (2020).

11. M. N. Al-Harere and M. A. Abdlhusein, *Pitchfork domination in graphs*, Discr. Math. Algorithm. Appl., 12(2), 2050025, (2020).

12. A. H. Alwan, *g-Small intersection graph of a module*, Baghdad Sci. J., 21, 2671-2680, (2024).

13. A. A. Alwan and A. A. Najim, *The mole plough domination in graphs*, Journal of Education for Pure Science, 14(1), 97-106, (2024).

14. L. K. Alzaki, M. A. Abdlhusein and A. K. Yousif , *Stability of (1,2)-total pitchfork domination*, Int. J. Nonlinear Anal. Appl., 12(2), 265-274, (2021).

15. Z. A. Hassan and M. A. Abdlhusein, *Disconnected multi-effect domination for several graphs constructed by some operations* , AIP Conf. Proc., 3282(1), 040024, (2025).

16. Z. A. Hassan and M. A. Abdlhusein, *Disconnected multi-effect domination in graphs*, Asia Pac. J. Math., 11 (76), (2024).

17. Z. A. Hassan, M. A. Abdlhusein, M. Farahani, M. Alaeiyan and M. Cancan, *Disconnected multi-effect domination for several graphs constructed by corona operation*, Journal of Education for Pure Science, 15(1), 27-32, (2025).

18. T. W. Haynes, S. T. Hedetniemi and P.J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker Inc., New York, (1998).

19. Z. N. Jwair and M. A. Abdlhusein, *Constructing new topological graph with several properties*, Iraqi J. Sci., 64, 2991-2999, (2023).

20. Z. M. Khalil and M. A. Abdlhusein, *New form of discrete topological graphs*, AIP Conf. Proc., 3282(1), 040026, (2025).

21. Z. M. Kalil, M. A. Abdlhusein and M. R. Farahani,*Some dominating applications on discrete topological graphs*, Journal of Education for Pure Science, 15(1), 40-46, (2025).

22. K. A. Mhawis and A. B. Attar, *A new kind of discrete topological graphs with some properties*, Journal of Education for Pure Science, 14(2), 132-137, (2024).

23. C. Y. Ponnappan, P. Surulinathan and S.B. Ahamed, *The perfect disconnected domination number in fuzzy graphs*, Int. J. IT,Eng. Appl. Sci. Res., 7 , 11-14, (2018).

24. S. J. Radhi, M. A. Abdlhusein and A. E. Hashoosh, *The arrow domination in graphs*, Int. J. Nonlinear Anal. Appl., 12(1), 473-480, (2021).

25. M. S. Rahman, *Basic graph theory*, Springer, (2017).

26. W. A. Rheem and M. A. Abdlhusein, *Pitchfork edge domination for complement graphs*, AIP Conf. Proc., 3282(1), 020021, (2025)

27. W. A. Rheem and M. A. Abdlhusein, *Pitchfork edge domination in graphs*, Asia Pac. J. Math., 11(66), (2024).

28. N. K. Tuama and A. A. Alwan, *Non-comaximal graphs of commutative semirings*, Journal of Education for Pure Science, 15(1), 132-137, (2025).

Ashraf L. Dahham,
Department of Mathematics,
College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar,
Iraq.
E-mail address: `Ashraf_lateef@utq.edu.iq`

and

Mohammed A. Abdlhusein,
College of Education for Women, Shatrah University, Thi-Qar, 64001, Iraq.
E-mail address: `mmhd@shu.edu.iq`