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# Constructing New Disconnected Parameter in Domination of Graphs

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ABSTRACT: In this paper, a disconnected domination model in graph theory, known as doubly disconnected domination, is introduced. Suppose that $G = (V, E)$ be a simple, undirected, finite and nontrivial graph. A subset $D \subset V$ is called doubly disconnected dominating set in $G$ if $D$ is dominating set and both the induced subgraphs $G[D]$ and $G[V-D]$ are disconnected subgraphs. The least cardinality among all doubly disconnected dominating sets of $G$ is the doubly disconnected domination number $\gamma_{dn}(G)$. This study investigates various bounds and properties of this domination parameter to give the relations between $\gamma_{dn}(G)$ and the maximum degree in $G$, minimum degree, the size and the order of the graph. Some results are given in this paper to explain and prove the doubly disconnected domination number $\gamma_{dn}(G)$ for any graph constructed by corona or join operations. Furthermore, $\gamma_{dn}(G)$ is discussed and evaluated for some well-known graphs, while other graphs are proved to be hasn't this type of domination.

Key Words: dominating set, doubly disconnected domination, induced subgraph.

## Contents



## 1. Introduction

Let $G = (V, E)$ be a graph has vertex set V and edge set E, where the number of edges is $m = |E|$ and the number of vertices is $n = |V|$. The degree of any vertex $v$ denoted as $deg(v)$, is the number of edges that incident on it. A vertex with degree zero is said an isolated vertex, while a vertex with degree one is a pendant or end vertex. The minimum and the maximum degrees over all vertices of the graph are denoted as $\delta(G)$ and $\Delta(G)$. A subgraph induced by a vertex set S with the edges incident between its vertices is denoted as $G[S]$. For more definitions and details of graph theory, see [26]. The study of domination sorts is one of the fastest-growing areas in graph theory, with a comprehensive analysis of fundamental domination concepts [18].

A set $D \subset V$ is a dominating set if every vertex out of it is adjacent with at least one vertex in it. A minimal dominating set is a dominating set that does not contains any proper subset that is also a dominating set. The domination number $\gamma(G)$ is the cardinality of the dominating set which has least order. Throughout the significance of domination in various applications, several variations of domination models have emerged, as discussed in several papers. Some parameters of domination putted a conditions on the dominating set such as [5,7,9,11,13,15,16,17,24,25]. Another dominating parameters suggested a conditions on the complement dominating set $V - D$ such as [8]. There are more papers discussed the stability of domination parameters when they remove vertex or when add or remove edge [1,3,14,19]. Other papers studied the domination on the edge set [27,28]. While, some types of dominating definitions assess a conditions on both sets, the dominating set $D$ and its complement set $V - D$, such papers as [2,4,6]. In fact, the domination of graph may be studied on the vertices or on the edges, where the elements

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of the dominating set D and its complement set $V - D$ can be contains vertices or edges. Some papers deals a relations between domination of graph and another branches of Mathematics such as formed a graph from certain modules [12,29] or transform special kinds of topological spaces into graphs say "topological graphs" with distinct properties and applied several dominating results on that topological graphs [10,20,21,22,23]. In this paper, the concepts of doubly disconnected domination is given as a new disconnected type of domination in graphs. This type of dominating parameter putting a conditions on $D$ and its complement set $V - D$. Specific constraints on the doubly disconnected domination number are explored concerning graph size and order, minimum degree in $G$, maximum degree in $G$, and other structural properties. Additionally, the doubly disconnected domination number is determined for certain well-known graphs and any graphs formed by join or addition operations.

## 2. Doubly Disconnected Domination in Graphs

Doubly disconnected domination parameter is defined here. Some bounds and more properties of this parameter are discussed. Several relations in the graph related with this type of domination are studied.

**Definition 2.1** *Suppose that $G = (V, E)$ be an undirected graph, finite, simple, and nontrivial. A subset $D \subset V$ is a doubly disconnected dominating set (DD-set) of $G$ if the set is dominating and satisfies both conditions $G[D]$ and $G[V - D]$ are disconnected subgraphs.*

**Definition 2.2** *Minimal DD-set is a DD-set of $G$ which is not contains doubly disconnected dominating subset. It is said minimum DD-set if it has the least order among all minimal DD-sets in $G$ denoted by $\gamma_{dn}$- set. Its order being the doubly disconnected domination number $\gamma_{dn}(G)$.*

**Example 2.1** *The following figures show the different between the $\gamma$-set and $\gamma_{dn}-$set.*


Figure 1: The $\gamma$-set and $\gamma_{dn}$- set

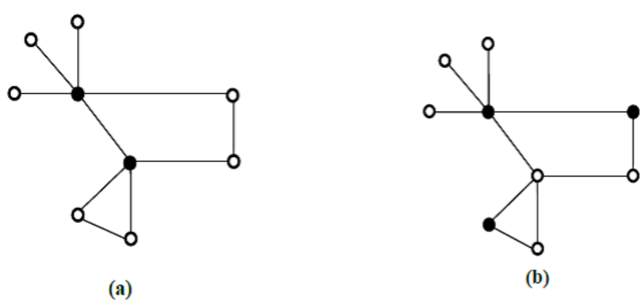


Figure 2: The $\gamma$-set and $\gamma_{dn}$- set

**Proposition 2.1** *For each graph $G = (n, m)$, that has a doubly disconnected dominating set $D$ and doubly disconnected domination number $\gamma_{dn}(G)$, the following holds:*

1. *The order of $G$ is $n \geq 4$.*

2. $|D| \geq 2$.

3. $|V - D| \geq 2$.

4. $\delta(G) \geq \begin{cases} 0 & \text{if } G \text{ has isolated vertex} \\ 1 & e.w \end{cases}$

5. $\Delta(G) \geq 1$.

6. For every $v \in D$, then $d(v) \geq \begin{cases} 0 & \text{if } G \text{ has isolated vertex} \\ 1 & e.w \end{cases}$

7. $\gamma(G) \leq \gamma_{dn}(G)$.

**Theorem 2.1** *Let $G$ be a graph and let $D$ be a doubly disconnected dominating set. If any of the given conditions is satisfied, hence $D$ forms a minimal doubly disconnected dominating set:*

1. $|N(u) \cap D| = 1$ *for all* $u \in V - D$.

2. $G[D]$ *is a null subgraph.*

3. $G[D]$ *has only disconnected support vertices.*

**Proof:** *Let $D$ be doubly disconnected dominating set in a graph $G$. If $D$ is not a minimal doubly disconnected dominating set. Then, there exists at least one vertex $v \in D$ such that the set $D - v$ is a minimal doubly disconnected dominating set. This scenario can be analyzed by considering various possible cases.*

***Case 1*** *Assume the first condition be satisfied. Hence, for each $u \in V - D$ that is dominated only by $v$, it is not dominated by any vertex in $D - \{v\}$. Consequently, $D - \{v\}$ does not form a doubly disconnected dominating set.*

***Case 2*** *When the second condition is satisfied. The vertex $v$ is not adjacent to any vertex in $D$, since the subgraph $G[D]$ is null . Therefore, $v$ isn't dominated by $D - \{v\}$. Therefore, $D - \{v\}$ is not doubly disconnected dominating set.*

***Case 3*** *When the third condition is satisfied. There exist at least one end vertex of $u$ is not dominated by any vertex of $D - \{v\}$. Thus, $D - \{v\}$ is not dominating set.*

□

**Theorem 2.2** *Let $G = (n, m)$ be any graph then the size of $G$ having doubly disconnected dominating set $D$ and doubly disconnected domination number $\gamma_{dn}$ $(G)$ is bounded as: $n - \gamma_{dn} < m < \frac{1}{2}(n^2 - 3n + 4)$.*

**Proof:** *Let $D$ be a $\gamma_{dn}$- set in $G$, then:*
***Case 1***: *Let $G[D]$ and $G[V - D]$ are two null subgraphs. From doubly disconnected domination definition, then from $V - D$ to $D$ there exist one edge from every vertex. Then, the number of edges from $D$ to $V - D$ equal to $|V - D| = n - \gamma_{dn}$ $(G)$. Therefore, in general $m \geq n - \gamma_{dn}(G)$ which is the lower bound.*
***Case 2***: *Let $G[D]$ be the union of complete graph and an isolated vertex, such that$G[D] = K_t \cup K_1$. Similarly,let $G[V - D]$ be the union of complete graph and an isolated vertex, where $G[V - D] = K_s \cup K_1$. consequently. Then, the number of edges in $G[D]$ and $G[V - D]$ are denoted by $m_1$ and $m_2$:*
$m_1 = \frac{(|D-1||D-2|)}{2} = \frac{(\gamma_{dn}-1)(\gamma_{dn}-2)}{2}$ *and* $m_2 = \frac{(|V-D-1||V-D-2|)}{2} = \frac{(n-\gamma_{dn}-1)(n-\gamma_{dn}-2)}{2}$ .
*Now, from doubly disconnected domination definition, there is $V - D$ edges at most joined each vertex of $D$ and $V - D$, where each vertex belong to $D$ dominates all the vertices of $V - D$. Thus, the maximum number of edges lies between $D$ and $V - D$ become $|D||V - D| = \gamma_{dn}(n - \gamma_{dn}) = m_3$, then the size of $G$ equals*
$m \leq m_1 + m_2 + m_3 = \frac{(\gamma_{dn}-1)(\gamma_{dn}-2)}{2} + \frac{(n-\gamma_{dn}-1)(n-\gamma_{dn}-2)}{2} + n\gamma_{dn} - \gamma_{dn}^2) = \frac{1}{2}(n^2 - 3n + 4)$. □

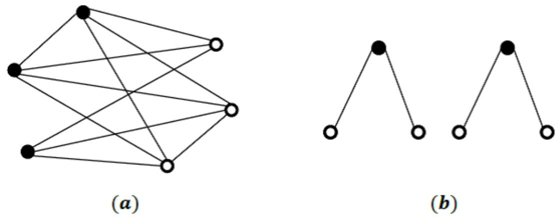


Figure 3: The lower and upper bounds

From above theorem, the upper and lower bounds are sharps with respect to the following figure.

**Theorem 2.3** *Let $G = (n, m)$ be any graph having doubly disconnected domination number $\gamma_{dn}$ $(G)$, then $2 \leq \gamma_{dn}(G) \leq n - 2$.*

**Proof:** *Let $D$ be $\gamma_{dn}$- set in $G$. Then, for the lower bound, by Proposition 2.1, since $|D| \geq 2$. Hence, $2 < \gamma_{dn}$ $(G)$. Now, since $G[D]$ and $G[V - D]$ are disconnected subgraphs, then $V - D$ must be contains two or more vertices to be dominated by the other vertices from $D$. Therefore, $\gamma_{dn}(G) \leq n - 2$.* □

The lower and upper bounds of above theorem will be sharp when $G = P_4$, where $gamma_{dn}(P_4) = 2$. For example, see Fig. 4.

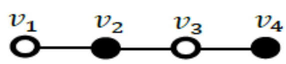


Figure 4: The lower and upper bounds

**Proposition 2.2** *A null graph $G = (n, m)$ has no doubly disconnected domination.*

**Proof:** *Since $G$ is null graph, then all vertices of $G$ is an isolated vertices. Therefore, every vertex in $G$ belongs to $D$. Such that $D = V(G)$. Then, $V - D = \phi$ this contradiction Proposition 2.1. Then, $G$ has no doubly disconnected dominating set.* □

## 3. Doubly Disconnected Domination in Corona and Join Operations

In this section, examines the effect of corona and join operations on the doubly disconnected domination number in graphs. The potential changes in these values are determined based on the properties of the underlying graph.

**Proposition 3.1** *For each graph $G = (n, m)$ having $\Delta(G) = n - 1$, hence $G$ has no doubly disconnected dominating set.*

**Proof:** *Since $\Delta(G) = n - 1$, there exist a vertex $v$ in $G$ such that $deg(v) = n - 1$. Therefore, the vertex $v$ is adjacent to all other vertices in $G$. Two cases need to be considered. If $v \in D$, then $v$ is adjacent with all other vertices of $D$. consequently, $G[D]$ is connected subgraph and $G$ does not have a doubly disconnected dominating set. If $v \in V - D$, thus $v$ is adjacent with all other vertices of $V - D$. As a result the subgraph $G[V - D]$ is connected and $G$ does not have a doubly disconnected dominating set.* □

**Theorem 3.1** *A connected graph $G_1$ and $G_2$ be any graph order $n$ and $m$ respectively $(n, m \geq 2)$, hence $\gamma_{dn}(G_1 \odot G_2) = (n - 1) + \gamma(G_2)$.*

**Proof:** *Since all vertices from each copy of $G_2$ are adjacent with one vertex of $G_1$. Let $D$ contains $n - 1$ vertices from $G_1$ and all vertices of the dominating set of $G_2$. Then, every vertex from $n - 1$ vertices in*

*$G_1$ dominates one copy of $G_2$. Since $G_1$ is connected graph. Therefore, we cannot choose the last vertex from $G_1$ in D. So, we will choose the dominating vertices from the last copy of $G_2$. Since each vertex in $G_2$ is adjacent to exactly one vertex in $G_1$, then $G[D]$ and $G[V-D]$ are disconnected subgraphs. Hence, $\gamma_{dn}(G_1 \odot G_2) = (n-1) + \gamma(G_2)$ and D is $\gamma_{dn}$- set. For example, see Fig.5.* □

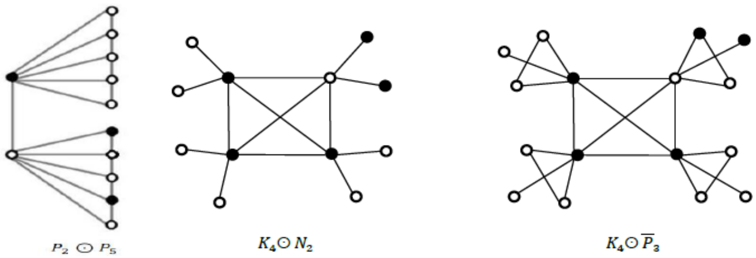


Figure 5: $\gamma_{dn}(G_1 \odot G_2) = (n-1) + \gamma(G_2)$

**Theorem 3.2** *Suppose that $G_1$ Serve as a disconnected graph and $G_2$ be any graph with order n and m respectively, such that $(n, m \geq 2)$, then $\gamma_{dn}(G_1 \odot G_2) = n$.*

**Proof:** *Since all vertices from each copy of$G_2$ are adjacent with one vertex of $G_1$. Then, assume that all vertices of $G_1$ belong to D such that $D = V(G_1)$. Therefore, every $v \in D$ dominates all vertices of one copy of $G_2$ and $G[D]$ is disconnected subgraph. Thus, D is $\gamma_{dn}$ set and $\gamma_{dn}(G_1 \odot G_2) = n$. For example, see Fig. 6.* □

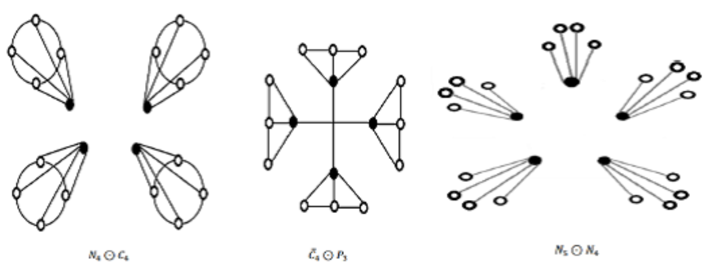

Figure 6: $\gamma_{dn}(G_1 \odot G_2) = n$

**Corollary 3.1** *Let $G_1 = (F_n, W_n, S_n)$ of order $n+1$ and $G_2$ be any graph of order m has $\gamma_{dn}$- set, then $\gamma_{dn}(G_1 \odot G_2) = n + \gamma(G_2)$.*

**Proof:** *Since $G_1$ has no doubly disconnected dominating set by Proposition 3.1. Then, each vertex of $G_1$ is adjacent with all the vertices of one copy in $G_2$. Let D contains n vertices of $G_1$ and all vertices of the dominating set of $G_2$, then every vertex from n vertices in $G_1$ dominates one copy of $G_2$, since $G_1$ connected graph. Therefore, we will choose the dominating vertices from one copy of $G_2$. Then $G[D]$ and $G[V-D]$ are disconnected subgraphs. Hence, $\gamma_{dn}(G_1 \odot G_2) = n + \gamma(G_2)$. and D is $\gamma_{dn}$- set. For example, see Fig. 7.* □

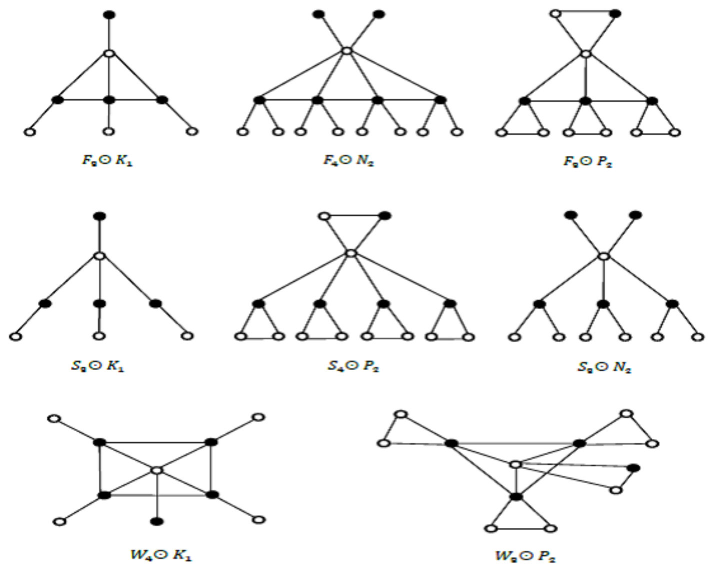

Figure 7: $\gamma_{dn}(G_1 \odot G_2) = n + \gamma(G_2)$

**Corollary 3.2** *A graph $G$ of order $n$, $(n \geq 2)$, we have $\gamma_{dn}(G \odot K_m) = n$.*

**Proof:** *Since $K_m$ has no doubly disconnected dominating set by Proposition 3.1. Then, all the vertices from every copy of $K_m$ are adjacent with on vertex of $G$. Since, every vertex of $G$ is adjacent with all other vertices in $K_m$. Thus, let $D$ contains $n-1$ vertices from $G$ and one vertex from one copy of $K_m$. Then, $G[D]$ and $G[V-D]$ are disconnected subgraphs. Therefore, $D$ is a $\gamma_{dn}$-set and$\gamma_{dn}(G \odot K_m) = n$. For example, see Fig. 8.* □

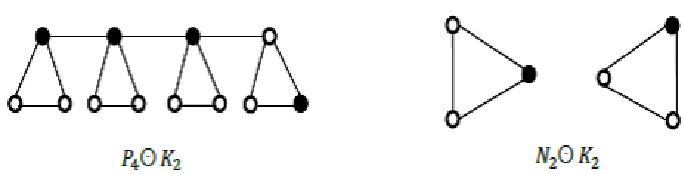


Figure 8: $\gamma_{dn}(G \odot K_m) = n$

**Proposition 3.2** *Any graph $G$ of order $m$, then $(K_1 \odot G)$ has no doubly disconnected domination.*

**Proof:** *Since $K_1$ contain one vertex say $v$ is adjacent with all vertices of $G$. Then, the degree of $v$ equals $m$, while the order of $K_1 \odot G$ equals $m+1$. Then, from Proposition 3.1. $K_1 \odot G$ has no doubly disconnected dominating set. For example, see Fig. 9.* □

**Theorem 3.3** *Let $G_1$ be a connected graph of order $n$, and $G_2$ be any graph of order $m$, then $G_1 + G_2$ has no doubly disconnected domination.*

**Proof:** *Since, all the vertices in $G_1$ are adjacent to each vertex of $G_2$. If $D = V(G_1)$, hence $G[D]$ is connected subgraph. If $D = V(G_2)$, then $G[V-D]$ is connected subgraph. If $D$ contains vertices from $G_1$ and $G_2$. Let $v_1, u_1 \in D$ where $v_1 \in V(G_1)$ along with $u_1 \in V(G_2)$. Then, $v_1u_1 \in E(G_1 + G_2)$ by the definition of join operation. Hence, $G[D]$ is connected subgraph and $G_1 + G_2$ has no doubly disconnected domination. Let $v_2, u_2 \in V - D$ where $v_2 \in V(G_1)$ and $u_2 \in V(G_2)$. Then, $v_2u_2 \in E(G_1 + G_2)$. Hence, $G[V-D]$ is connected subgraph which is contradiction. For example, see Fig. 10.* □

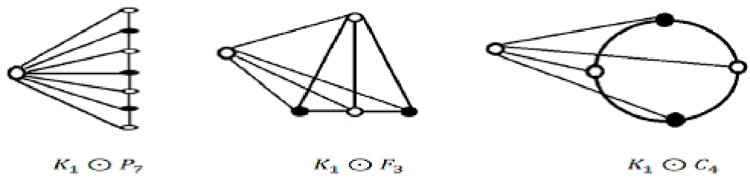


Figure 9: $(K_1 \odot G)$ has no doubly disconnected dominating set

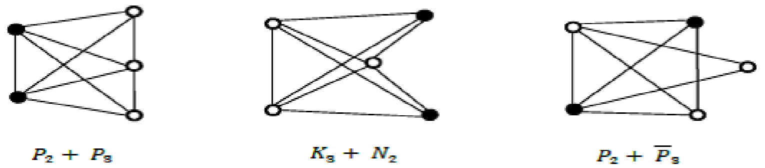


Figure 10: $(G_1 + G_2)$ has no doubly disconnected domination

**Corollary 3.3** *Let $G_1$ be any graph and $G_2$ be connected graph. Then, $G_1+G_2$ has no doubly disconnected domination.*

**Proof:** *The proof follows a similar approach to that of Theorem 3.3* □

**Theorem 3.4** *Let $G_1$ and $G_2$ are two disconnected graphs of order $n$ and $m$ respectively. Then $G_1 + G_2$ has doubly disconnected domination. Such that: $\gamma_{dn}(G_1 + G_2) = min\{n, m\}$*

**Proof:** *Since, all the vertices in $G_1$ are adjacent to every vertex of $G_2$.If $n \leq m$ let $D = V(G_1)$. Since, all the vertices in $G_1$ are adjacent to each vertex of $G_2$. Then, every vertex in $G_1$ dominate on all vertices of $G_2$. As, $G_1$ and $G_2$ are disconnected graphs. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs. If $m \leq n$ let $D = V(G_2)$. Since, all vertices of $G_1$ are adjacent to every vertex. Then, every vertex in $G_2$ dominate on all vertices of $G_1$. As $G_1$ and $G_2$ are disconnected graphs. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs. Suppose that $D$ contain from $G_1$ and $G_2$. Then, $D$ is not doubly disconnected dominating set. As, each vertex belong to $D$ in $G_1$ is adjacent to all vertices of $D$ in $G_2$. Then, by the definition join operation. Then, $G[D]$ is connected graph. Then, $G_1 + G_2$ has doubly disconnected dominating set. $\gamma_{dn}(G_1 + G_2) = min\{n, m\}$. For example, see Fig. 11.* □

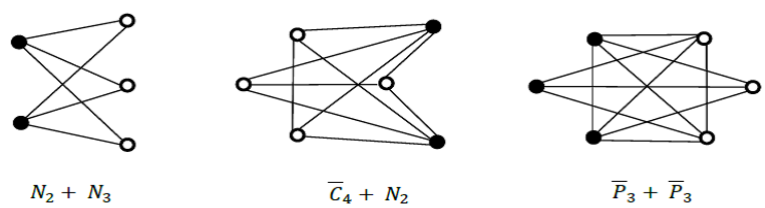


Figure 11: $\gamma_{dn}(G_1 + G_2) = n$

## 4. Doubly Disconnected Domination of Special Graphs

The doubly disconnected domination parameter is applied on some special graphs such as: path, complete bipartite graph , wheel graph, star graph, helm graph, barbell graph, big helm graph, the corresponding barbell graph, and fan graph.

**Proposition 4.1** *A path graph $P_n, (n \leq 3)$ has no doubly disconnected domination.*

**Proof:** *Since $deg(v_i) \leq 2, \forall v \in p_i, i = 2, 3$. If $n = 2$ then, $D$ has one vertex and $V - D$ has one vertex. Then, $G[D]$ and $G[V - D]$ are connected subgraphs. If $n = 3$, then $D$ has one vertex and $V - D$ has two vertices but $G[D]$ is connected. Hence, $P_n$ has no doubly disconnected domination.* □

**Theorem 4.1** *For each path graph $P_n, (n \geq 4)$ there is $\gamma_{dn}(P_n) = \lceil n/3 \rceil$.*

**Proof:** *The vertices $v_1, v_2, \cdots, v_n$ of a path graph of order $n$ and let $D \subset V(P_n)$ defined as:*

$$D = \begin{cases} \{v_{3i-1}, i = 1, 2, ..., \frac{n}{3}\} & \text{if } n \equiv 0 \ (mod\ 3) \\ \{v_{3i-1}, i = 1, 2, 3, \ldots, \frac{n-1}{3}\} \cup \{v_n\} & \text{if } n \equiv 1 \ (mod\ 3) \\ \{v_{3i-1}, i = 1, 2, 3, \ldots, \frac{n-2}{3}\} \cup \{v_n\} & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

*To prove $D$ is the doubly disconnected dominating set in path graph, we will examine three cases:*
***Case 1**: If $n \equiv 0$ (mod 3). Let $D = \{v_{3i-1}, i = 1, 2, ..., n/3\}$, we divide the vertices of $P_n$ in to $n/3$ sets such that each set contains three vertices. Then, we choose the second vertex of each set to be in $D$ which dominates two vertices of $V - D$. Hence, $G[D]$ and $G[V - D]$ are disconnected subgraphs. Thus, $P_n$ has doubly disconnected dominating set and $\gamma_{dn}(P_n) = n/3$.*
***Case 2**: If $n \equiv 1$ (mod3). Let $D = \{v_{3i-1}, i = 1, 2, 3, \ldots, (n-1)/3\} \cup \{v_n\}$, we divide the vertices of $P_n$ in to $\lceil n/3 \rceil$ sets such that each set contains three vertices unless the last set contains one vertex. Then, we choose the second vertex of each set to be in $D$ which dominates two vertices in $V - D$. And the last set contains one vertex belongs to $D$ which dominates only one vertex. Therefore, $G[D]$ and $G[V - D]$ are disconnected subgraphs and $P_n$has doubly disconnected dominating set where $\gamma_{dn}(P_n) = \lceil n/3 \rceil$.*
***Case 3**: If $n \equiv 2$ (mod3). Let $D = v_{3i-1}, i = 1, 2, 3, \cdots, \frac{n-2}{3} \cup \{v_n\}$, we divide the vertices of $P_n$ in to $\lceil n/3 \rceil$ sets such that each set contains three vertices unless the last set contains two vertices. Then, we choose the second vertex of each set to be in $D$ and dominates two vertices in $V - D$. And the last set the second vertex dominates one vertex in $V - D$. Then, $G[D]$ and $G[V - D]$ are disconnected subgraphs and $P_n$ has doubly disconnected dominating set where $\gamma_{dn}(P_n) = \lceil n/3 \rceil$.*
*Now, we prove that $D$ is a minimum doubly disconnected dominating set in all above cases. Let $D'$ is a doubly disconnected dominating set in $G$, such that $|D'| < |D|$. In this case there must one or more exist vertices of $V - D$ that are not dominated by any vertex of $D'$. This contradicts the definition of a doubly disconnected dominating set. Then, $D'$ cannot be a doubly disconnected dominating set and $D$ must be the minimum doubly disconnected dominating set. See Fig. 12.* □

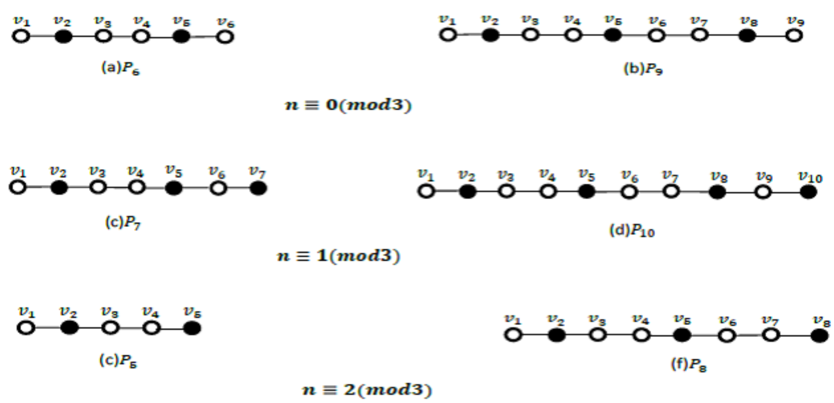


Figure 12: A minimum $DD - set$ of $P_n$.

**Proposition 4.2** *Any star graph $S_n, (n \geq 3)$ has no doubly disconnected dominating set.*

**Proof:** *Let $u \in S_n$ be a root vertex adjacent to all the pendants vertices $v_1, v_2, ..., v_n$. if $u \in D$, it will dominate three or more of the terminal vertices, and $G[D]$ will be a connected subgraph.On the other* *hand, If $u \notin D$, then $u \in V - D$ and all other vertices belongs to $D$, in this case, every vertex among the $n \geq 3$ terminal vertices will only dominate the vertex $u$. However, $G[V - D]$ is connected. Therefore, $S_n$ hasn't DD-set. See Fig. 13.* □

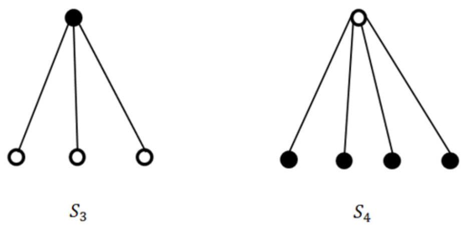


Figure 13: $S_n$ has no doubly disconnected domination

**Proposition 4.3** *If $G$ is a fan graph $F_n$. Then,$F_n$ has no doubly disconnected dominating set.*

**Proof:** *Since the fan graph $F_n = P_n + K_1$. If $D = V(K_1)$, then one vertex in $D$ dominates all other vertices of $V - D$, then $G[D]$ is connected graph, therefore $F_n$ has no doubly disconnected dominating set. If $D \subset V(P_n)$, then the vertices of $V - D$ are union of the vertex of $K_1$ and other vertices of $P_n$, then $G[V - D]$ is connected graph. Thus, $F_n$ hasn't DD-set. For example, see Fig. 14.* □

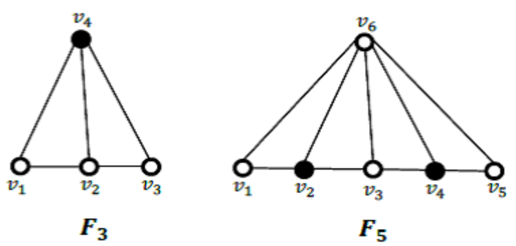


Figure 14: $F_n$ has no doubly disconnected dominating set.

**Proposition 4.4** *The wheel graph $W_n$ has no doubly disconnected dominating set.*

**Proof:** *Since the wheel graph $W_n = C_n + K_1$, and the vertex $v \in K_1$ is adjacent with all other vertices of $C_n$. Thus, if $D$ contains the single vertex of $K_1$, then $G[D]$ is connected subgraph, therefore $W_n$ has no doubly disconnected dominating set. If $V - D$ contains the single vertex of $k_1$, then $G[V - D]$ is connected subgraph. Thus, $W_n$ has no doubly disconnected dominating set. For example, see Fig. 15 .* □

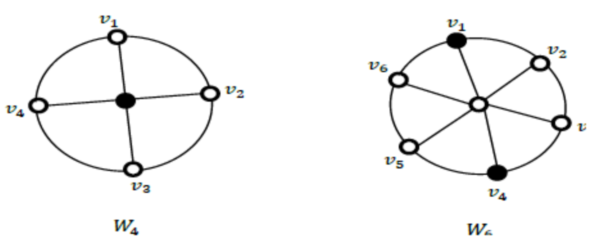


Figure 15: A wheel graph has no doubly disconnected dominating set

**Theorem 4.2** *$K_{n,m}$ has doubly disconnected domination, if and only if, $n \geq 2$ and $m \geq 2$, where $\gamma_{dn}(K_{n,m}) = min\{n, m\}$*

**Proof:** *Let $U_1$ as $U_2$ be the two sets of vertices of $K_{n,m}$ such that $|U_1| = n$ and $|U_2| = m$. Then, there are some cases as follows:*
***Case 1**: When $n < m$, then $D$ contains all vertices of $U_1$ where each vertex in$U_1$ will dominates all the $m$ vertices of $U_2$, that belongs to $V - D$. So, $G[D]$ and $G[V - D]$ are disconnected subgraphs. Hence, $\gamma_{dn}(K_{n,m}) = n$.*
***Case 2**: When $m < n$, then $D$ contains, all vertices of $U_2$ where every vertex in $D$ will dominates all vertices in $U_1$, that belongs to $V - D$. Where $G[D]$ and $G[V - D]$are disconnected subgraphs. Hence,$\gamma_{dn}(K_{n,m}) = m$.*
***Case 3**: When $n = m$, then $D$ contains all vertices of $U_1$ or all vertices of $U_2$ where every vertex in $D$ will dominates all vertices in $V - D$. Where $G[D]$ and $G[V - D]$ are disconnected subgraphs. Hence, $\gamma_{dn}(K_{n,m}) = n$ or $m$. The set $D$ is a minimum doubly disconnected dominating set in all three previous cases, the proof is similar to the proof of Theorem 4.1. For example, see Fig. 16.* □

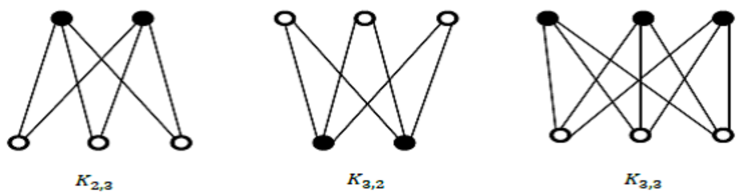


Figure 16: A minimum doubly disconnected dominating set

## 5. Conclusion

This study introduces a new type of domination in graph theory, known as doubly disconnected domination. Highlighting the connections between the doubly disconnected domination number and fundamental graph properties such as the number of edges, vertices, maximum degree, and minimum degree. We explored the construction of various conventional graphs and their modified versions, allowing for precise calculation of the domination number. Through this research, we have demonstrated the significance of doubly disconnected domination in enhancing our understanding of domination properties in several graphs.

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