Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–12.
ISSN-0037-8712
doi:10.5269/bspm.76248

Finite Integral Involving Incomplete Aleph-Functions, Generalized Extended Hurwitz's Zeta Function, Incomplete Gamma Function and Elliptic Integrals of the First Kind

Dinesh Kumar* and Frédéric Ayant

ABSTRACT: In this study, we derive a general finite integral that incorporates the generalized Hurwitz-Lerch Zeta function of two variables, the incomplete Gamma function, elliptic integrals of the first kind, and incomplete Aleph-functions. The paper concludes with a discussion of several corollaries and observations.

Key Words: Incomplete Gamma function, incomplete Aleph-function, Mellin-Barnes integrals contour, generalized Hurwitz's-Lerch Zeta function, incomplete Gamma function, elliptic integrals of first kind.

Contents

1. Introduction and Preliminaries

The development of incomplete special functions is closely tied to the evolution of special functions, which play a crucial role in various fields such as mathematics, physics, and engineering. These functions often arise as solutions to differential or integral equations, as well as in the analysis of probability and statistics. Incomplete special functions extend classical functions by truncating or limiting the domain of integration or summation.

The incomplete Gamma function and incomplete hypergeometric function were analyzed by Srivastava et al. [32], focusing on their specific characteristics. Recently, Srivastava et al. [33] introduced and analyzed the incomplete H-function and incomplete $\bar{H}$-function. Further contributions to the study of the incomplete $\aleph$-function, incomplete I-function, and integrals involving the incomplete H-function have been made by several researchers, including Bansal et al. [5], Bansal and Kumar [3], and Bansal et al. [4]. In more recent studies, Kumar et al. [13] examined improper integrals related to the incomplete Aleph-functions, while Kumar et al. [18] studied the Boros integral involving a class of polynomials and incomplete $\aleph$-functions. Additionally, Kumar et al. [14] investigated the Boros integral with the generalized multi-index Mittag-Leffler function and incomplete I-functions.

In the present study, we investigate a generalized finite integral that involves the generalized Hurwitz-Lerch Zeta function of two variables, the incomplete Gamma function, elliptic integrals of the first kind, and incomplete Aleph-functions.

The Gamma function, first introduced by Leonhard Euler in 1729 as an extension of the factorial function [10], is represented by Euler's integral formula, given as

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^u \,\mathrm{d}u \quad (\Re(\alpha) > 0), \tag{1.1}$$

* Corresponding author.
2020 *Mathematics Subject Classification:* Primary: 33B10, 33E20, 33F05; Secondary: 26A33, 33C05, 33C60.
Submitted March 22, 2025. Published December 29, 2025

Typeset by $\mathcal{B}^{S}\mathcal{P}_{\mathcal{M}}$ style.
© Soc. Paran. de Mat.

this was subsequently generalized to the incomplete Gamma function, defined as

$$\gamma\left(\alpha,x\right)=\int_0^x u^{\alpha-1}e^{-u}\,\mathrm{d}u \quad \left(\Re(\alpha)>0;\ x\geqslant 0\right), \tag{1.2}$$

and

$$\Gamma\left(\alpha,x\right)=\int_x^\infty u^{\alpha-1}\,e^{-u}\,\mathrm{d}u \quad \left(x\geqslant 0;\,\Re(\alpha)>0 \text{ when } x=0\right). \tag{1.3}$$

We have the following relation:

$$\gamma\left(\alpha,x\right)+\Gamma\left(\alpha,x\right)=\Gamma\left(\alpha\right) \quad \left(Re\left(\alpha\right)>0\right). \tag{1.4}$$

Now, we give the expression of the incomplete Aleph-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)$ defined by Bansal et al. [5] (see also, [16]).

$$\begin{aligned}{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z,x\right) &= {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z\,\middle|\,\begin{array}{c}\left(a_1,A_1,x\right),\left(a_j,A_j\right)_{2,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \\ \left(g_j,G_j\right)_{1,m},\left[\tau_i\left(g_{ji},G_{ji}\right)\right]_{m+1,q_i;r}\end{array}\right)\\ &=\frac{1}{2\pi\omega}\int_L\frac{\Gamma\left(1-a_1-A_1s,x\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-A_js\right)\prod_{j=1}^{m}\Gamma\left(g_j+G_js\right)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma\left(1-g_{ji}-G_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\right]}z^{-s}\mathrm{d}s,\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}{}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z,x\right) &= {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left(z\,\middle|\,\begin{array}{c}\left(a_1,A_1,x\right),\left(a_j,A_j\right)_{2,n},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{n+1,p_i;r}\\ \\ \left(g_j,G_j\right)_{1,m},\left[\tau_i\left(g_{ji},G_{ji}\right)\right]_{m+1,q_i;r}\end{array}\right)\\ &=\frac{1}{2\pi\omega}\int_L\frac{\gamma\left(1-a_1-A_1s,x\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-A_js\right)\prod_{j=1}^{m}\Gamma\left(g_j+G_js\right)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma\left(1-g_{ji}-G_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)\right]}z^{-s}\mathrm{d}s.\end{aligned} \tag{1.6}$$

The incomplete $\aleph$-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)$, as defined above, exist for $x\geq 0$ under the following validity conditions:

The contour L lies in the s-plane and extends from $\sigma-i\infty$ to $\sigma+i\infty$ where σ is a real number. A loop may be introduced, if necessary, to ensure that the poles of $\Gamma(1-a_j-A_js)$ for $j=2,\cdots,n$ lie to the right of the contour L, and the poles of $\Gamma(g_j+G_js)$, $j=1,\cdots,m$ lie to the left of it. The parameters τ_i,m,n,p_i and q_i are positive numbers that satisfy the conditions $0\leq n\leq p_i$, $0\leq m\leq q_i$. The values of a_j,g_j and a_{ji},g_{ji} are complex numbers. It is assumed that the poles of the integrand are simple. The following conditions apply:

$$\Omega_i>0,|\arg(z)|<\frac{\pi}{2}\Omega_i \quad (i=1,\cdots,r), \tag{1.7}$$

$$\Omega_j\geq 0,|\arg(z)|<\frac{\pi}{2}\Omega_i \text{ and } \Re(\zeta_i)+1<0, \tag{1.8}$$

where

$$\Omega_i=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}G_j-\tau_i\max_{1\leq i\leq r}\left(\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=m+1}^{q_i}G_{ji}\right) \tag{1.9}$$

and

$$\zeta_i=\sum_{j=1}^{m}b_j-\sum_{j=1}^{n}a_j+\tau_i\left(\sum_{j=m+1}^{q_i}b_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right)+\frac{p_i-q_i}{2} \quad (i=1,\cdots,r). \tag{1.10}$$

We can easily derive the following relation:

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)+{}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)=\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z), \tag{1.11}$$

where $\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z)$ is the Aleph-function [2,12,17,19,22,23,27] (see also, [15,20,34]).
Taking $\tau_i \to 1$, then the incomplete Aleph-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r}(z,x)$ reduce respectively to incomplete I-functions ${}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z,x)$ and ${}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z,x)$, defined by

$$\begin{aligned} {}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z,x) &= {}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}\left(z \left| \begin{array}{c} (a_1,A_1,x), (a_j,A_j)_{2,n}, (a_{ji},A_{ji})_{n+1,p_i;r} \\ \\ (g_j,G_j)_{1,m}, (g_{ji},G_{ji})_{m+1,q_i;r} \end{array} \right. \right) \\ &= \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}\, z^{-s}\mathrm{d}s, \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} {}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z,x) &= {}^{(\gamma)}I^{m,n}_{p_i,q_i;r}\left(z \left| \begin{array}{c} (a_1,A_1,x), (a_j,A_j)_{2,n}, (a_{ji},A_{ji})_{n+1,p_i;r} \\ \\ (g_j,G_j)_{1,m}, (g_{ji},G_{ji})_{m+1,q_i;r} \end{array} \right. \right) \\ &= \frac{1}{2\pi\omega}\int_L \frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}s)\right]}\, z^{-s}\mathrm{d}s, \end{aligned} \tag{1.13}$$

under the same conditions as stated earlier, with $\tau_i \to 1$, we now consider $r = 1$. In this case, the incomplete I-functions ${}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z,x)$ and ${}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z,x)$ simplify to incomplete H-functions, namely ${}^{(\Gamma)}H^{m,n}_{p,q}(z,x)$ and ${}^{(\gamma)}H^{m,n}_{p,q}(z,x)$, respectively. Given as

$$\begin{aligned} {}^{(\Gamma)}H^{m,n}_{p,q}(z,x) &= {}^{(\Gamma)}H^{m,n}_{p,q}\left(z \left| \begin{array}{c} (a_1,A_1,x), (a_j,A_j)_{2,p} \\ \\ (g_j,G_j)_{1,q} \end{array} \right. \right) \\ &= \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}\, z^{-s}\mathrm{d}s \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} {}^{(\gamma)}H^{m,n}_{p,q}(z,x) &= {}^{(\gamma)}H^{m,n}_{p,q}\left(z \left| \begin{array}{c} (a_1,A_1,x), (a_j,A_j)_{2,p} \\ \\ (g_j,G_j)_{1,q} \end{array} \right. \right) \\ &= \frac{1}{2\pi\omega}\int_L \frac{\gamma(1-a_1-A_1s,x)\prod_{j=2}^{n}\Gamma(1-a_j-A_js)\prod_{j=1}^{m}\Gamma(g_j+G_js)}{\prod_{j=m+1}^{q}\Gamma(1-g_j-G_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}\, z^{-s}\mathrm{d}s, \end{aligned} \tag{1.15}$$

under the same conditions confirmed by the incomplete I-functions when $r = 1$.
By using the formula (1.11), we have the following relations:

$$ {}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}(z,x) + {}^{(\gamma)}I^{m,n}_{p_i,q_i;r}(z,x) = I^{m,n}_{p_i,q_i;r}(z), \tag{1.16}$$

the function $I^{m,n}_{p_i,q_i,r}(z)$ being the function defined by Saxena [28], and

$$ {}^{(\Gamma)}H^{m,n}_{p,q}(z,x) + {}^{(\gamma)}H^{m,n}_{p,q}(z,x) = H^{m,n}_{p,q}(z). \tag{1.17}$$

Using the incomplete Gamma function as defined in equation 1.2, we express it in terms of the hypergeometric function [8, p.656, Eq. 20]:

$$\gamma(\mu,z) = \frac{z^{\upsilon}}{\upsilon}\, {}_1F_1\left[1;\upsilon+1:z\right]. \tag{1.18}$$

The complete elliptic integrals of first kind is defined by (see, Whittaker and Watson [35, p. 515])

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2 sin^2\theta}} = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1.19}$$

Apostol [1, p. 89] studied a generalization of the Riemann Zeta function, defined as follows:

$$Z(s,a) = \sum_{k=0}^{\infty} \frac{1}{(a+s)^k} \qquad (\Re(s) > -1,\ a \notin \{0,-1,-2,\cdots\}). \tag{1.20}$$

A new generalization of the Zeta function was introduced by Erdélyi et al. [9, p. 27, Eq. 1.11(1)]. This function is referred to as the generalized Hurwitz-Lerch Zeta function of one variable. The expression for this function is:

$$\phi(x,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+s)^k} \qquad (|z|<1,\ s\in\mathbb{C},\ a \notin \{0,-1,-2,\cdots\}). \tag{1.21}$$

Goyal and Ladha [11, p. 100, Eq. (15)] presented another generalization of the Hurwitz-Lerch Zeta function of one variable, defined by:

$$\phi_{\mu}^{*}(x,s,a) = \sum_{k=0}^{\infty} \frac{(\mu)_k\, z^k}{(a+m)^s\, k!}, \tag{1.22}$$

where $a \notin \{0,-1,-2,\cdots\}$, $\mu \geq 1$ and either $|z|<1$, $\Re(s)>0$ and $\Re(s)>\mu$.
We will note

$$A_{\mu}^{*}(k,s,a) = \frac{(\mu)_k}{(a+m)^s\, k!}. \tag{1.23}$$

Recently Parmar et al. [26, p. 179, Eq.(2,1)] gave an extension of the Hurwitz's-Lerch Zeta function of one variable, defined as follows:

$$\phi_{\lambda,\mu,\upsilon}^{(\rho,\sigma)}(z,s,a,m) = \sum_{k=0}^{\infty} \frac{(\lambda)_k\, B_p^{(\rho,\sigma)}(\mu+k,\upsilon-\mu)}{B(\upsilon,\upsilon-\mu)} \frac{z^k}{(a+s)^k}, \tag{1.24}$$

provided that $p, \Re(\rho), \Re(\sigma) \geq 0,\ \lambda,\mu \in \mathbb{C},\ \upsilon, a \setminus \mathbb{Z}_0^-, s \in \mathbb{C}$ when $|z|<1$; $\Re(s+\upsilon-\lambda-\mu)>1$ for $z=1$.
We have

$$A_{\lambda,\mu,\upsilon}^{(\rho,\sigma)}(k,s,a,m) = \frac{(\lambda)_k\, B_p^{(\rho,\sigma)}(\mu+k,\upsilon-\mu)}{B(\upsilon,\upsilon-\mu)} \frac{1}{(a+s)^k}, \tag{1.25}$$

where the extended Beta function $B_p(.)$ is defined by Özergin et al. [25], defines as:

$$B_p^{(\rho,\sigma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, {}_1F_1\left[\rho;\sigma;-\frac{p}{t(1-t)}\right]\mathrm{d}t, \tag{1.26}$$

and $\Re(p), \min\{\Re(x),\Re(y)\} > 0$.

Batra and Rai [6, p. 1551, Eq.(5)] introduced a novel extension of the Hurwitz-Lerch Zeta function for two variables, defined as follows:

$$\Phi_{\alpha,\beta,\beta',\gamma,\gamma'}(z,t,s,a) = \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_i(\gamma')_j\, i!j!} \frac{z^i t^j}{(a+i+j)^s}, \tag{1.27}$$

provided that $\alpha,\beta,\beta' \in \mathbb{C}$, and $\gamma, a \notin \{0,-1,-2,\cdots\}$, with $s,z,t \in \mathbb{C}$, and $\Re(s)>0$ when $|z|,|t|<1$, or alternatively, we have $\Re(\gamma-s-\alpha-\beta-\beta')>1$ when $|z|=|t|=1$.
We will denote

$$A_{\alpha,\beta,\beta',\gamma,\gamma'}(z,t,s,a) = \frac{(\alpha)_{i+j}(\beta)_i(\beta')_j}{(\gamma)_i(\gamma')_j\, i!j!} \frac{1}{(a+i+j)^s}. \tag{1.28}$$

Recently, Batra and Rai [7] provided a generalization of the extended Hurwitz-Lerch zeta function for two variables, defined as follows:

$$\begin{aligned}&\Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(z,t,s,a)\\&=\sum_{i,j=0}^{\infty}\frac{(\alpha)_{i+j}\,B_p^{(\rho,\sigma,r)}(\beta+i,\gamma-\beta)\,B_p^{(\rho',\sigma',r)}(\beta'+j,\gamma'-\beta')}{B(\beta,\gamma-\beta)\,B(\beta',\gamma'-\beta')\,i!j!}\frac{z^it^j}{(a+i+j)^s},\end{aligned}\tag{1.29}$$

assuming that $p\geq 0$, $\Re(\rho),\Re(\sigma),\Re(\rho'),\Re(\sigma')>0,\alpha,\beta,\beta'\in\mathbb{C},\gamma,\gamma',a\mathbb{C}\setminus\mathbb{Z}_0^-$, and $s,z,\in\mathbb{C}$ with $\Re(s)>0$ when $|z|,|t|<1$, and $\Re(s+\gamma+\gamma'-\beta-\beta'-\alpha)>1$ when $|z|=|t|=1$.
Let us denote

$$\begin{aligned}&A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s,a)\\&=\frac{(\alpha)_{i+j}\,B_p^{(\rho,\sigma,r)}(\beta+i,\gamma-\beta)\,B_p^{(\rho',\sigma',r)}(\beta'+j,\gamma'-\beta')}{B(\beta,\gamma-\beta)\,B(\beta',\gamma'-\beta')\,i!j!}\frac{1}{(a+i+j)^s},\end{aligned}\tag{1.30}$$

where, the generalized Beta function, as outlined in [24, p. 189, Eq. (1.13)], is given by

$$B_p^{(\alpha,\beta,m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left[\alpha;\beta;-\frac{p}{t^m(1-t)^m}\right]\mathrm{d}t,\tag{1.31}$$

where, $\Re(p)\geq 0,\min\{\Re(x),\Re(y),\Re(\alpha),\Re(\beta)\}>0$.

1.1. Required integral

We present a generalized finite integral, as seen in Brychkov [8, 4.1.1 Eq. 2, p. 276]. The result is as follows:

Lemma 1.1

$$\begin{aligned}&\int_0^1 x^s e^{ax}\gamma(\upsilon,ax)\,K\left(\sqrt{1-x}\right)\mathrm{d}x\\&=\frac{\pi a^{\upsilon}\Gamma^2(s+\upsilon)}{2\upsilon\Gamma^2\left(s+\upsilon+\frac{1}{2}\right)}\,{}_3F_3\left(a\left|\begin{matrix}1,s+\upsilon,s+\upsilon\\ \\ \upsilon+1,s+\upsilon+\frac{1}{2},s+\upsilon+\frac{1}{2}\end{matrix}\right.\right),\end{aligned}\tag{1.32}$$

provided that $\Re(s+\upsilon)>0$.

2. Main Integral

In the following section, we give a general formula with $x'>0$.

Theorem 2.1

$$\begin{aligned}&\int_0^1 x^s e^{ax}\gamma(\upsilon,ax)\,K\left(\sqrt{1-x}\right)\,{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r'}\left(zx^A,x'\right)\,\Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B,Yx^C,s',a'\right)\mathrm{d}x\\&=\frac{\pi a^{\upsilon}}{2}\sum_{i,j,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'}Z^iY^jA^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s,a)\\&\times{}^{(\Gamma)}\aleph^{m,n+2}_{p_i+2,q_i+2,\tau_i;r'}\left(z\left|\begin{matrix}(a_1,A_1,x'),(1-s-\upsilon-Bi-Cj-n';A),\\ \\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i;r'},\end{matrix}\right.\right.\\&\qquad\left.\begin{matrix}(1-s-\upsilon-Bi-Cj-n';A),(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r'}\\ \\ \left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right),\left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right)\end{matrix}\right),\end{aligned}\tag{2.1}$$

provided that $\Re\left(s+\upsilon'\right)>0,\ A,B,C>0,\ x'>0,$

$$\Re\left(s+\upsilon+Bi+Cj\right)-A\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right)>-1,$$

and $\Omega_i>0$, *with* $|\arg(z)|<\frac{\pi}{2}\Omega_i$ *for* $i=1,\cdots,r'$ *or* $\Omega_i\geq 0$ *and* $|\arg(z)|<\frac{\pi}{2}\Omega_i$, *while also satisfying* $\Re\left(\zeta_i\right)+1<0$, *where* Ω_i *and* ζ_i *are defined by* (1.9) *and* (1.10), *and also satisfy the conditions in* (1.29) *and* (1.31).

Proof: To prove the theorem, we begin by expressing an extension of the generalized Hurwitz's-Lerch Zeta function for two variables, as defined by Batra and Rai [7]. Using the series representation from (1.29) and the modified incomplete Gamma Aleph-function in the Mellin-Barnes contour integral, as described in (1.5), we interchange the order of integration and summation. This step is justified by the absolute convergence of the integrals involved in the process. Finally, by collecting the powers of x, we obtain $\mathcal{I}$, as expressed by

$$\begin{aligned}\mathcal{I}=&\int_0^1 x^s e^{ax}\gamma\left(\upsilon,ax\right)\,K(\sqrt{(1-x)}\,{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r'}\left(zx^A,x'\right)\\&\times\Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B,Yx^C,s',a'\right)\mathrm{d}x\\&=\frac{\pi a^{\upsilon}}{(2)}\sum_{i,j=0}^{\infty}\frac{1}{2\pi\omega}\int_L\frac{\Gamma\left(1-a_1-A_1t,x'\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-A_jt\right)\prod_{j=1}^{m}\Gamma\left(g_j+G_jt\right)}{\sum_{i=1}^{r'}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma\left(1-g_{ji}-G_{ji}t\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}t\right)\right]}z^{-t}\\&\times A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(i,j,s',a'\right)\int_0^1 x^{s-At+Bi+Cj}\,e^{ax}\gamma\left(\upsilon,ax\right)\,K\left(\sqrt{1-x}\right)\mathrm{d}t.\end{aligned}\tag{2.2}$$

With the lemma 1.1, this leads to

$$\begin{aligned}\mathcal{I}=&\int_0^1 x^s e^{ax}\gamma\left(\upsilon,ax\right)\,K\left(\sqrt{1-x}\right)\,{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r'}\left(zx^A,x'\right)\\&\times\Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B,Yx^C,s',a'\right)\mathrm{d}x\\&=\frac{\pi a^{\upsilon}}{2}\sum_{i,j=0}^{\infty}\frac{1}{2\pi\omega}\int_L\frac{\Gamma\left(1-a_1-A_1t,x'\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-A_jt\right)\prod_{j=1}^{m}\Gamma\left(g_j+G_jt\right)}{\sum_{i=1}^{r'}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma\left(1-g_{ji}-G_{ji}t\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}t\right)\right]}z^{-t}\\&\times A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(i,j,s,a\right)\frac{\Gamma^2\left(s+Bl+Cj-At\right)}{\Gamma^2\left(s+\frac{1}{2}+Bl+Cj-At\right)}\\&\times{}_3F_3\left(a\left|\begin{matrix}1,s+\upsilon+Bi+Cj-At,s+\upsilon+Bi+Cj-At\\ \upsilon+1,s+\upsilon+Bi+Cj-At+\frac{1}{2},s+Bi+Cj-At+\upsilon+\frac{1}{2}\end{matrix}\right.\right)\mathrm{d}t.\end{aligned}\tag{2.3}$$

We substitute the Gauss hypergeometric function with the series $\sum_{n=0}^{\infty}$ (see, Slater [31]).Under the assumption that we can interchange the series and the t-integrals, we obtain

$$\begin{aligned}\mathcal{I}=&\int_0^1 x^s e^{ax}\gamma\left(\upsilon,ax\right)\,K\left(\sqrt{1-x}\right)\,{}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r'}\left(zx^A,x'\right)\\&\times\Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B,Yx^C,s',a'\right)\mathrm{d}x=\frac{\pi a^{\upsilon}}{2}\sum_{i,j,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}\left(a\right)^{n'}Z^iY^j\\&\times\frac{1}{2\pi\omega}\int_L\frac{\Gamma\left(1-a_1-A_1t,x'\right)\prod_{j=2}^{n}\Gamma\left(1-a_j-A_jt\right)\prod_{j=1}^{m}\Gamma\left(g_j+G_jt\right)}{\sum_{i=1}^{r'}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma\left(1-g_{ji}-G_{ji}t\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}t\right)\right]}z^{-t}\\&\times A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(i,j,s',a'\right)\frac{\Gamma^2\left(s+Bl+Cj-At\right)}{\Gamma^2\left(s+\frac{1}{2}+Bl+Cj-At\right)}\frac{\left[\left(s+Bl+Cj-At\right)_{n'}\right]^2}{\left[\left(s+\frac{1}{2}+Bl+Cj-At\right)_{n'}\right]^2}\mathrm{d}t.\end{aligned}\tag{2.4}$$

By applying the formula $\Gamma(a)(a)_n = \Gamma(a+n)$, this gives

$$\begin{aligned}
\mathcal{I} = &\int_0^1 x^s e^{ax}\gamma(\upsilon, ax)\, K\left(\sqrt{1-x}\right)\, {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r'}\left(zx^A, x'\right)\\
&\times \Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B, Yx^C, s', a'\right)\mathrm{d}x = \frac{\pi a^\upsilon}{2}\sum_{i,j,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'} Z^i Y^j\\
&\times \frac{1}{2\pi\omega}\int_L \frac{\Gamma(1-a_1-A_1t, x')\prod_{j=2}^{n}\Gamma(1-a_j-A_jt)\prod_{j=1}^{m}\Gamma(g_j+G_jt)}{\sum_{i=1}^{r'}\tau_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-g_{ji}-G_{ji}t)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+A_{ji}t)\right]} z^{-t}\\
&\times A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s',a')\,\frac{\Gamma^2(s+Bl+Cj+n'-At)}{\Gamma^2\left(s+\frac{1}{2}+Bl+Cj+n'-At\right)}\mathrm{d}t,
\end{aligned} \tag{2.5}$$

by interpreting the contour integral (2.5) with the help of the definition of the Incomplete Aleph-function (1.5), we arrive at the desired formula. □

Theorem 2.2

$$\begin{aligned}
&\int_0^1 x^s e^{ax}\gamma(\upsilon, ax)\, K\left(\sqrt{1-x}\right)\, {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\tau_i;r'}\left(zx^A, x'\right)\Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B, Yx^C, s', a'\right)\mathrm{d}x\\
&= \frac{\pi a^\upsilon}{2}\sum_{i,j,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'} Z^i Y^j A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s',a')\\
&\times {}^{(\gamma)}\aleph^{m,n+2}_{p_i+2,q_i+2,\tau_i;r'}\left(z\left|\begin{array}{l}(a_1,A_1,x'),(1-s-\upsilon-Bi-Cj-n';A),\\ (g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i;r'},\end{array}\right.\right.\\
&\qquad\left.\begin{array}{l}(1-s-\upsilon-Bi-Cj-n';A),(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r'}\\ \left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right),\left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right)\end{array}\right),
\end{aligned} \tag{2.6}$$

under the same notation and conditions as in Theorem 2.1, the proof follows similarly to that of (2.1).

3. Special Cases

In this section, we present several specific cases. Initially, we discuss the particular incomplete I- and H-functions. Subsequently, we reference the special cases of the Hurwitz-Lerch Zeta function, both for two variables and for one variable, as considered in this paper.

Corollary 3.1

$$\begin{aligned}
&\int_0^1 x^s e^{ax}\gamma(\upsilon, ax)\, K\left(\sqrt{1-x}\right)\, {}^{(\Gamma)}I^{m,n}_{p_i,q_i,\tau_i;r'}\left(zx^A, x'\right)\, \Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B, Yx^C, s', a'\right)\mathrm{d}x\\
&= \frac{\pi a^\upsilon}{2}\sum_{i,j,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'} Z^i Y^j A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s',a')\\
&\times {}^{(\Gamma)}I^{m,n+2}_{p_i+2,q_i+2;r'}\left(z\left|\begin{array}{l}(a_1,A_1,x'),(1-s-\upsilon-Bi-Cj-n';A),\\ (g_j,G_j)_{1,m},(g_{ji},G_{ji})_{m+1,q_i;r'},\end{array}\right.\right.\\
&\qquad\left.\begin{array}{l}(1-s-\upsilon-Bi-Cj-n';A),(a_j,A_j)_{2,n},(a_{ji},A_{ji})_{n+1,p_i;r'}\\ \left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right),\left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right)\end{array}\right),
\end{aligned} \tag{3.1}$$

provided that $A, B, C > 0$, *and*

$$\Re(s+\upsilon') > 0\,\Re(s+\upsilon+Bi+Cj) - A\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) > -1.$$

Also verified the conditions given in (1.29) and (1.31), and $\Omega_i > 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$ *for* $i = 1, \cdots, r'$ *or* $\Omega_i \geq 0, |\arg(z)| < \frac{\pi}{2}\Omega_i$ *and* $\Re(\zeta_i) + 1 < 0$*, where* Ω_i *and* ζ_i *are given by (1.9) and (1.10) respectively.*

Corollary 3.2

$$\begin{aligned}
&\int_0^1 x^s e^{ax} \gamma(\upsilon, ax)\ K\left(\sqrt{1-x}\right)\ {}^{(\gamma)}I^{m,n}_{p_i,q_i;r'}\left(zx^A, x'\right)\ \Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B, Yx^C, s', a'\right) \mathrm{d}x \\
&= \frac{\pi a^\upsilon}{2} \sum_{i,j,n'=0}^{\infty} \frac{1}{(\upsilon)_{n'}} (a)^{n'} Z^i Y^j A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s',a') \\
&\times {}^{(\gamma)}I^{m,n+2}_{p_i+2,q_i+2;r'}\left(z \left| \begin{array}{l} (a_1, A_1, x'), (1-s-\upsilon-Bi-Cj-n';A), \\ \qquad (g_j, G_j)_{1,m}, (g_{ji}, G_{ji})_{m+1,q_i;r'}, \end{array}\right.\right. \\
&\qquad \left.\begin{array}{c} (1-s-\upsilon-Bi-Cj-n';A), (a_j, A_j)_{2,n}, (a_{ji}, A_{ji})_{n+1,p_i;r'} \\ \left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right), \left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right) \end{array}\right), \qquad (3.2)
\end{aligned}$$

under the same conditions and notations as in Corollary 3.1.

Corollary 3.3

$$\begin{aligned}
&\int_0^1 x^s e^{ax} \gamma(\upsilon, ax)\ K\left(\sqrt{1-x}\right)\ {}^{(\Gamma)}H^{m,n}_{p,q}\left(zx^A, x'\right)\ \Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B, Yx^C, s', a'\right) \mathrm{d}x \\
&= \frac{\pi a^\upsilon}{2} \sum_{i,j,n'=0}^{\infty} \frac{1}{(\upsilon)_{n'}} (a)^{n'} Z^i Y^j A^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s',a') \\
&\times {}^{(\Gamma)}H^{m,n+2}_{p+2,q+2}\left(z \left| \begin{array}{l} (a_1, A_1, x'), (1-s-\upsilon-Bi-Cj-n';A), \\ (g_j, G_j)_{1,q}, \left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right), \end{array}\right.\right. \\
&\qquad \left.\begin{array}{c} (1-s-\upsilon-Bi-Cj-n';A), (a_j, A_j)_{2,p} \\ \left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right) \end{array}\right), \qquad (3.3)
\end{aligned}$$

provided that

$$\Re(s+\upsilon') > 0,\ \Re(s+\upsilon+Bi+Cj) - A \min_{1\leqslant j\leqslant m} \Re\left(\frac{g_j}{G_j}\right) > -1,\ A, B, C > 0.$$

The conditions specified in (1.29) and (1.31) are satisfied. Additionally, either $\Omega > 0$ *and* $|\arg(z)| < \frac{\pi}{2}\Omega$*, or* $\Omega \geq 0$, $|\arg(z)| < \frac{\pi}{2}\Omega$*, and* $\Re(\zeta) + 1 < 0$*, where*

$$\Omega = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} G_j - \left(\sum_{j=n+1}^{p} A_j + \sum_{j=m+1}^{q} G_j\right), \qquad (3.4)$$

and

$$\zeta = \sum_{j=1}^{m} G_j - \sum_{j=1}^{n} A_j + \left(\sum_{j=m+1}^{q} g_j - \sum_{j=n+1}^{p} a_j\right) + \frac{p-q}{2}. \qquad (3.5)$$

Corollary 3.4

$$\int_0^1 x^s e^{ax}\gamma(\upsilon,ax)\,K\left(\sqrt{1-x}\right)\,{}^{(\gamma)}H_{p,q}^{m,n}\left(zx^A,x'\right)\,\Phi^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}\left(ZX^B,Yx^C,s',a'\right)\mathrm{d}x$$
$$=\frac{\pi a^{\upsilon}}{2}\sum_{i,j,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'}Z^iY^jA^{*,\rho,\sigma,\rho',\sigma',r}_{\alpha,\beta,\beta',\gamma,\gamma';r}(i,j,s',a')$$
$${}^{(\gamma)}H_{p+2,q+2}^{m,n+2}\left(z\left|\begin{matrix}(a_1,A_1,x'),(1-s-\upsilon-Bi-Cj-n';A),\\(g_j,G_j)_{1,q},\left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right),\end{matrix}\right.\begin{matrix}(1-s-\upsilon-Bi-Cj-n';A),(a_j,A_j)_{2,p}\\\left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right)\end{matrix}\right),\tag{3.6}$$

Next, We refer to the Hurwitz-Lerch Zeta function with two variables, as introduced by Batra and Rai [6, p. 1551, Eq.(5)]. Then, we arrive at the following result:

Corollary 3.5

$$\int_0^1 x^s e^{ax}\gamma(\upsilon,ax)\,K\left(\sqrt{1-x}\right)\,{}^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r'}^{m,n}\left(zx^A,x'\right)\,\Phi_{\alpha,\beta,\beta',\gamma,\gamma'}\left(Zx^B,Yx^C,s',a'\right)\mathrm{d}x$$
$$=\frac{\pi a^{\upsilon}}{2}\sum_{i,j,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'}Z^iY^jA_{\alpha,\beta,\beta',\gamma,\gamma'}(i,j,s',a')$$
$$\times\,{}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r'}^{m,n+2}\left(z\left|\begin{matrix}(a_1,A_1,x'),(1-s-\upsilon-Bi-Cj-n';A),\\(g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i;r'},\end{matrix}\right.\begin{matrix}(1-s-\upsilon-Bi-Cj-n';A),(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r'}\\\left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right),\left(\frac{1}{2}-s-\upsilon-Bi-Cj-n';A\right)\end{matrix}\right),\tag{3.7}$$

under the conditions confirmed by Theorem 2.1, and when the conditions in (1.26) and (1.27) are met instead of those in (1.29) and (1.31).

Next, we consider an extension of the Hurwitz-Lerch Zeta function with one variable, as defined by Parmar et al. [26, p. 179, Eq.(2.1)]. This leads to

Corollary 3.6

$$\int_0^1 x^s e^{ax}\gamma(\upsilon,ax)\,K\left(\sqrt{1-x}\right)\,\phi^{(\rho,\sigma)}_{\lambda,\mu,\upsilon}\left(Sx^B,s,a,m\right)\,{}^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r'}^{m,n}\left(zx^A,x'\right)$$
$$=\frac{\pi a^{\upsilon}}{2}\sum_{i,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'}Z^iA^{(\rho,\sigma)}_{\lambda,\mu,\upsilon}(i,s',a,m)$$
$$\times\,{}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r'}^{m,n+2}\left(z\left|\begin{matrix}(a_1,A_1,x'),(1-s-\upsilon-Bi-n';A),(1-s-\upsilon-Bi-n';A),\\(g_j,G_j)_{1,m},[\tau_i(g_{ji},G_{ji})]_{m+1,q_i;r'},\left(\frac{1}{2}-s-\upsilon-Bi-n';A\right),\end{matrix}\right.\begin{matrix}(a_j,A_j)_{2,n},[\tau_i(a_{ji},A_{ji})]_{n+1,p_i;r'}\\\left(\frac{1}{2}-s-\upsilon-Bi-n';A\right)\end{matrix}\right),\tag{3.8}$$

under the conditions confirmed by Theorem 2.1, and when the conditions in (1.24) and (1.26) are met instead of those in (1.29) and (1.31).

Now, we examine an extension of the Hurwitz-Lerch Zeta function with one variable, as defined by Goyal and Ladha [11, p. 100, Eq. (15)]. Then, we obtain the following result:

Corollary 3.7

$$\int_0^1 x^s e^{ax}\gamma(\upsilon, ax)\ K\left(\sqrt{1-x}\right)\ \phi_\mu^*\left(Zx^B, s', a'\right)\ {}^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r'}^{m,n}\left(zx^A, x'\right)$$
$$= \frac{\pi a^\upsilon}{2}\sum_{i,n'=0}^{\infty}\frac{1}{(\upsilon)_{n'}}(a)^{n'}\,Z^i\,A_\mu^*(i, s', a')$$
$$\times\ {}^{(\Gamma)}\aleph_{p_i+2,q_i+2,\tau_i;r'}^{m,n+2}\left(z\,\middle|\,\begin{matrix}(a_1, A_1, x'),(1-s-\upsilon-Bi-n'; A),\\ \\ (g_j, G_j)_{1,m},[\tau_i(g_{ji}, G_{ji})]_{m+1,q_i;r'},\end{matrix}\right.$$
$$\left.\begin{matrix}(1-s-\upsilon-Bi-n'; A),(a_j, A_j)_{2,n},[\tau_i(a_{ji}, A_{ji})]_{n+1,p_i;r'}\\ \\ \left(\frac{1}{2}-s-\upsilon-Bi-n'; A\right),\left(\frac{1}{2}-s-\upsilon-Bi-n'; A\right)\end{matrix}\right), \tag{3.9}$$

assuming that

$$\Re(s+\upsilon') > 0,\ \Re(s+\upsilon+Bi) - A\min_{1\leqslant j\leqslant m}\Re\left(\frac{g_j}{G_j}\right) > -1,\ A, B > 0,$$

and that for $i = 1, \cdots, r'$, *either* $\Omega_i > 0$ *and* $|\arg(z)| < \frac{\pi}{2}\Omega_i$, *or* $\Omega_i \geq 0$, $|\arg(z)| < \frac{\pi}{2}\Omega_i$ *and* $\Re(\zeta_i)+1 < 0$, *with* Ω_i *and* ζ_i *defined by* (1.9) *and* (1.10), *while* $a' \notin \{0, -1, -2, \dots\}$, $\mu \geq 1$, *and either* $|z| < 1$, $\Re(s) > 0$, *and* $\Re(s) > \mu$.

Remark 3.1 For the three preceding corollaries, we obtain identical formulas involving the incomplete gamma Aleph function ${}^{(\gamma)}\aleph_{p_i,q_i,\tau_i;r'}^{m,n}(z, x')$, the incomplete Gamma I-function ${}^{(\Gamma)}I_{p_i,q_i;r'}^{m,n}(.)$, the incomplete gamma I-function ${}^{(\gamma)}I_{p_i,q_i,r'}^{m,n}(.)$, along with the incomplete H-functions ${}^{(\Gamma)}H_{p,q}^{m,n}(.)$ and ${}^{(\gamma)}H_{p,q}^{m,n}(.)$. These are the same generalized finite integrals that involve the extension of the generalized Hurwitz-Lerch Zeta function of two and one variables, along with the Aleph function [29,30], the I-function defined by Saxena [28], and Fox's H-function [21].

4. Concluding Remarks

The significance of all our results stems from their broad generality. Firstly, by specializing the various parameters and variables in the incomplete Aleph-functions ${}^{(\Gamma)}\aleph_{p_i,q_i,\tau_i;r'}^{m,n}(z)$ and ${}^{(\gamma)}\aleph_{p_i,q_i,\tau_i;r'}^{m,n}(z)$, we derive numerous results that encompass a wide range of useful special functions (or products of such functions), which can be expressed in terms of the I-function defined by Saxena [28], the H-function, Meijer's G-function, the E-function, and the hypergeometric function of one variable. Secondly, by specializing the parameters in the integrals of several variables involving the incomplete gamma function, we can obtain a variety of integrals involving the incomplete Aleph-functions. Thirdly, by adjusting the parameters or variables of the extended Hurwitz-Lerch Zeta function of one or two variables discussed in this paper, we generate a substantial number of new integrals.

5. Compliance with Ethical Standards

Conflict of Interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical approval: This article does not contain any studies with human participants performed by any of the authors.

References

1. T.M. Apostol, Introduction to analytic number theory, *Springer-Verlag*, New York, 1976.
2. F.Y. Ayant and D. Kumar, A unified study of Fourier series involving the Aleph-function and the Kampé de Fériet's function, *Int. J. Math. Trend. Technol.* **35**(1) (2016), 40–48.
3. M.K. Bansal and D. Kumar, On the integral operators pertaining to a family of incomplete I-functions, *AIMS Math.* 5(2) (2020), 1247-1259.
4. M.K. Bansal, D. Kumar, I. Khan, J. Singh and K.S. Nisar, Certain unified integrals associated with product of M-series and incomplete H-functions, *Mathematics* **7**(12):1191 (2019), 1-11.
5. M.K. Bansal, D. Kumar, K.S. Nisar and J. Singh, Certain fractional calculus and integral transform results of incomplete $\aleph$-functions with applications, *Math. Methods Appl. Sci.* **43**(8) (2020), 5602–5614.
6. S. Batra and P. Rai, A further extension of generalized Hurwitz's-Lerch zeta function of two variables, *Int. J. Pure Appl. Math.* **119**(18) (2018), 1551–1556.
7. S. Batra and P. Rai, A further generalization of extended Hurwitz's-Lerch zeta function of two variables, *Adv. Stud. Contemp. Math.* **30**(3) (2020), 391–399.
8. Y.A. Brychkov, Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas (1st ed.), *Chapman and Hall/CRC*, New York, 2008.
9. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, **Vol. I**, *McGraw-Hill Book Inc.*, New York, Toronto and London, 1953.
10. L. Euler, De progressionibus transcendenti bus, *Novi Comment. Acad. Sci. Imp. Petropol.* **5** (1729), 61–95.
11. S.P. Goyal and R.K. Ladha, On the generalization Zeta function and generalized Lambert's function, *Ganita Sandesh* **11** (1997), 99–108.
12. D. Kumar, Generalized fractional differintegral operators of the Aleph-function of two variables, *J. Chem. Biol. Phys. Sci., Sect. C* **6**(3) (2016), 1116–1131.
13. D. Kumar, F.Y. Ayant and M.S. Bajrolia, Improper integrals involving the incomplete Aleph-functions, *Sohag J. Math.* **11**(2), (2024), 23–27.
14. D. Kumar, F.Y. Ayant, P. Nirwan and D.L. Suthar, Boros integral involving the generalized multi-index Mittag-Leffler function and incomplete I–functions, *Res. Math.* **9**(1), (2022), 1–7.
15. D. Kumar, F.Y. Ayant, K.S. Nisar and D.L. Suthar, On fractional q–integral operators involving the basic analogue of multivariable Aleph-function, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.* **93**(2) (2023), 211–218.
16. D. Kumar, F.Y. Ayant and D.S. Sachan, Certain unified infinite integrals involving incomplete special functions, *Int. J. Appl. Comput. Math.* **11**, 173 (2025).
17. D. Kumar, F.Y. Ayant, A. Singh and P.K. Banerji, Finite integral formula involving Aleph-function and generalized Mittag-Leffler function, *Probl. Anal. Issues Anal.* **9(27)**(1) (2020), 96–109.
18. D. Kumar, F.Y. Ayant, D.L. Suthar, P. Nirwan and M. Kumari, Boros integral involving the class of polynomials and incomplete $\aleph$–functions, *Proc. Indian Natl. Sci. Acad.* **91** (2025), 325–332.
19. D. Kumar, F.Y. Ayant and F. Uçar, Integral involving Aleph-function and the generalized incomplete hypergeometric function, *TWMS J. App. and Eng. Math.* **10**(3) (2020), 650–656.
20. D. Kumar, D. Baleanu, F.Y. Ayant and N. Südland, On transformation involving basic analogue to the Aleph-function of two variables, *Fractal Fract.* **6**(2): 71 (2022), 1–8.
21. D. Kumar and J. Daiya, Fractional calculus pertaining to generalized H-functions, *Glob. J. Sci. Front. Res. F. Math. Decis. Sci.* **14**(3) (2014), 25–36.
22. D. Kumar, R.K. Gupta, B.S. Shaktawat and J. Choi, Generalized fractional calculus formulas involving the product of Aleph-function and Srivastava polynomials, *Proc. Jangjeon Math. Soc.* **20**(4) (2017), 701–717.
23. D. Kumar, R.K. Saxena and J. Ram, Finite integral formulas involving Aleph function, *Bol. Soc. Parana. Mat.* **36**(1) (2018), 177–193.
24. D.M. Lee, A.K. Rathie, R.K. Parmar and Y.S. Kim, Generalization of extended Beta function, hypergeometric and confluent hypergeometric functions, *Honam Math. J.* **33** (2011), 187–206.
25. E. Özergin, M.A. Özarslan and A. Altin, Extension of Gamma, Beta and hypergeometric functions, *J. Comput. Appl. Math.* **235** (16) (2011), 4601–4610.
26. R.K. Parmar, J. Choi and S.D. Purohit, Further generalization of the extended Hurwitz's-Lerch Zeta functions, *Bol. Soc. Paran. Mat.* **37**(1) (2019), 177–190.
27. J. Ram and D. Kumar, Generalized fractional integration of the $\aleph$ –function, *J. Rajasthan Acad. Phys. Sci.* **10**(4) (2011), 373–382.
28. V.P. Saxena, The I-function, *Anamaya Publishers*, New Delhi, 2008.

29. R.K. Saxena, J. Ram and D. Kumar, Generalized fractional differentiation for Saigo operators involving Aleph functions, *J. Indian Acad. Math.* **34** (1) (2012), 109–115.

30. R.K. Saxena, J. Ram and D. Kumar, Generalized fractional integral of the product of two Aleph-functions, *Appl. Appl. Math.* **8**(2) (2013), 631–646.

31. L.J. Slater, Generalized hypergeometric functions, *Cambridge University Press*, 1966.

32. H.M. Srivastava, M.A. Chaudhary and R.P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, *Integral Transforms Spec. Funct.* **23** (2012), 659–683.

33. H.M. Srivastava, R.K. Saxena and R.K. Parmar, Some families of the incomplete H-function and the incomplete $\bar{H}$-functions and associated integrals transforms and operators of fractional calculus with applications, *Russ. J. Math. Phys.* **25** (2018), 116–138.

34. D.L. Suthar, S. Agarwal and D. Kumar, Certain integrals involving the product of Gaussian hypergeometric function and Aleph function, *Honam Math. J.* **41** (1) (2019), 1–17.

35. E.T. Whittaker and G.N. Watson, A course of modern analysis, *Cambridge [Eng.], The University Press*, The Macmillan Company, New York, 1943.

D. Kumar,

Department of Applied Sciences, College of Agriculture - Jodhpur,

Agriculture University Jodhpur, Jodhpur–342304 (Raj.), INDIA

E-mail address: dinesh_dino03@yahoo.com

and

F. Y. Ayant,

[1]*Collége Jean L'herminier, Allée des Nymphéas, 83500 La Seyne-sur-Mer, FRANCE*

[2]*Department VAR, Avenue Joseph Raynaud Le parc Fleuri, 83140 Six-Fours-les-Plages, FRANCE*

E-mail address: fredericayant@gmail.com