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Nonlinear Impulsive Hybrid Differential Equations with Linear and Nonlinear Perturbations: A Study Involving the $\psi-$ Caputo Fractional Derivative

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ABSTRACT: In this work, we prove the existence and uniqueness of solutions to a system of impulsive coupled nonlinear hybrid fractional differential equations that involve the ψ-Caputo fractional derivative, with both linear and nonlinear perturbations. The results are established using the nonlinear alternative of Leray-Schauder type and Banach's fixed-point theorem. A practical example is provided to illustrate the applicability of the obtained results.

Key Words: $\psi-$fractional integral, $\psi-$Caputo fractional derivative, Coupled system, Impulsive hybrid fractional differential equations and fixed point theorems.

Contents

1. Introduction

Fractional calculus has garnered significant attention in recent years and continues to be a rapidly expanding field due to its vast applications across various engineering and scientific domains. Its relevance spans disciplines such as signal processing, nonlinear control theory, viscoelasticity, optimization theory, controlled thermonuclear fusion, chemistry, biological systems, mechanics, electrical networks, fluid dynamics, diffusion processes, oscillation, relaxation phenomena, turbulence, stochastic dynamical systems, plasma and polymer physics, chemical physics, astrophysics, and economics. Given its broad scope, fractional calculus warrants a dedicated theoretical framework that runs parallel to the well-established theory of ordinary differential equations(see [15]).

Nonlinear differential equations play a fundamental role in describing complex real-world phenomena across numerous scientific and technological fields. They frequently arise in the study of fluid dynamics, atmospheric motion, and various other physical processes. The study of nonlinear differential equations often necessitates the development of various analytical techniques to establish the existence and fundamental properties of solutions. Despite significant advancements in this area, numerous open problems persist regarding the solvability of nonlinear systems, making it a continuously evolving field of research.

Perturbation methods play a crucial role in nonlinear analysis, particularly for examining dynamical systems modeled by linear differential and integral equations. In many cases, differential equations representing such systems do not admit explicit analytical solutions, making perturbation techniques a valuable approach. Perturbed differential equations are classified into different categories, among which hybrid differential equations hold particular importance. A hybrid differential equation typically refers to a quadratic perturbation of a linear differential equation.

Another significant class of differential equations is the impulsive differential equation, which serves as a powerful mathematical tool for modeling processes characterized by sudden state changes. Many real-world systems exhibit impulsive behavior, including pendulum clocks, mechanical systems experiencing impacts, population dynamics with periodic stocking or harvesting, spacecraft maneuvering through thrust impulses, and even biological processes such as heart function. For an introduction to impulsive

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differential equations, we refer to [16]. Impulsive phenomena commonly arise in various applications, such as disturbances in cellular neural networks, damper operations influenced by percussive effects, valve shutter speed transitions, external impulsive effects on pendulum systems, vibrational percussive systems, relaxation oscillations in electromechanical systems, dynamic systems with automatic regulation, and satellite orbit control via radial acceleration, among others.

In recent years, hybrid fractional differential equations have also garnered considerable attention from researchers. These equations incorporate fractional derivatives in hybrid systems where the nonlinearity depends on the unknown function. Several studies have contributed to this field, with notable results presented in works such as [17]-[18]. Hybrid differential equations form a broad and active area of research, particularly due to their formulation as quadratic perturbations of nonlinear differential equations. Their increasing popularity stems from their wide-ranging applications across multiple disciplines. For further details, we refer to [21], [19] and [20].

In [22], Hannabou, Bouaouid, and Hilal investigated the existence and uniqueness of solutions for the following impulsive coupled system of nonlinear hybrid fractional differential equations:

$$\begin{cases} D^{\alpha}\{u(t)h_1(t,u,v)-p_1(t,u,v)\}=s_1(t), & 0<\alpha<1,\ t\in Q=[0,1],\\ D^{\beta}\{u(t)h_2(t,u,v)-p_2(t,u,v)\}=s_2(t), & 0<\beta<1,\ t\in Q=[0,1],\\ u(t_i^+)=u(t_i^-)+I_i(u(t_i^-)),\\ u(0)h_1(0,u(0),v(0))-p_1(0,u(0),v(0))=\phi(u),\\ v(0)h_2(0,u(0),v(0))-_2(0,u(0),v(0))=\varphi(v), \end{cases} \tag{1.1}$$

where D^{α}, D^{β} denote the Caputo fractional derivatives of order α and β, respectively, $s_i \in C(Q\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $p_i, s_i \in C(Q\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ for $i=1,2$, and $\varphi,\psi : C(J,\mathbb{R})\to\mathbb{R}$ are continuous functions defined by

$$\varphi(u)=\sum_{i=1}^{n}a_iu(x_i), \quad \psi(v)=\sum_{j=1}^{m}b_jv(y_j),$$

where $x_i, y_j \in (0,1)$ for $i=1,2,\dots,n$, and $j=1,2,\dots,m$. Also, $I_k:\mathbb{R}\to\mathbb{R}$ and

$$u(t_k^+)=\lim_{\epsilon\to 0^+}u(t_k+\epsilon), \quad u(t_k^-)=\lim_{\epsilon\to 0^-}u(t_k+\epsilon)$$

represent the right and left limits of $u(t)$ at $t=t_k$, for $k=i,j$.

In this work, we extend and generalize their results by considering the ψ-Caputo fractional derivative, leading to the following more general system:

$$\begin{cases} {}^CD^{\alpha,\psi}\{u(t)f_1(t,u,v)-g_1(t,u,v)\}=h_1(t), & 0<\alpha<1,\ t\in J=[0,T],\\ {}^CD^{\beta,\psi}\{u(t)f_2(t,u,v)-g_2(t,u,v)\}=h_2(t), & 0<\beta<1,\ t\in J=[0,T],\\ u(t_i^+)=u(t_i^-)+I_i(u(t_i^-)),\\ u(0)f_1(0,u(0),v(0))-g_1(0,u(0),v(0))=\phi(u),\\ v(0)f_2(0,u(0),v(0))-g_2(0,u(0),v(0))=\varphi(v), \end{cases} \tag{1.2}$$

with ${}^CD^{\alpha,\psi}$, ${}^CD^{\beta,\psi}$ denote the $\psi-$Caputo fractional derivatives of order α and β, respectively, $f_i \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g_i, h_i \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ for $i=1,2$, and $\phi,\varphi : C(J,\mathbb{R})\to\mathbb{R}$ are continuous functions defined by

$$\phi(u)=\sum_{i=1}^{n}\lambda_iu(\xi_i), \quad \varphi(v)=\sum_{j=1}^{m}\delta_jv(\eta_j),$$

where $\xi_i, \eta_j \in (0,T)$ for $i=1,2,\dots,n$, and $j=1,2,\dots,m$. Also, $I_k:\mathbb{R}\to\mathbb{R}$ and

$$u(t_k^+)=\lim_{h\to 0^+}u(t_k+h), \quad u(t_k^-)=\lim_{h\to 0^-}u(t_k+h)$$

represent the right and left limits of $u(t)$ at $t=t_k$, for $k=i,j$.

This paper is organized as follows: Section 2 provides a review of essential tools related to fractional calculus, along with some necessary results. The main results are presented in Section 3, and finally, Section 4 summarizes the conclusions.

2. Preliminaries

We provide initial context that will be referenced throughout this paper.
Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], ..., J_{n-1} = (t_{n-1}, t_n]$ and $J_n = (t_n, 1]$ for all $n \in \mathbb{N}$ and $n > 1$.
For $t_i \in (0, T)$ such that $t_1 < t_2 < ... < t_n$, we define the following spaces:
$I' = I \setminus \{t_1, t_2, ..., t_n\}$,

$$X = \left\{u \in \mathcal{C}([0,T], \mathbb{R}) : u \in \mathcal{C}(I') \text{ and there exist } u(t_i^+) \text{ and } u(t_i^-),\ i = 1, ..., n \text{ such that } u(t_i^-) = u(t_i)\right\}.$$

with the norm $||u||_X = \sup_{t\in[0,T]} |u(t)|$. Then it is clearly that $(X, ||.||_X)$ is a Banach space.
Similarly for $t_j \in (0, T)$ such that $t_1 < t_2 < ... < t_n$, we define the following spaces:
$J' = J \setminus \{t_1, t_2, ..., t_m\}$,

$$Y = \left\{v \in \mathcal{C}([0,T], \mathbb{R}) : v \in J' \text{ and there exist } v(t_j^+) \text{ and } v(t_j^-),\ j = 1, ..., m \text{ such that } v(t_j^-) = v(t_j)\right\}.$$

with the norm $||v||_Y = \sup_{t\in[0,T]} |v(t)|$. Then it is clearly that $(Y, ||.||_Y)$ is a Banach space.
Consequently, the product $X \times Y$ is a Banach space under the norms $||(u, v)|| = ||u|| + ||v||$ and $||(u, v)|| = \max\{||u||, ||v||\}$.
We now provide a few results and properties from the theory of fractional calculus.

Definition 2.1 *[11](ψ-Riemann-Liouville fractional integral)*
Let $q > 0$, f be an integrable function defined on $[a, b]$ and $\psi : [a, b] \to \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$.
The Ψ-Riemann-Liouville fractional integral operator of order q of a function f is defined by

$$I_a^{q,\psi} f(t) = \frac{1}{\Gamma(q)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{q-1} f(s) ds.$$

Definition 2.2 *[11](ψ-Riemann-Liouville fractional derivative)*
Let $n \in \mathbb{N}$, $f, \psi \in \mathcal{C}^n([a, b])$ be two functions such that ψ is increasing with $\psi'(t) \neq 0$, for all $t \in [a, b]$.
ψ-Riemann-Liouville fractional derivative of order q of a function f is defined by

$$\begin{aligned} D_a^{q,\psi} f(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n (I_a^{n-q,\psi} f(t)) \\ &= \frac{1}{\Gamma(n-q)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-q-1} f(s) ds, \end{aligned}$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3 *[11] Let $n \in \mathbb{N}$, $f, \psi \in \mathcal{C}^n([a, b])$ be two functions such that ψ is increasing with $\psi'(t) \neq 0$, for all $t \in [a, b]$.*
ψ-Caputo fractional derivative of order q of a function f is defined by

$$\begin{aligned} {}^C D_a^{q,\psi} f(t) &= (I_a^{n-q,\psi} f_\psi^{[n]})(t) \\ &= \frac{1}{\Gamma(n-q)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-q-1} f_\psi^{[n]}(s) ds, \end{aligned}$$

where $n = [q] + 1$, for $\alpha \notin \mathbb{N}$. And $f_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n f(t)$ on $[a, b]$.

From the definition, it is clear that when $\alpha = n \in \mathbb{N}$, we have

$${}^C D_a^{q,\psi} f(t) = f_\psi^{[n]}(t).$$

We note that if $f \in \mathcal{C}^n([a, b])$. The ψ-Caputo fractional derivative of order q of f is determined as

$${}^C D_a^{q,\psi} f(t) = D_a^{q,\psi} \left(f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a^+)}{k!} (\psi(t) - \Psi(a))^k \right).$$

Theorem 2.1 *[11]*

$$I_a^{q,\psi}\ {}^C D_a^{q,\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a^+)}{k!}(\psi(t)-\psi(a))^k.$$

In particular, given $q \in (0,1)$ we have:

$$I_a^{q,\psi}\ {}^C D_a^{q,\psi} f(t) = f(t) - f(a).$$

3. Main Results

Lemma 3.1 *Let assume that hypothesis (H_1) and (H_3) hold, let $\alpha \in (0,1)$ and $h : J \to \mathbb{R}$ be continuous. A function u is a solution to the fractional integral equation*

$$u(t) = \frac{1}{f_1(t,u,v)}\left\{\varphi(u) + g_1(t,u,v) + \Delta(t)\sum_{i=1}^{n}[I_i(u(t_i^-))f_1(t_i,u(t_i),v(t_i)) \right.$$
$$\left. + + \frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i} \psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}h_1(s)ds] \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\}, \quad (3.1)$$

where

$$\Delta(t) = \begin{cases} 0, & \text{if } t \in [0,t_1[, \\ 1, & \text{if } t \in [t_i,t_{i+1}|,\ i = 1,...,n, \end{cases}$$

if and only if u is a solution of the following problem:

$$\begin{cases} {}^C D^{\alpha,\psi}\{u(t)f_1(t,u,v) - g_1(t,u,v)\} = h_1(t), \quad t \in J = [0,T], \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), \\ u(0)f_1(0,u(0),v(0)) - g_1(0,u(0),v(0)) = \varphi(u). \end{cases} \quad (3.2)$$

Proof: For $t \in [0,t_1[$, we apply $I^{\alpha,\psi}$ in the first equation of (3.2). We have

$$u(t) = \frac{1}{f_1(t,u,v)}\left\{c_0 + g_1(t,u,v) + I^{\alpha,\psi}h_1(t)\right\}.$$

When $t = 0$, we obtain that $c_0 = \phi(u)$. Then

$$u(t) = \frac{1}{f_1(t,u,v)}\left\{\phi(u) + g_1(t,u,v) + I^{\alpha,\psi}h_1(t)\right\}$$

For $t \in [t_1,t_2[$, we apply again $I^{\alpha,\psi}$ in the first equation of (3.2), we have

$$u(t) = \frac{1}{f_1(t,u,v)}\left\{b_0 + g_1(t,u,v) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\}.$$

We have $u(t_i^+) = u(t_i^-) + I_i(u(t_i^-))$, then

$$u(t_1^+) = \frac{1}{f_1(t_1,u(t_1),v(t_1))}\left\{b_0 + g_1(t,u,v)\right\},$$

and

$$u(t_1^-) = \frac{1}{f_1(t_1,u(t_1),v(t_1))}\left\{\phi(u) + g_1(t_1,u(t_1),v(t_1)) + I^{\alpha,\psi}h_1(t_1)\right\}.$$

So

$$b_0 = \phi(u) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}h_1(s)ds + I_1(u(t_1^-)) \times f_1(t_1,u(t_1),v(t_1)).$$

Therefore

$$u(t)=\frac{1}{f_1(t,u,v)}\left\{\phi(u)+g_1(t,u,v)+I_1(u(t_1^-))\times f_1(t_1,u(t_1),v(t_1))\right.$$
$$\left.+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}h_1(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\},\ t\in[t_1,t_2[.$$

The same thing when $t\in[t_2,t_3[$, we obtain that

$$u(t)=\frac{1}{f_1(t,u,v)}\left\{d_0+g_1(t,u,v)+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\}.$$

Also, we have $u(t_2^+)=u(t_2^-)+I_2(u(t_2^-))$, so

$$d_0=\phi(u)+I_1(u(t_1^-))\times f_1(t_1,u(t_1),v(t_1))+I_2(u(t_2^-))\times f_2(t_2,u(t_2),v(t_2))$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}h_1(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}h_1(s)ds.$$

Then

$$u(t)=\frac{1}{f_1(t,u,v)}\left\{\phi(u)+g_1(t,u,v)+I_1(u(t_1^-))\times f_1(t_1,u(t_1),v(t_1))+I_2(u(t_2^-))\times f_2(t_2,u(t_2),v(t_2))\right.$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}h_1(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}h_1(s)ds$$
$$\left.+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\}.$$

Repeating the process in this way, the solution $u(t)$ for $t\in[t_i,t_{i+1}[$ can be written as:

$$u(t)=\frac{1}{f_1(t,u,v)}\left\{\phi(u)+g_1(t,u,v)+\sum_{i=1}^{n}[I_i(u(t_i^-))\times f_i(t_i,u(t_i),v(t_i))\right.$$
$$\left.++\frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}h_1(s)ds]+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\}.$$

Conversely, assume that u satisfies (3.1). If $t\in[0,t_1[$, then we have

$$\begin{aligned}f_1(t,u,v)u(t)&=\phi(u)+g_1(t,u,v)+I^{\alpha,\psi}h_1(t)\quad(*)\\&\Leftrightarrow f_1(t,u,v)u(t)-g_1(t,u,v)-\phi(u)=I^{\alpha,\psi}h_1(t)\\&\Leftrightarrow f_1(t,u,v)u(t)-g_1(t,u,v)-c_0=I^{\alpha,\psi}h_1(t)\\&\Leftrightarrow I^{\alpha,\psi}\{{}^C\mathcal{D}^{\alpha,\psi}(f_1(t,u,v)u(t)-g_1(t,u,v))\}=I^{\alpha,\psi}h_1(t).\end{aligned}$$

We apply $\mathcal{D}^{\alpha,\psi}$ in the above equation, we have

$${}^C\mathcal{D}^{\alpha,\psi}(f_1(t,u,v)u(t)-g_1(t,u,v))=h_1(t).$$

Substitute $t=0$ in $(*)$, we obtain

$$f_1(0,u(0),v(0))u(0)-g_1(0,u(0),v(0))=\phi(u).$$

If $t\in[t_1,t_2[$, we have

$$u(t)=\frac{1}{f_1(t,u,v)}\left\{\phi(u)+g_1(t,u,v)+I_1(u(t_1^-))\times f_1(t_1,u(t_1),v(t_1))\right.$$
$$\left.+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}h_1(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\}.$$

Then

$$\begin{aligned} u(t)f_1(t,u,v) - b_0 &= I^{\alpha,\psi} h_1(t) \\ \Leftrightarrow I^{\alpha,\psi}\{{}^C\mathcal{D}^{\alpha,\psi}(f_1(t,u,v)u(t) - g_1(t,u,v))\} &= I^{\alpha,\psi} h_1(t). \end{aligned}$$

We apply $\mathcal{D}^{\alpha,\psi}$ in the above equation, we have

$${}^C\mathcal{D}^{\alpha,\psi}(f_1(t,u,v)u(t) - g_1(t,u,v)) = h_1(t).$$

It remains to check that $u(t_1^+) = u(t_1^-) + I_1(u(t_1^-))$. So

$$\begin{aligned} u(t_1^+) - u(t_1^-) &= \frac{1}{f_1(t_1,u(t_1),v(t_1))}\left\{\phi(u) + g_1(t_1,u(t_1),v(t_1)) + I_1(u(t_1^-)) \times f_1(t_1,u(t_1),v(t_1))\right. \\ &\left.+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1}h_1(s)ds\right\} \\ &- \frac{1}{f_1(t_1,u(t_1),v(t_1))}\left\{\phi(u) + g_1(t_1,u(t_1),v(t_1)) + I^{\alpha,\psi}h_1(t_1)\right\} \\ &= I_1(u(t_1^-)). \end{aligned}$$

Similarly, for $t \in [t_i, t_{i+1}[, i = 2, ..., n$. We get

$$\begin{cases} {}^C D^{\alpha,\psi}\{u(t)f_1(t,u,v) - g_1(t,u,v)\} = h_1(t), \quad t \in [t_i, t_{i+1}[, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)). \end{cases}$$

This complete the proof. □

Lemma 3.2 *Let h_1, h_2 are continuous. Then $(u,v) \in X \times Y$ is a solution of (1.2) if an only if (u,v) is a solution of the integral equations*

$$\begin{aligned} u(t) = \frac{1}{f_1(t,u,v)}&\left\{\phi(u) + g_1(t,u,v) + \Delta(t)\sum_{i=1}^{n}[I_i(u(t_i^-)) \times f_i(t_i,u(t_i),v(t_i))\right. \\ &\left.+\frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}h_1(s)ds] + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s)ds\right\} \text{ for } t \in [t_i, t_{i+1}[, \end{aligned}$$

$$\begin{aligned} v(t) = \frac{1}{f_2(t,u,v)}&\left\{\varphi(v) + g_2(t,u,v) + \Sigma(t)\sum_{j=1}^{m}[I_j(u(t_j^-)) \times f_j(t_j,u(t_j),v(t_j))\right. \\ &\left.+\frac{1}{\Gamma(\beta)}\int_{t_{j-1}}^{t_j}\psi'(s)(\psi(t_j)-\psi(s))^{\beta-1}h_2(s)ds] + \frac{1}{\Gamma(\beta)}\int_{t_j}^{t}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}h_2(s)ds\right\}. \end{aligned}$$

Where

$$\Delta(t) = \begin{cases} 0, & \text{if } t \in [0, t_1[, \\ 1, & \text{if } t \in [t_i, t_{i+1}|,\ i = 1, ..., n, \end{cases}$$

and

$$\Sigma(t) = \begin{cases} 0, & \text{if } t \in [0, t_1[, \\ 1, & \text{if } t \in [t_j, t_{j+1}|,\ j = 1, ..., m. \end{cases}$$

We will now present our main result concerning the existence of solutions of problem (1.2). Let us introduce the following hypotheses:
(H_1): The function $u \mapsto uf_1(t,u,v)$ is increasing in $\mathbb{R}$ for every $t \in [0,T]$.
(H_2): The function $v \mapsto vf_2(t,u,v)$ is increasing in $\mathbb{R}$ for every $t \in [0,T]$.

(H_3): (a) The functions f_i and g_i are continuous, bounded and there exist positive numbers $\nu_{fi} > 0$ and $\mu_{gi} > 0$ such that:

$$|f_i(t,u,v)| \geq \nu_{fi} \text{ and } |g_i(t,u,v)| \leq \mu_{gi} \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}\ (i=1,2).$$

(b) There exist positive numbers $M_{fi} > 0$ and $M_{gi} > 0$ such that:

$$|f_i(t,u,v) - f_i(t,\overline{u},\overline{v})| \leq M_{fi}\{||u-\overline{u}|| + ||v-\overline{v}||\}(i=1,2),$$

and

$$|g_i(t,u,v) - g_i(t,\overline{u},\overline{v})| \leq M_{gi}\{||u-\overline{u}|| + ||v-\overline{v}||\}(i=1,2),$$

for all $u, \overline{u}, v, \overline{v} \in \mathbb{R}$ and $t \in [0,T]$.
(H_4): There exist positive numbers $M_{hi} > 0$ such that:

$$|h_i(t,u,v) - h_i(t,\overline{u},\overline{v})| \leq M_{hi}\{||u-\overline{u}|| + ||v-\overline{v}||\}(i=1,2),$$

for all $u, \overline{u}, v, \overline{v} \in \mathbb{R}$ and $t \in [0,T]$.
(H_5): There exist constants $A, B > 0$ such that:

$$|I_i(u) - I_i(\overline{u})| < A|u-\overline{u}|,\ i=1,2,...,n \text{ and for all } u,\overline{u} \in \mathbb{R},$$

and

$$|I_j(v) - I_j(\overline{v})| < B|v-\overline{v}|,\ j=1,2,...,m \text{ and for all } v,\overline{v} \in \mathbb{R}.$$

(H_6): There exist constants $K_\phi > 0$ and $K_\varphi > 0$ such that:

$$|\phi(u) - \phi(v)| \leq K_\phi ||u-v|| \text{ for all } u,v \in \mathcal{C}([0,T],\mathbb{R}),$$

and

$$|\varphi(u) - \varphi(v)| \leq K_\varphi ||u-v|| \text{ for all } u,v \in \mathcal{C}([0,T],\mathbb{R}).$$

(H_7): There exist constants $M_\phi, M_\varphi > 0$ and $N_u, N_v > 0$ such that

$$|\phi(u)| \leq M_\phi ||u|| \text{ and } |\varphi(v)| \leq M_\varphi ||v|| \text{ for all } u,v \in \mathcal{C}([0,T],\mathbb{R}).$$

And

$$|I_i(u)| \leq N_u|u|,\ \ |I_j(v)| \leq N_v|v|,\ i=1,...,n \text{ and } j=1,...,m.$$

(H_8): There exist constants $C, D > 0$ such that:

$$|I_i(u_i)| \leq C,\ \ |I_j(v_j)| \leq D \text{ for all } u,v \in \mathbb{R},\ \ i=1,...,n \text{ and } j=1,...,m.$$

(H_9): There exist constants $\rho, \mu > 0$ such that:

$$|\phi_(u)| < \rho,\ \ |\varphi(v)| < \mu \text{ for all } u \in X \text{ and } v \in Y.$$

(H_{10}): There exist constants $\rho_0, \delta_0 > 0$ and $\rho_i, \delta_i > 0$ (i = 1, 2), such that:

$$|h_1(t,u,v)| < \rho_0 + \rho_1||u|| + \rho_2||v||,$$

and

$$|h_2(t,u,v)| < \delta_0 + \delta_1||u|| + \delta_2||v||,$$

for all $(u,v) \in X \times Y$ and $t \in [0,T]$.
Let us set

$$\pi_1 = \frac{1}{\nu_{f_1}}\left\{K_\phi + M_{g_1} + n(CM_{g_1} + A) + \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}(\lambda_1+\epsilon_1)(1+n)\right\},$$
$$\pi_2 = \frac{1}{\nu_{f_2}}\left\{K_\varphi + M_{g_2} + m(DM_{g_2} + B) + \frac{(\psi(T))^\beta}{\Gamma(\beta+1)}(\lambda_2+\epsilon_2)(1+m)\right\}. \tag{3.3}$$

We define the operator $S : X \times Y \to X \times Y$

$$S(u,v)(t) = (S_1(u,v)(t), S_2(u,v)(t)),$$

where

$$S_1(u,v)(t) = \frac{1}{f_1(t,u,v)}\Bigg\{\phi(u) + g_1(t,u,v) + \Delta(t)\sum_{i=1}^{n} I_i(u(t_i^-))f_1(t_i,u(t_i),v(t_i)) + \frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}h_1(s,u,v)ds + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h_1(s,u,v)ds\Bigg\},$$

and

$$S_2(u,v)(t) = \frac{1}{f_2(t,u,v)}\Bigg\{\varphi(v) + g_2(t,u,v) + \Sigma(t)\sum_{j=1}^{m} I_j(u(t_j^-)) \times f_j(t_j,u(t_j),v(t_j)) + \frac{1}{\Gamma(\beta)}\int_{t_{j-1}}^{t_j}\psi'(s)(\psi(t_j)-\psi(s))^{\beta-1}h_2(s,u,v)ds + \frac{1}{\Gamma(\beta)}\int_{t_j}^{t}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}h_2(s,u,v)ds\Bigg\}.$$

The following theorem on the existence of solutions of problem (1.2) will now be proven

Theorem 3.1 *Suppose that* $(H_1)-(H_7)$ *hold and* $h_1, h_2 : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ *are continuous functions. In addition, there exist positive constants* $\lambda_i, \epsilon_i, i = 1, 2$

$$|h_1(t,u,v) - h_1(t,\overline{u},\overline{v})| \le \lambda_1|u-\overline{u}| + \epsilon_1|v-\overline{v}|,$$
$$|h_2(t,u,v) - h_2(t,\overline{u},\overline{v})| \le \lambda_2|u-\overline{u}| + \epsilon_2|v-\overline{v}|.$$

If $\max(\pi_1,\psi_2) < 1$, π_1, π_2 *given by* (3.3). *Then the impulsive coupled system* (1.2) *has a unique solution.*

Proof: Set $\sup_{t\in J} h_1(t,0,0) = k_1 < \infty$ and $\sup_{t\in J} h_2(t,0,0) = k_2 < \infty$ and let define a closed balls

$$B_r = \{(u,v)\in X\times Y : ||(u,v)|| \le r\},$$

where

$$r \ge \max\left\{\frac{\mu_{g_1} + \frac{k_1(\psi(T))^{\alpha}(1+n)}{\Gamma(\alpha+1)}}{\nu_{f_1} - M_{\varphi} + nN_u + \frac{(\psi(T))^{\alpha}(1+n)(\lambda_1+\epsilon_1)}{\Gamma(\alpha+1)}}, \frac{\mu_{g_2} + \frac{k_2(\psi(T))^{\beta}(1+M)}{\Gamma(\beta+1)}}{\nu_{f_2} - M_{\phi} + mN_v + \frac{(\psi(T))^{\beta}(1+m)(\lambda_2+\epsilon_2)}{\Gamma(\beta+1)}}\right\}.$$

* Show that $SB_r \subset B_r$. For $(u,v)\in B_r$, we obtain

$$\begin{aligned}
||S_1(u,v)(t)|| \le{}& \frac{1}{\nu_{f_1}}\Bigg\{M_{\phi}||u|| + \mu_{g_1} + nN_u||u|| + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}|h_1(s,u,v)-h_1(s,0,0)|ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}|h_1(s,0,0)|ds + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}\\
&\times|h_1(s,u,v)-h_1(s,0,0)|ds + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}|h_1(s,0,0)ds\Bigg\}\\
\le{}& \frac{1}{\nu_{f_1}}\Bigg\{\mu_{g_1} + M_{\phi}||u|| + nN_u||u|| + r\frac{\lambda_1+\epsilon_1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}(\psi(t_i)-\psi(t_{i-1}))^{\alpha} + \frac{k_1}{\Gamma(\alpha+1)}\sum_{i=1}^{n}(\psi(t_i)-\psi(t_{i-1}))^{\alpha}\\
&+ r\frac{\lambda_1+\epsilon_1}{\Gamma(\alpha+1)}(\psi(t)-\psi(t_i))^{\alpha} + \frac{k_1}{\Gamma(\alpha+1)}(\psi(t)-\psi(t_i))^{\alpha}\Bigg\}\\
\le{}& \frac{1}{\nu_{f_1}}\left\{\mu_{g_1} + (M_{\phi}+nN_u)r + \frac{(\psi(T))^{\alpha}}{\Gamma(\alpha+1)}[(\lambda_1+\epsilon_1)nr + k_1n + (\lambda_1+\epsilon_1)r + k_1]\right\}\\
={}& \frac{1}{\nu_{f_1}}\left\{\mu_{g_1} + (M_{\phi}+nN_u)r + \frac{(\psi(T))^{\alpha}}{\Gamma(\alpha+1)}[(\lambda_1+\epsilon_1)(n+1)r + k_1(1+n)]\right\}.
\end{aligned}$$

Then

$$||S_1(u,v)(t)|| \leq \frac{1}{\nu_{f_1}}\left\{\mu_{g_1} + (M_\phi + nN_u)r + \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}[(1+n)(k_1+(\lambda_1+\epsilon_1)r)]\right\},$$

and

$$||S_2(u,v)(t)|| \leq \frac{1}{\nu_{f_2}}\left\{\mu_{g_2} + (M_\varphi + mN_v)r + \frac{(\psi(T))^\beta}{\Gamma(\beta+1)}[(1+m)(k_2+(\lambda_2+\epsilon_2)r)]\right\}.$$

So

$$||S(u,v)|| = \max\{||S_1(u,v)||, ||S_2(u,v)||\} \leq r.$$

It remains to prove that S is a contraction on $X \times Y$.
Let $(u,v), (\overline{u},\overline{v}) \in X \times Y$ and $t \in [0,T]$, we have

$$\begin{aligned}
|S_1(u,v)(t) - S_1(\overline{u},\overline{v})(t)| \leq{}& \frac{1}{\nu_{f_1}}\Bigg\{|\phi(u)-\phi(\overline{u})| + |g_1(t,u,v)-g_1(t,\overline{u},\overline{v})| + \sum_{i=1}^{n}\Big[|I_i(u(t_i^-))f_1(t_i,u(t_i),v(t_i)) \\
&-I_i(\overline{u}(t_i^-))f_1(t_i,\overline{u}(t_i),\overline{v}(t_i))| + \frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\psi'(s)(\psi(t_i)-\psi(s))^{\alpha-1}|h_1(s,u,v)-h_1(s,\overline{u},\overline{v})|ds\Big] \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|h_1(s,u,v)-h_1(s,\overline{u},\overline{v})|ds\Bigg\} \\
&\leq \frac{1}{\nu_{f_1}}\{K_\phi||u-\overline{u}|| + M_{g_1}[||u-\overline{u}|| + ||v-\overline{v}||] + n(CM_{g_1}(||u-\overline{u}|| + ||v-\overline{v}||) + A||u-\overline{u}||) \\
&+\frac{n(\psi(T))^\alpha}{\Gamma(\alpha+1)}(\lambda_1||u-\overline{u}|| + \epsilon_1||v-\overline{v}||) + \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}(\lambda_1||u-\overline{u}|| + \epsilon_1||v-\overline{v}||)\Big\} \\
&\leq \frac{1}{\nu_{f_1}}\left\{K_\phi + M_{g_1} + n(CM_{g_1}+A) + \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}(\lambda_1+\epsilon_1)(1+n)\right\}(||u-\overline{u}|| + ||v-\overline{v}||) \\
&= \pi_1(||u-\overline{u}|| + ||v-\overline{v}||),
\end{aligned}$$

with

$$\pi_1 = \frac{1}{\nu_{f_1}}\left\{K_\phi + M_{g_1} + n(CM_{g_1}+A) + \frac{(\psi(T))^\alpha}{\Gamma(\alpha+1)}(\lambda_1+\epsilon_1)(1+n)\right\}.$$

Similarly, we can also show that

$$|S_2(u,v)(t) - S_2(\overline{u},\overline{v})(t)| \leq \pi_2(||u-\overline{u}|| + ||v-\overline{v}||),$$

with

$$\pi_2 = \frac{1}{\nu_{f_2}}\left\{K_\varphi + M_{g_2} + m(DM_{g_2}+B) + \frac{(\psi(T))^\beta}{\Gamma(\beta+1)}(\lambda_2+\epsilon_2)(1+m)\right\}.$$

Then, we deduce that

$$\begin{aligned}
||S(u,v)(t) - S(\overline{u},\overline{v})(t)|| &\leq \max(\pi_1,\pi_2)(||u-\overline{u}|| + ||v-\overline{v}||) \\
&\leq ||u-\overline{u}|| + ||v-\overline{v}||,
\end{aligned}$$

because of $\max(\pi_1,\pi_2) < 1$.
Therefore, Banach's fixed point theorem applies that the operator S has a unique fixed point. Thus implies that our problem (1.2) has a unique mild solution on J. □

Now, we discuss the existence of solution of (1.2) by means of Leray-Schauder alternative.
Let set

$$\wedge_1 = \frac{(\psi(T))^\alpha}{\nu_{f_1}\Gamma(\alpha+1)}, \quad \wedge_2 = \frac{(\psi(T))^\beta}{\nu_{f_2}\Gamma(\beta+1)}, \tag{3.4}$$

and

$$\wedge_0 = \min\{1-[(1+n)\wedge_1\rho_1 + [(1+m)\wedge_2\delta_1], 1-[(1+n)\wedge_1\rho_2 + [(1+m)\wedge_2\delta_2]\}.$$

Lemma 3.3 *[?](Leray-Schauder alternative) Let $\psi : G \to G$ be a completely continuous operator (i.e. a map that is restricted to any bounded set in G is compact).*
Let $P(\pi) = \{u \in G, u = \lambda\pi u \,\text{for some}\, 0 < \lambda < 1\}$, then either the set $P(\pi)$ is unbounded or π has at least one fixed point.

Theorem 3.2 *Let assume that conditions $(H_1) - (H_3)$ and $(H_8) - (H_{10})$ hold.*
If $\wedge_1\rho_1 + [(1+m)\wedge_2 \delta_1 < 1$ and $\wedge_1\rho_2 + [(1+m)\wedge_2 \delta_2 < 1$, where $\wedge_1$ and $\wedge_2$ are given by (3.4). Then the problem (1.2) has at least one solution.

Proof: We will prove that $S : X \times Y \to X \times Y$ satisfies all assumptions of Lemma (3.3).
Step 1. We prove that S is completely continuous.
It is easy to see that S is a continuous operator. Since $G \subset X \times Y$ be bounded, then we can find positive constants Ω_1 and Ω_2 such that $|h_1(t, u, v)| < \Omega_1$ and $|h_2(t, u, v)| < \Omega_2$ for all $(u, v) \in G$. Then for any $u, v \in G$, we can get

$$||S_1(u,v)|| \leq \frac{1}{\nu_{f_1}}\left\{\rho + \mu_{g_1} + nC + \frac{n\Omega_1}{\Gamma(\alpha+1)}(\psi(T))^{\alpha} + \frac{\Omega_1}{\Gamma(\alpha+1)}(\psi(T))^{\alpha}\right\},$$

and

$$||S_2(u,v)|| \leq \frac{1}{\nu_{f_2}}\left\{\mu + \mu_{g_2} + mD + \frac{m\Omega_1}{\Gamma(\beta+1)}(\psi(T))^{\beta} + \frac{\Omega_2}{\Gamma(\beta+1)}(\psi(T))^{\beta}\right\}.$$

Then the operator S is uniformly bounded.
Step 2. We show that the operator S is equicontinuous.
Let $t_1, t_2 \in [0, T]$, with $t_1 < t_2$.

$$\begin{aligned}
|S_1(u(t_2), v(t_2)) - S_1(u(t_1), v(t_1))| &\leq \frac{1}{\nu_{f_1}}\{|g_1(t_2, u(t_2), v(t_2)) - g_1(t_2, u(t_2), v(t_2))| + (\Delta(t_2) - \Delta(t_1)) \\
&\times \sum_{i=1}^{n}\Big[|I_i(u(t_i^-))||f_1(t_i, u(t_i), v(t_i))| \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\psi'(s)(\psi(t_i) - \psi(s))^{\alpha-1}|h_1(s,u,v)|ds\Big] \\
&+ \frac{1}{\Gamma(\alpha)}\Big|\int_{t_i}^{t_2}\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}|h_1(s,u,v)|ds \\
&- \int_{t_i}^{t_1}\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}|h_1(s,u,v)|ds\Big|\Big\} \\
&\leq \frac{1}{\nu_{f_1}}\{|g_1(t_2, u(t_2), v(t_2)) - g_1(t_2, u(t_2), v(t_2))| + (\Delta(t_2) - \Delta(t_1)) \\
&\times \sum_{i=1}^{n}\left[|I_i(u(t_i^-))||f_1(t_i, u(t_i), v(t_i))|\right] + \frac{n\Omega_1}{\Gamma(\alpha+1)}(\psi(T))^{\alpha} \\
&+ \frac{\Omega_1}{\Gamma(\alpha+1)}[(\psi(t_2) - \psi(t_i))^{\alpha} - (\psi(t_1) - \psi(t_i))^{\alpha}]\Big\} \\
&\longrightarrow 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

Also, we have

$$
\begin{aligned}
||S_2(u(t_2),v(t_2)) - S_2(u(t_1),v(t_1))|| \leq \frac{1}{\nu_{f_2}} & \{|g_2(t_2,u(t_2),v(t_2)) - g_2(t_2,u(t_2),v(t_2))| + (\varepsilon(t_2)-\varepsilon(t_1)) \\
& \times \sum_{j=1}^{m} \left[|I_j(u(t_j^-))||f_2(t_j,u(t_j),v(t_j))|\right] + \frac{m\Omega_2}{\Gamma(\beta+1)}(\psi(T))^{\beta} \\
& + \frac{\Omega_2}{\Gamma(\beta+1)}[(\psi(t_2)-\psi(t_i))^{\beta} - (\psi(t_1)-\psi(t_i))^{\beta}] \Big\} \\
& \longrightarrow 0 \text{ as } t_1 \to t_2.
\end{aligned}
$$

Thus, the operator $S(u,v)$ is completely continuous.
Step 3. We prove that $P = \{(u,v) \in X \times Y, (u,v) = \lambda S(u,v),\ 0 < \lambda < 1\}$ is a bounded set. Let $(u,v) \in P$. Then $u(t) = \lambda S_1(u,v)(t)$ and $v(t) = \lambda S_2(u,v)(t)$ for $0 < \lambda < 1$.
Thus

$$
\begin{aligned}
|u(t)| &= |\lambda S_1(u,v)(t)| = |\lambda||S_1(u,v)(t)| \\
& \leq \frac{\lambda}{\nu_{f_1}} \left\{ \rho + \mu_{g_1} + nC + (1+n)\frac{\rho_0 + \rho_1||u|| + \rho_2||v||}{\Gamma(\alpha+1)}(\psi(T))^{\alpha} \right\},
\end{aligned}
$$

it means that

$$
||u|| \leq \frac{\lambda}{\nu_{f_1}} \{\rho + \mu_{g_1} + nC\} + \wedge_1(1+n)(\rho_0 + \rho_1||u|| + \rho_2||v||).
$$

The same,we have

$$
||v|| \leq \frac{\lambda}{\nu_{f_2}} \{\mu + \mu_{g_2} + mD\} + \wedge_2(1+m)(\delta_0 + \delta_1||u|| + \delta_2||v||).
$$

In consequence, we have

$$
\begin{aligned}
||u|| + ||v|| \leq & \frac{\lambda}{\nu_{f_1}}(\rho + \mu_{g_1} + nC) + \frac{\lambda}{\nu_{f_2}}(\mu + \mu_{g_2} + mD) + \wedge_1(1+n)\rho_0 + \wedge_2(1+m)\delta_0 \\
& + (\wedge_1(1+n)\rho_1 + \wedge_2(1+m)\delta_1)||u|| + (\wedge_1(1+n)\rho_2 + \wedge_2(1+m)\delta_2)||v||.
\end{aligned}
$$

Then

$$
||(u,v)|| \leq \frac{\frac{\lambda}{\nu_{f_1}}(\rho + \mu_{g_1} + nC) + \frac{\lambda}{\nu_{f_2}}(\mu + \mu_{g_2} + mD) + \wedge_1(1+n)\rho_0 + \wedge_2(1+m)\delta_0}{\wedge_0}.
$$

Hence, all conditions of Lemma (3.3) are satisfied and consequently S has at least one fixed point which corresponds for a solution of system (1.2). □

4. Conclusion

This paper explores the existence of solutions for an impulsive coupled system of nonlinear hybrid fractional differential equations governed by the $\psi-$Caputo fractional derivative, incorporating both linear and nonlinear perturbations. The findings presented here extend and refine several known results in the literature. The developed theoretical approach can be further applied to broader classes of impulsive fractional differential equations (IFDEs) with diverse perturbation structures. Additionally, the fixed-point techniques employed in this study may contribute to the existence theory of other IFDEs involving different types of fractional derivatives, including Hilfer and Hadamard derivatives.

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