Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–19.
ISSN-0037-8712
doi:10.5269/bspm.76256

Remediability problem in linear time-varying systems with disturbances

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ABSTRACT: In this work, a class of time-varying linear perturbed systems in finite dimensions are considered. Examining the compensation problem for linear time-varying disturbed systems is the main goal of this paper. The primary objective of remediability or compensation is to reject the effects of disturbances on the system. We present the characterisation results of remediability, and we demonstrate that a specific rank condition provides a sufficient criterion to ensure the remediability of our system. For the analytical case, remediability is described. The relationship between controllability and remediability is also given, and various situations are examined for different examples.

Key Words: Dynamical systems, disturbance, remediability, control, observation.

Contents

1. Introduction

Significant advancements have been made in the control of linear time-invariant systems. On the other hand, a lot of systems in the actual world change with time. These systems are frequently used to describe and approximate dynamical systems seen in the real world, such as autonomous cars and robotics. These systems are much more difficult to control when internal or external disruptions are introduced. In these systems, the capacity to recover or lessen the system's performance loss as a result of these disturbances is known as the remediability problem.

Remediability is a complex and multifaceted issue that arises in various domains, including economics, human resources, and organisational management. And many researchers have studied the concept of the compensation and its application; see, for example, [4,6,9,12,13,14]. The concept of remediability is rooted in the idea that certain problems or issues can be effectively addressed and corrected through appropriate measures or remedies. It is commonly applied in various fields, including environmental science, engineering, medicine, and law.

In control theory, compensation refers to the process of designing and adding control elements to a system to achieve certain performance objectives. The goal of compensation in disturbed linear systems is to mitigate the impact of uncertainties and disturbances in order to enhance the system's performance and stability. Disturbances can be external forces or inputs that affect the system's behaviour, and they can lead to undesired outcomes or deviations from the desired behaviour.

The remediability of parabolic, hyperbolic, and discrete systems can vary, with parabolic systems generally being fully remediable, hyperbolic systems being partially remediable, and discrete systems having varied remediable behaviour based on their specific properties. Regional cases and asymptotic cases involve additional complexities, making remediable behaviour dependent on the specific characteristics of the problem at hand (see [1], [8] and [12]).

Submitted March 23, 2025. Published September 01, 2025

In [11], Silverman and Meadows investigated controllability and observability in time-variable linear systems. Chang [2] given an algebraic characterisation of controllability for a linear time-varying system.
The topic of remediability has not yet been covered in relation to linear time-varying systems in finite dimensions. This work aims to discuss the remediability issue for such systems. We provided necessary and sufficient conditions for the remediability, we investigated the possibility of eliminating the disturbance effect with an appropriate choice of the control operator, and we discussed the remediability for the analytical case. Also, a comparison of controllability and remediability is shown, demonstrating that in finite dimensional space, this concept remains lower than controllability. To demonstrate this in our paper, numerous examples are provided.
We organise this paper as follows: We present the model of disturbed time-varying systems and define the problem statement in Section 2. Controllability and remediability are then determined and described. In Section 3, we present results that characterise the compensation system, offer examples, and illustrate the achieved outcomes. We examine the analytical case in Section 4. In Section 5, we look at how remediability and controllability relate to one another. Finally, Section 6 summarises the conclusions.

2. Statement of the problem

Let's consider the time-variable linear systems expressed by

$$\begin{cases} \dot{z}(s) = A(s)z(s) + B(s)u(s) + K(s)\ ;\ 0 < s < \Theta \\ z(0) = z_0 \end{cases} \tag{2.1}$$

where $A \in C^\infty([0,\Theta], M_n(\mathbb{R}))$, $B \in C^\infty([0,\Theta], M_{n,p}(\mathbb{R}))$, $u \in L^2(0,\Theta;\mathbb{R}^p)$ and $K \in L^2(0,\Theta;\mathbb{R}^n)$.

The corresponding output is given by

$$y(s) = D(s)z(s)\ ;\ \ 0 < s < \Theta, \tag{2.2}$$

with $D \in C^\infty([0,\Theta], M_{q,n}(\mathbb{R}))$, we have

$$z(s) = \Psi(s,0)z_0 + H_s u + G_s K,$$

where Ψ is the resolvent of the time-varying linear system $\dot{x} = A(s)x$. Then

$$y(s) = D(s)\Psi(s,0)z_0 + D(s)H_s u + D(s)G_s K,$$

where

$$\begin{array}{rccc} H_s : & L^2(0,s;\mathbb{R}^p) & \longrightarrow & \mathbb{R}^n \\ & u & \longrightarrow & \displaystyle\int_0^s \Psi(s,r)B(r)u(r)dr, \end{array} \tag{2.3}$$

and

$$\begin{array}{rccc} G_s : & L^2(0,s;\mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \\ & K & \longrightarrow & \displaystyle\int_0^s \Psi(s,r)K(r)dr. \end{array} \tag{2.4}$$

Let us define that

$\mathcal{R}(.)$ indicates the range of a operator.

$\mathcal{N}(.)$ indicates the null space of a operator.

We give the characterization of controllability for linear time-varying system.

Definition 2.1 *The system*

$$\begin{cases} \dot{z}(s) = A(s)z(s) + B(s)u(s)\ ;\ 0 < s < \Theta \\ z(0) = z_0 \end{cases} \tag{2.5}$$

is controllable on $[0, \Theta]$ *if and only if*

$$\mathcal{R}(H_\Theta) = \mathbb{R}^n,$$

or also, the matrix

$$\Delta(\Theta) = \int_0^\Theta \Psi(\Theta, r)B(r)B(r)^*\Psi(\Theta, r)^* dr,$$

is invertible.

We introduce a sequence of maps $(B_i(s))_{0 \leq i \leq n-1}$ in the following way

$$B_0(s) = B(s), \qquad B_i(s) = -A(s)B_{i-1}(s) + \dot{B}_{i-1}(s), \qquad \forall\, i = 1, ..., n-1, \qquad \forall\, s \in [0, \Theta], \tag{2.6}$$

then one has the following result (see, in particular, the paper [11]).

Theorem 2.1 *Assume that, for some* $\bar{s} \in [0, \Theta]$*,*

$$rank \begin{pmatrix} B_0(\bar{s}) & B_1(\bar{s}) & ... & B_{n-1}(\bar{s}) \end{pmatrix} = n,$$

then the system (2.5) is controllable on $[0, \Theta]$*.*

Let us give the following definition.

Definition 2.2 *The system (2.1) with the output function (2.2), is said to be remediable on* $[0, \Theta]$*, if for any* $K \in L^2(0, \Theta; \mathbb{R}^n)$*, there exists a control* $u \in L^2(0, \Theta; \mathbb{R}^p)$ *such that*

$$D(\Theta)H_\Theta u + D(\Theta)G_\Theta K = 0.$$

3. Characterization results

Proposition 3.1 *The subsequent attributes are analogous*

i) *The system (2.1) with observation (2.2) is remediable on* $[0, \Theta]$*.*

ii) $\mathcal{R}(D(\Theta)H_\Theta) = \mathcal{R}(D(\Theta))$.

iii) $\mathcal{N}(H_\Theta^* D(\Theta)^*) = \mathcal{N}(G_\Theta^* D(\Theta)^*)$.

iv) $\mathcal{N}(B(.)^* G_\Theta^* D(\Theta)^*) = \mathcal{N}(G_\Theta^* D(\Theta)^*)$.

v) $\exists \gamma > 0$, $\forall \alpha \in \mathbb{R}^q$,

$$\| \Psi(\Theta, .)^* D(\Theta)^* \alpha\|_{L^2(0,\Theta;\mathbb{R}^n)} \leq \gamma\, \|B(.)^* \Psi(\Theta, .)^* D(\Theta)^* \alpha\|_{L^2(0,\Theta;\mathbb{R}^p)}. \tag{3.1}$$

Proof: Infer from the definition that

$$\mathcal{N}(H_\Theta^* D(\Theta)^*) = \mathcal{N}(B(.)^* \Psi(\Theta, .)^* D(\Theta)^*),$$

$$\mathcal{N}(G_\Theta^* D(\Theta)^*) = \mathcal{N}(\Psi(\Theta, .)^* D(\Theta)^*),$$

and also the theorem 3.3, page 60 [3]. □

We define the remediability Gramian of the system (2.1) with observation (2.2).

Definition 3.1 *The remediability Gramian of the system (2.1) with observation (2.2) is the symmetric* $q \times q$*-matrix*

$$\Gamma(\Theta) = D(\Theta)H_\Theta H_\Theta^* D(\Theta)^* = \int_0^\Theta D(\Theta)\Psi(\Theta, r)B(r)B(r)^*\Psi(\Theta, r)^* D(\Theta)^* dr. \tag{3.2}$$

Remark 3.1 *Note that, for every $\beta \in \mathbb{R}^q$, we have*

$$\beta^*\Gamma(\Theta)\beta = \int_0^\Theta \|B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\beta\|^2 dr.$$

This theorem offers a characterisation of the remediability notion.

Theorem 3.1 *Let $\bar{\Gamma}(\Theta) = \Gamma_{|_{\mathcal{R}(D(\Theta))}}$, (2.1)+ (2.2) is remediable on $[0,\Theta]$ if and only if, the matrix $\bar{\Gamma}(\Theta)$ is invertible in $\mathcal{R}(D(\Theta))$.*

Proof: To prove the theorem, suppose that $\bar{\Gamma}(\Theta)$ is invertible in $\mathcal{R}(D(\Theta))$ and prove that (2.1) with observation (2.2) is remediable on $[0,\Theta]$. Let $u \in L^2(0,\Theta;\mathbb{R}^p)$ is given by:

$$u(r) = B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\bar{\Gamma}(\Theta)^{-1}(-D(\Theta)G_\Theta K), \qquad r \in [0,\Theta].$$

Then

$$\begin{aligned} y(\Theta) &= D(\Theta)\Psi(\Theta,0)z_0 + \\ &\int_0^\Theta D(\Theta)\Psi(\Theta,r)B(r)B(r)^*\Psi(\Theta,r)^*D(\Theta)^*dr\Gamma(\Theta)^{-1}(-D(\Theta)G_\Theta K) + \\ &D(\Theta)G_\Theta K \;=\; D(\Theta)\Psi(\Theta,0)z_0. \end{aligned}$$

Hence (2.1)+ (2.2) is remediable on $[0,\Theta]$.
Conversely, we suppose that $\bar{\Gamma}(\Theta)$ is not invertible in $\mathcal{R}(D(\Theta))$. Then

$$\exists \beta \in \mathcal{R}(D(\Theta))\backslash\{0\}, \qquad \bar{\Gamma}(\Theta)\beta = 0.$$

In particular, $\beta^*\bar{\Gamma}(\Theta)\beta = 0$, that is,

$$\int_0^\Theta \beta^*D(\Theta)\Psi(\Theta,r)B(r)B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\beta dr = 0. \tag{3.3}$$

Then (3.3) is equal to

$$\int_0^\Theta \|B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\beta\|^2 dr.$$

Hence (3.3) implies that

$$\beta^*D(\Theta)\Psi(\Theta,r)B(r) = 0, \qquad r \in [0,\Theta],$$

then

$$B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\beta = 0.$$

Thus, (3.1) is not verified. □

From this demonstration we can conclude the following lemma.

Lemma 3.1 *(2.1)+ (2.2) is not remediable on $[0,\Theta]$ if and only if*

$$\exists \beta \in \mathbb{R}^q\backslash\{0\}, \qquad \beta^*D(\Theta)\Psi(\Theta,r)B(r) = 0, \qquad \forall r \in [0,\Theta].$$

The proposition that follows gives a sufficient condition ensuring the remediability of (2.1)+ (2.2) on $[0,\Theta]$.

Proposition 3.2 *Assume that, for some $\bar{s} \in [0,\Theta]$,*

$$rank\left(\; V(\bar{s}) \;\right) = q,$$

with

$$V(\bar{s}) = \left[\; D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s}) \quad D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s}) \quad \dots \quad D(\Theta)\Psi(\Theta,\bar{s})B_{n-1}(\bar{s}) \;\right],$$

then (2.1)+ (2.2) is remediable on $[0,\Theta]$.

Proof: We suppose that for some $\bar{s} \in [0, \Theta]$

$$rank\left(\ V(\bar{s})\ \right) = q,$$

and (2.1)+ (2.2) is not remediable on $[0, \Theta]$. Hence, by lemma 3.1, there exists $\beta \in \mathbb{R}^q \backslash \{0\}$ such that

$$\beta^* D(\Theta)\Psi(\Theta, r)B(r) = 0, \qquad \forall r \in [0, \Theta].$$

We note that the row vector $\beta^* D(\Theta)\Psi(\Theta, s)$ represents the solution to the differential equation

$$\left\{ \begin{array}{lll} \dot{x}(s) & = & -x(s)A(s)\ ;\ 0 < s < \Theta \\ x(\Theta) & = & \beta^* D(\Theta) \in \mathbb{R}^n \backslash \{0\} \end{array} \right. \tag{3.4}$$

where $x(s)$ is line vector, one gets

$$x(r) = \beta^* D(\Theta)\Psi(\Theta, r), \qquad \forall r \in [0, \Theta],$$

then

$$x(r)B(r) = 0, \qquad \forall r \in [0, \Theta]. \tag{3.5}$$

Differentiating (3.5) and using (3.4) and the definition of the B_i in (2.6), one gets

$$0 = \dot{x}B + x\dot{B} = -xAB + x\dot{B} = xB_1 \tag{3.6}$$

Repeated differentiation of (3.6) yields,

$$x(r)B_i(r) = 0; \qquad i = 0, 1, ..., n-1, \qquad \forall r \in [0, \Theta],$$

one gets

$$\beta^* D(\Theta)\Psi(\Theta, r)B_i(r) = 0; \qquad i = 0, 1, ..., n-1, \qquad \forall r \in [0, \Theta]. \tag{3.7}$$

But for all $r \in [0, \Theta]$, $\beta \neq 0$, so (3.7) contradicts the assumption for some $\bar{s} \in [0, \Theta]$

$$rank\left(\ V(\bar{s})\ \right) = q,$$

this concludes our proof of proposition 3.2. □

The following result gives another sufficient condition.

Proposition 3.3 *We denote*

$$\lambda_n(s) = B(s), \qquad \forall\, s \in [0, \Theta],$$

$$\lambda_i(s) = \frac{d\lambda_{i+1}}{ds} - A(s)\lambda_{i+1}, \qquad i = 1, ..., n-1, \qquad \forall\, s \in [0, \Theta].$$

If $rank\left(\ W(\Theta)\ \right) = q$, *where*

$$W(\Theta) = \left[\ D(\Theta)\lambda_1(\Theta) \quad D(\Theta)\lambda_2(\Theta) \quad \ldots \quad D(\Theta)\lambda_n(\Theta)\ \right],$$

then (2.1)+(2.2) is remediable on $[0, \Theta]$.

Proof: We first assume that

$$rank\left(\ W(\Theta)\ \right) = q,$$

and (2.1)+ (2.2) is not remediable on $[0, \Theta]$. Let $r \in [0, \Theta]$ and let the matrix $B(r)$ be partitioned into columns, i.e.

$$B(r) = \left[b_1(r), b_2(r), ..., b_p(r)\right],$$

then $B(r)^*$ is

$$\begin{bmatrix} b_1(r)^* \\ b_2(r)^* \\ \vdots \\ b_p(r)^* \end{bmatrix}.$$

Hence $B(r)B(r)^*$ can be expressed

$$B(r)B(r)^* = [b_1(r)b_1(r)^* + b_2(r)b_2(r)^* + ... + b_p(r)b_p(r)^*] = \sum_{i=1}^{p} b_i(r)b_i(r)^*. \tag{3.8}$$

Since (2.1)+ (2.2) is not remediable on $[0, \Theta]$, then $\bar{\Gamma}(\Theta)$ is not invertible. Consequently,

$$\exists \beta \in \mathbb{R}^q \backslash \{0\}, \qquad \bar{\Gamma}(\Theta)\beta = 0.$$

One has, $\beta^* \bar{\Gamma}(\Theta)\beta = 0$, using (3.8), one gets

$$\sum_{i=1}^{p} \int_0^{\Theta} \beta^* D(\Theta)\Psi(\Theta, r)b_i(r)b_i(r)^* \Psi(\Theta, r)^* D(\Theta)^* \beta dr = 0. \tag{3.9}$$

But the left hand side of (3.9) is equal to

$$\sum_{i=1}^{p} \int_0^{\Theta} \|b_i(r)^* \Psi(\Theta, r)^* D(\Theta)^* \beta\|^2 dr.$$

Hence (3.9) implies that

$$\beta^* D(\Theta)\Psi(\Theta, r)b_i(r) = 0; \qquad r \in [0, \Theta], \qquad i = 1, ..., p, \tag{3.10}$$

if equation (3.10) is differentiated $n - 1$ times with respect to r, then

$$\beta^* D(\Theta) \frac{d^j}{dr^j} [\Psi(\Theta, r)b_i(r)] = 0; \qquad j = 1, 2, ..., n - 1 \qquad i = 1, ..., p. \tag{3.11}$$

Define $b_i(r) = \lambda_n^i(r)$, and examine the vector

$$\frac{d^j}{dr^j} [\Psi(\Theta, r)b_i(r)] = \frac{d^j}{dr^j} [\Psi(\Theta, r)\lambda_n^i(r)].$$

Indeed, for $j = 1$, we have

$$\frac{d}{dr} [\Psi(\Theta, r)\lambda_n^i(r)] = \frac{d}{dr} [\Psi(\Theta, r)]\lambda_n^i(r) + \Psi(\Theta, r) \frac{d\lambda_n^i}{dr}(r),$$

we obtain

$$-\Psi(\Theta, r)A(r)\lambda_n^i(r) + \Psi(\Theta, r) \frac{d\lambda_n^i}{dr}(r) = \Psi(\Theta, r) \left(\frac{d\lambda_n^i}{dr}(r) - A(r)\lambda_n^i(r) \right).$$

If

$$\lambda_{n-1}^i(r) = \frac{d\lambda_n^i}{dr}(r) - A(r)\lambda_n^i(r),$$

then

$$\begin{aligned} \frac{d^2}{dr^2} [\Psi(\Theta, r)\lambda_n^i(r)] &= \frac{d}{dr} [\Psi(\Theta, r)\lambda_{n-1}^i(r)] \\ &= \Psi(\Theta, r) \left(\frac{d\lambda_{n-1}^i}{dr}(r) - A(r)\lambda_{n-1}^i(r) \right) \\ &= \Psi(\Theta, r)\lambda_{n-2}^i(r). \end{aligned}$$

And more generally,
$$\frac{d^j}{dr^j}[\Psi(\Theta,r)\lambda_n^i(r)] = \Psi(\Theta,r)\lambda_{n-j}^i(r).$$
Equation (3.11) can then be expressed
$$\beta^* D(\Theta)\Psi(\Theta,r)\lambda_{n-j}^i(r) = 0; \qquad j = 0,1,...,n-1, \qquad i = 1,2,...,p. \tag{3.12}$$
Now define a set of matrices from the column vectors $\lambda_{n-j}^i(r)$ as follows
$$\lambda_{n-j}(r) = (\lambda_{n-j}^1(r), \lambda_{n-j}^2(r), ..., \lambda_{n-j}^p(r)); \qquad j = 0,1,...,n-1,$$
then equation (3.12) can be expressed in vector matrix form as
$$\beta^* D(\Theta)\Psi(\Theta,r)(\lambda_1(r), \lambda_2(r), ..., \lambda_n(r)) = 0^*, \qquad \forall r \in [0,\Theta].$$
For $r = \Theta$, we get
$$\beta^* D(\Theta)(\lambda_1(\Theta), \lambda_2(\Theta), ..., \lambda_n(\Theta)) = 0^*, \tag{3.13}$$
so (3.13) contradicts the assumption, which completes the proof. □

We have the following remarks.

Remark 3.2 **i)** *One can has the property for some* $\bar{s} \in [0,\Theta]$
$$rank\left(\; V(\bar{s}) \;\right) = q,$$
without the system (2.5) being controllable on $[0,\Theta]$.

ii) *One can also has*
$$rank\left(\; W(\Theta) \;\right) = q,$$
even if the system (2.5) is not controllable on $[0,\Theta]$.

iii) *(2.1) with (2.2) may be remediable on* $[0,\Theta]$ *without having the condition for some* $\bar{s} \in [0,\Theta]$
$$rank\left(\; V(\bar{s}) \;\right) = q.$$

iv) *(2.1) with (2.2) may be remediable on* $[0,\Theta]$ *without having*
$$rank\left(\; W(\Theta) \;\right) = q.$$

v) *One can has the property for some* $\bar{s} \in [0,\Theta]$
$$rank\left(\; V(\bar{s}) \;\right) = q,$$
without having
$$rank\left(\; W(\Theta) \;\right) = q.$$

In the following example we illustrate these various situations examined.

Example 3.1 *i) Let us define* $A(s)$*, where* $n = 2$*, by*
$$A(s) = \begin{pmatrix} 0 & 0 \\ f(s) & 0 \end{pmatrix},$$
where
$$f(s) = \begin{cases} 0 & if \quad s \le 0 \\ g(s-1) & if \quad 0 \le s \le 2 \\ 0 & if \quad 2 \le s \le 4 \\ g(s-5) & if \quad 4 \le s \le 6 \\ 0 & if \quad 6 \le s \end{cases} \qquad g(s) = \begin{cases} e^{\frac{-1}{1-s^2}} & if \quad |s| \le 1 \\ 0 & if \quad |s| \ge 1. \end{cases} \tag{3.14}$$

Computing the resolvent of $\dot{z}(s) = A(s)z(s)$*, one has*

$$\Psi(\Theta, \bar{s}) = \begin{pmatrix} 1 & 0 \\ F(\bar{s}) & 1 \end{pmatrix},$$

where

$$F(\bar{s}) = \int_{\bar{s}}^{\Theta} f(r)dr.$$

We have $p = 1$*,* $q = 1$*,* $\Theta = 8$ *and*

$$B(s) = \begin{pmatrix} h(s) \\ 0 \end{pmatrix}; \quad D(s) = \begin{pmatrix} 0 & s \end{pmatrix},$$

where

$$h(s) = f(s-2).$$

The controllability matrix is given for any $\bar{s} \in [0, \Theta]$

$$\begin{pmatrix} B_0(\bar{s}) & B_1(\bar{s}) \end{pmatrix} = \begin{pmatrix} h(\bar{s}) & \dot{h}(\bar{s}) \\ 0 & -f(\bar{s})h(\bar{s}) \end{pmatrix} = \begin{pmatrix} h(\bar{s}) & \dot{h}(\bar{s}) \\ 0 & 0 \end{pmatrix},$$

has rank $1 < 2$*. Then, the corresponding system is not controllable on* $[0, \Theta]$*. However, we have*

$$\begin{pmatrix} D(\Theta)\Psi(\Theta, \bar{s})B_0(\bar{s}) & D(\Theta)\Psi(\Theta, \bar{s})B_1(\bar{s}) \end{pmatrix} = \begin{pmatrix} \Theta F(\bar{s})h(\bar{s}) & \Theta F(\bar{s})\dot{h}(\bar{s}) \end{pmatrix},$$

where $\bar{s} = 3$*, its rank is* $1 = q$*, hence (2.1)+ (2.2) is remediable on* $[0, \Theta]$*.*

ii) Let us new consider

$$A(s) = \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix}.$$

We have $p = 1$*,* $q = 1$ *and*

$$B(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad D(s) = \begin{pmatrix} s & 0 \end{pmatrix}.$$

The controllability matrix is given for $\bar{s} \in [0, \Theta]$

$$\begin{pmatrix} \lambda_1(\bar{s}) & \lambda_2(\bar{s}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

has rank is $1 < 2$*. Then, the corresponding system is not controllable on* $[0, \Theta]$*. However*

$$\begin{pmatrix} D(\Theta)\lambda_1(\Theta) & D(\Theta)\lambda_2(\Theta) \end{pmatrix} = \begin{pmatrix} 0 & \Theta \end{pmatrix},$$

has rank $1 = q$*, consequently (2.1)+ (2.2) is remediable on* $[0, \Theta]$*.*

iii) Let us define A*, where* $n = 2$*, by*

$$A(s) = \begin{pmatrix} 0 & 0 \\ f(s) & 0 \end{pmatrix},$$

where $f(s)$ *is defined in (3.14). Computing the resolvent of* $\dot{z}(s) = A(s)z(s)$*, one has*

$$\Psi(\Theta, r) = \begin{pmatrix} 1 & 0 \\ F(r) & 1 \end{pmatrix},$$

where

$$F(r) = \int_{r}^{\Theta} f(t)dt.$$

We have $p=1$, $q=2$ *and*

$$B(s)=\begin{pmatrix}1\\0\end{pmatrix};\quad D(s)=\begin{pmatrix}s&0\\s&0\end{pmatrix};\quad \alpha=\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix},$$

we have

$$\begin{aligned}\Psi(\Theta,r)^*D(\Theta)^*\alpha &= \begin{pmatrix}1&F(r)\\0&1\end{pmatrix}\begin{pmatrix}\Theta&\Theta\\0&0\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\\ &= \begin{pmatrix}\Theta(\alpha_1+\alpha_2)\\0\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\alpha &= \begin{pmatrix}1&0\end{pmatrix}\begin{pmatrix}1&F(r)\\0&1\end{pmatrix}\begin{pmatrix}\Theta&\Theta\\0&0\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\\ &= \Theta(\alpha_1+\alpha_2),\end{aligned}$$

then

$$\|\Psi(\Theta,r)^*D(\Theta)^*\alpha\|^2_{L^2(0,\Theta;\mathbb{R}^2)} = \int_0^\Theta \Theta^2(\alpha_1+\alpha_2)^2dr,$$

and

$$\|B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\alpha\|^2_{L^2(0,\Theta;\mathbb{R})} = \int_0^\Theta \Theta^2(\alpha_1+\alpha_2)^2dr.$$

Hence

$$\|\Psi(\Theta,r)^*D(\Theta)^*\alpha\|_{L^2(0,\Theta;\mathbb{R}^2)} \le \|B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\alpha\|_{L^2(0,\Theta;\mathbb{R})}$$

with $\gamma=1$, *and consequently, (2.1)+ (2.2) is remediable on* $[0,\Theta]$ *even if for any* $\bar{s}\in[0,\Theta]$

$$rank\begin{pmatrix}D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s}) & D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s})\end{pmatrix} = rank\begin{pmatrix}\Theta&0\\\Theta&0\end{pmatrix}=1\neq 2.$$

iv) Let us define A *where* $n=2$ *by*

$$A(s)=\begin{pmatrix}2s&0\\0&2s\end{pmatrix},$$

compute the resolvent of $\dot{z}(s)=A(s)z(s)$. *One has*

$$\Psi(\Theta,r)=\begin{pmatrix}e^{\Theta^2-r^2}&0\\0&e^{\Theta^2-r^2}\end{pmatrix}.$$

We have $p=1$, $q=2$ *and*

$$B(s)=\begin{pmatrix}1\\1\end{pmatrix};\quad D(s)=\begin{pmatrix}s&s\\1&1\end{pmatrix};\quad \alpha=\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix},$$

we obtain

$$\begin{aligned}\Psi(\Theta,r)^*D(\Theta)^*\alpha &= \begin{pmatrix}e^{\Theta^2-r^2}&0\\0&e^{\Theta^2-r^2}\end{pmatrix}\begin{pmatrix}\Theta&1\\\Theta&1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\\ &= \begin{pmatrix}e^{\Theta^2-r^2}(\Theta\alpha_1+\alpha_2)\\e^{\Theta^2-r^2}(\Theta\alpha_1+\alpha_2)\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}B(r)^*\Psi(\Theta,r)^*D(\Theta)^*\alpha &= \begin{pmatrix}1&1\end{pmatrix}\begin{pmatrix}e^{\Theta^2-r^2}&0\\0&e^{\Theta^2-r^2}\end{pmatrix}\begin{pmatrix}\Theta&1\\\Theta&1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\\ &= 2e^{\Theta^2-r^2}(\Theta\alpha_1+\alpha_2).\end{aligned}$$

Then

$$\| \Psi(\Theta,r)^* D(\Theta)^* \alpha\|^2_{L^2(0,\Theta;\mathbb{R}^2)} = 2\int_0^\Theta e^{2(\Theta^2-r^2)}(\Theta\alpha_1+\alpha_2)^2 dr,$$

and

$$\| B(r)^*\Psi(\Theta,r)^* D(\Theta)^* \alpha\|^2_{L^2(0,\Theta;\mathbb{R})} = 4\int_0^\Theta e^{2(\Theta^2-r^2)}(\Theta\alpha_1+\alpha_2)^2 dr.$$

Hence

$$\| \Psi(\Theta,r)^* D(\Theta)^* \alpha\|_{L^2(0,\Theta;\mathbb{R}^2)} \leq \sqrt{2}\,\| B(r)^*\Psi(\Theta,r)^* D(\Theta)^* \alpha\|_{L^2(0,\Theta;\mathbb{R})}$$

and consequently, (2.1)+ (2.2) is remediable on $[0,\Theta]$*, even if*

$$rank \begin{pmatrix} D(\Theta)\lambda_1(\Theta) & D(\Theta)\lambda_2(\Theta) \end{pmatrix} = rank \begin{pmatrix} -4\Theta^2 & 2\Theta \\ -4\Theta & 2 \end{pmatrix} = 1 \neq 2.$$

v) We define A by

$$A(s) = \begin{pmatrix} 0 & 0 \\ f(s) & 0 \end{pmatrix},$$

where $f(s)$ *is defined in (3.14).*
Computing the resolvent of $\dot{z}(s) = A(s)z(s)$*, one has*

$$\Psi(\Theta,\bar{s}) = \begin{pmatrix} 1 & 0 \\ \displaystyle\int_{\bar{s}}^\Theta f(r)dr & 1 \end{pmatrix}.$$

We consider $p = 1$*,* $q = 1$*,* $\Theta = 7$ *and*

$$B(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad D(s) = \begin{pmatrix} 0 & h(s) \end{pmatrix},$$

where

$$h(s) = f(s-2),$$

hence

$$\begin{aligned} rank \begin{pmatrix} D(\Theta)\lambda_1(\Theta) & D(\Theta)\lambda_2(\Theta) \end{pmatrix} &= rank \begin{pmatrix} -h(\Theta)f(\Theta) & 0 \end{pmatrix} \\ &= rank \begin{pmatrix} 0 & 0 \end{pmatrix}, \end{aligned}$$

and its rank is 0*. However*

$$\begin{aligned} rank \begin{pmatrix} D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s}) & D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s}) \end{pmatrix} &= rank \begin{pmatrix} h(\Theta)F(\bar{s}) & -h(\Theta)f(\bar{s}) \end{pmatrix} \\ &= rank \begin{pmatrix} h(\Theta)F(\bar{s}) & 0 \end{pmatrix}. \end{aligned}$$

For $\bar{s} = 3$ *its rank is* $1 = q$*. Therefore (2.1)+ (2.2) is remediable on* $[0,\Theta]$*.* ■

4. Analytical case

If we assume that $A(s), B(s)$ and $D(s)$ are analytic, and since derivatives, sums, and products of analytic functions are analytic, the matrix $B_i(s)$ is composed of analytic elements only, the matrix $\Psi(\Theta,s)$ has analytic elements. Apparently, then, the product matrix $D(\Theta)\Psi(\Theta,s)B_i(s)$ also has only analytic elements. Now, for a constant vector β the elements of the vector $\beta^* D(\Theta)\Psi(\Theta,s)B_i(s)$ are analytic. We have :

i) If they are zero for an interval of time of positive length, they are zero for all time $0 \leq s < +\infty$.

ii) If $rank\,(J(\bar{s})) = q$ for some $\bar{s} \in [0, \Theta]$, with

$$J(\bar{s}) = [D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s})\ D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s})\ \ldots\ D(\Theta)\Psi(\Theta,\bar{s})B_i(\bar{s})]\,,$$

then the functions are independent on the interval. Conversely, if they are linearly independent over a positive interval, they are linearly independent for all time $0 \leq s < +\infty$.

The subsequent theorem provides a necessary and sufficient rank condition for the remediability.

Theorem 4.1 *Suppose $A(s), B(s)$ and $D(s)$ are analytic, (2.1)+ (2.2) is remediable on $[0, \Theta]$ if and only if for any $s \in [0, \Theta]$, we have for some $i \in \mathbb{N}$*

$$rank\,(J(s)) = rank\,(D(\Theta))\,,$$

where

$$J(s) = [D(\Theta)\Psi(\Theta,s)B_0(s)\ D(\Theta)\Psi(\Theta,s)B_1(s)\ \ldots\ D(\Theta)\Psi(\Theta,s)B_i(s)]\,.$$

Proof: First, we assume that for some $\bar{s} \in [0, \Theta]$ and for any $i \in \mathbb{N}$,

$$rank\,(J(\bar{s})) \neq rank\,(D(\Theta))\,,$$

and prove that (2.1)+ (2.2) is not remediable on $[0, \Theta]$. We have for some $\bar{s} \in [0, \Theta]$ and for any $i \in \mathbb{N}$,

$$rank\,(J(\bar{s})) \neq rank\,(D(\Theta))\,,$$

then there exists $\beta \in \mathbb{R}^q$ such that for any $i \in \mathbb{N}$

$$\beta \in \quad \mathcal{N}\begin{pmatrix} (D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s}))^* \\ (D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s}))^* \\ \vdots \\ (D(\Theta)\Psi(\Theta,\bar{s})B_i(\bar{s}))^* \end{pmatrix} \setminus \mathcal{N}\,(D(\Theta)^*)\,,$$

i.e

$$\begin{pmatrix} (D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s}))^* \\ (D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s}))^* \\ \vdots \\ (D(\Theta)\Psi(\Theta,\bar{s})B_i(\bar{s}))^* \end{pmatrix} \beta = 0, \qquad \forall i \in \mathbb{N},$$

and

$$D(\Theta)^*\beta \neq 0.$$

We have

$$\frac{d^i(\beta^* D(\Theta)\Psi(\Theta,s)B(s))}{ds^i}(\bar{s}) = \beta^* D(\Theta)\Psi(\Theta,\bar{s})B_i(\bar{s}) = 0, \qquad i = 0, 1, \ldots$$

Since $A(s), B(s)$ and $D(s)$ are analytic, therefore $\beta^* D(\Theta)\Psi(\Theta,s)B(s)$ is also analytic. One gets

$$B(s)^*\Psi(\Theta,s)^*D(\Theta)^*\beta = 0, \qquad \forall s \in [0, \Theta],$$

one has $\beta \in \mathcal{N}(H_\Theta^* D(\Theta)^*)$ but $\beta \notin \mathcal{N}(D(\Theta)^*)$, then $\mathcal{N}(H_\Theta^* D(\Theta)^*) \neq \mathcal{N}\,(D(\Theta)^*)\,,$ from which we get (2.1)+ (2.2) is not remediable on $[0, \Theta]$.
Conversely, we assume that for any $s \in [0, \Theta]$ and for some $i \in \mathbb{N}$

$$rank\,(J(s)) = rank\,(D(\Theta))\,.$$

By duality

$$\mathcal{N}\begin{pmatrix} (D(\Theta)\Psi(\Theta,s)B_0(s))^* \\ (D(\Theta)\Psi(\Theta,s)B_1(s))^* \\ \vdots \\ (D(\Theta)\Psi(\Theta,s)B_i(s))^* \end{pmatrix} = \mathcal{N}(D(\Theta)^*).$$

Let $\beta \in \mathcal{N}(H_\Theta^* D(\Theta)^*)$, then

$$\begin{pmatrix} (D(\Theta)\Psi(\Theta,s)B_0(s))^* \\ (D(\Theta)\Psi(\Theta,s)B_1(s))^* \\ \vdots \\ (D(\Theta)\Psi(\Theta,s)B_i(s))^* \end{pmatrix} \beta = 0.$$

Hence

$$\mathcal{N}\left[G_\Theta^* D(\Theta)^*\right] = \mathcal{N}\left(D(\Theta)^*\right),$$

therefore (2.1) with (2.2) is remediable on $[0,\Theta]$. Which completes the proof. □

The next theorem gives a characterization result.

Theorem 4.2 *Suppose $A(s), B(s)$ and $D(s)$ are analytic, (2.1) with (2.2) is remediable on $[0,\Theta]$ if and only if for some $\bar{s} \in [0,\Theta]$, we have*

$$rank\left(V(\bar{s})\right) = rank\left(D(\Theta)\right).$$

Proof: The proof of theorem 4.2 a consequence of the following lemma [2].

Lemma 4.1 *Suppose $A(s)$ and $B(s)$ are analytic, let*

$$Q_j(s) = [B_0(s), B_1(s), ..., B_j(s)], \qquad j = 0, 1, ...$$

Then there exists $i \leq n-1$, and non-empty open set $O \subset [0,M]$, where $M > 0$, such that for each $s \in O$,

$$rank\left(Q_i(s)\right) = rank\left(Q_{i+j}(s)\right), \qquad j = 1, 2, ...$$

We first assume that for any $s \in [0,\Theta]$,

$$rank\left(V(s)\right) \neq rank\left(D(\Theta)\right),$$

with

$$V(s) = \left[D(\Theta)\Psi(\Theta,s)B_0(s)\ D(\Theta)\Psi(\Theta,s)B_1(s)\ \ldots\ D(\Theta)\Psi(\Theta,s)B_{n-1}(s)\right],$$

and prove that (2.1)+ (2.2) is not remediable on $[0,\Theta]$.

For any $s \in [0,\Theta]$, we have

$$rank\left(V(s)\right) \neq rank\left(D(\Theta)\right),$$

then there exists $\beta_s \in \mathbb{R}^q$ such that

$$\beta_s \in \quad \mathcal{N}\begin{pmatrix} (D(\Theta)\Psi(\Theta,s)B_0(s))^* \\ (D(\Theta)\Psi(\Theta,s)B_1(s))^* \\ \vdots \\ (D(\Theta)\Psi(\Theta,s)B_{n-1}(s))^* \end{pmatrix} \setminus \mathcal{N}\left(D(\Theta)^*\right),$$

i.e

$$\begin{pmatrix} (D(\Theta)\Psi(\Theta,s)B_0(s))^* \\ (D(\Theta)\Psi(\Theta,s)B_1(s))^* \\ \vdots \\ (D(\Theta)\Psi(\Theta,s)B_{n-1}(s))^* \end{pmatrix} \beta_s = 0, \qquad \forall s \in [0,\Theta],$$

and

$$D(\Theta)^*\beta_s \neq 0.$$

Lemma 4.1 implies that the columns of $B_j(s)$ for all $j > n-1$ are, for every $s \in O$, expressible as linear combinations of the columns of $[B_0(s), B_1(s), ..., B_{n-1}(s)]$.
Let O be the set in Lemma 4.1 and choose $s_1 \in O$. We have

$$\frac{d^j(\beta^* D(\Theta)\Psi(\Theta,s)B(s))}{ds^j}(s_1) = \beta^* D(\Theta)\Psi(\Theta,s_1)B_j(s_1) = 0, \qquad j = 0,1,...$$

Since $A(s), B(s)$ and $D(s)$ are analytic, then $\beta^* D(\Theta)\Psi(\Theta,s)B(s)$ is also analytic. One gets

$$B(s)^*\Psi(\Theta,s)^*D(\Theta)^*\beta = 0,$$

one has $\beta \in \mathcal{N}(H_\Theta^* D(\Theta)^*)$ but $\beta \notin \mathcal{N}(D(\Theta)^*)$, then $\mathcal{N}(H_\Theta^* D(\Theta)^*) \neq \mathcal{N}\left(D(\Theta)^*\right)$, from which we get (2.1)+ (2.2) is not remediable on $[0,\Theta]$.
In order to prove the converse it suffices to assume that

$$rank\left(V(\bar{s})\right) = rank\left(D(\Theta)\right).$$

Let $\beta \in \mathcal{N}(H_\Theta^* D(\Theta)^*)$, then

$$\begin{pmatrix} (D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s}))^* \\ (D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s}))^* \\ \vdots \\ (D(\Theta)\Psi(\Theta,\bar{s})B_{n-1}(\bar{s}))^* \end{pmatrix}\beta = 0.$$

Since

$$rank\left(V(\bar{s})\right) = rank\left(D(\Theta)\right),$$

and

$$\mathcal{N}\left[(G_\Theta)^* D(\Theta)^*\right] = \mathcal{N}\left(D(\Theta)^*\right).$$

Hence

$$\mathcal{N}(H_\Theta^* D(\Theta)^*) \subset \mathcal{N}(G_\Theta^* D(\Theta)^*),$$

consequently (2.1) with (2.2) is remediable on $[0,\Theta]$. Which completes the proof. □

Remark 4.1 *Suppose $A(s), B(s)$ and $D(s)$ are analytic, (2.1) with (2.2) is remediable on $[0,\Theta]$ if and only if for any $s \in [0,\Theta]$*

$$rank\left(V(s)\right) = rank\left(D(\Theta)\right),$$

except in finite number of points in $[0,\Theta]$.

Corollary 4.1 *If*

$$rank\begin{pmatrix} D(\Theta)B_0(\Theta) & D(\Theta)B_1(\Theta) & \dots & D(\Theta)B_{n-1}(\Theta) \end{pmatrix} = rank\begin{pmatrix} D(\Theta) \end{pmatrix},$$

then (2.1) with (2.2) is remediable on $[0,\Theta]$.

Remark 4.2 *We can have the property for some $\bar{s} \in [0,\Theta]$*

$$rank\left(V(\bar{s})\right) = rank\left(D(\Theta)\right),$$

without having

$$rank\begin{pmatrix} D(\Theta)B_0(\Theta) & D(\Theta)B_1(\Theta) & \dots & D(\Theta)B_{n-1}(\Theta) \end{pmatrix} = rank\begin{pmatrix} D(\Theta) \end{pmatrix}.$$

Example 4.1 *We define A by*

$$A(s) = \begin{pmatrix} 0 & 0 \\ f(s) & 0 \end{pmatrix},$$

where

$$f(s)=\begin{cases}0 & if \quad s\leq 4\\ g(s-5) & if \quad 4\leq s\leq 6\\ 0 & if \quad 6\leq s\end{cases}\qquad g(s)=\begin{cases}e^{\frac{-1}{1-s^2}} & if \quad |s|\leq 1\\ 0 & if \quad |s|\geq 1.\end{cases}$$

Compute the resolvent of $\dot{z}(s)=A(s)z(s)$. *One has*

$$\Psi(\Theta,\bar{s})=\begin{pmatrix}1 & 0\\ \int_{\bar{s}}^{\Theta} f(r)dr & 1\end{pmatrix}.$$

We have $p=1$, $q=1$, $\Theta=7$ *and*

$$B(s)=\begin{pmatrix}1\\0\end{pmatrix};\quad D(s)=\begin{pmatrix}0 & h(s)\end{pmatrix},$$

where

$$h(s)=f(s-2),$$

hence

$$\begin{aligned} rank\begin{pmatrix}D(\Theta)B_0(\Theta) & D(\Theta)B_1(\Theta)\end{pmatrix} &= rank\begin{pmatrix}0 & -h(\Theta)f(\Theta)\end{pmatrix}\\ &= rank\begin{pmatrix}0 & 0\end{pmatrix}=0\neq 1=rank\begin{pmatrix}D(\Theta)\end{pmatrix}.\end{aligned}$$

However

$$\begin{aligned} rank\begin{pmatrix}D(\Theta)\Psi(\Theta,\bar{s})B_0(\bar{s}) & D(\Theta)\Psi(\Theta,\bar{s})B_1(\bar{s})\end{pmatrix} &= rank\begin{pmatrix}h(\Theta)F(\bar{s}) & -h(\Theta)f(\bar{s})\end{pmatrix}\\ &= rank\begin{pmatrix}h(\Theta)F(\bar{s}) & 0\end{pmatrix}.\end{aligned}$$

For $\bar{s}=5$, *its rank is* $1=rank\begin{pmatrix}D(\Theta)\end{pmatrix}$, *therefore (2.1)+ (2.2) is remediable on* $[0,\Theta]$. ■

5. Remediability and controllability

We propose in this section the involvements between the notions of controllability and remediability, we have the following proposition.

Proposition 5.1 *i) If the system (2.5) is controllable on* $[0,\Theta]$, *then (2.1)+ (2.2) is remediable on* $[0,\Theta]$.

ii) The reciproque is false.

Proof: We suppose that the system (2.5) is controllable on $[0,\Theta]\Longleftrightarrow\mathcal{R}(H_\Theta)=\mathbb{R}^n$, one has

$$\mathcal{R}\left(D(\Theta)H_\Theta\right)=\mathcal{R}(D(\Theta)),$$

then (2.1)+ (2.2) is remediable on $[0,\Theta]$.

Counter-example: In this case we consider the matrix A is defined by

$$A(s)=\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix}\in\mathcal{M}_2(\mathbb{R}).$$

One has

$$\Psi(\Theta,r)=\begin{pmatrix}e^{\frac{\Theta^2-r^2}{2}} & 0\\ 0 & e^{\frac{\Theta^2-r^2}{2}}\end{pmatrix}.$$

We consider $p = q = 1$ and

$$B(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad D(s) = \begin{pmatrix} s & 0 \end{pmatrix},$$

we have

$$\begin{aligned} \Psi(\Theta, r)^* D(\Theta)^* \alpha &= \begin{pmatrix} e^{\frac{\Theta^2 - r^2}{2}} & 0 \\ 0 & e^{\frac{\Theta^2 - r^2}{2}} \end{pmatrix} \begin{pmatrix} \Theta \\ 0 \end{pmatrix} \alpha \\ &= \begin{pmatrix} \Theta e^{\frac{\Theta^2 - r^2}{2}} \alpha \\ 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} B(r)^* \Psi(\Theta, r)^* D(\Theta)^* \alpha &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{\Theta^2 - r^2}{2}} & 0 \\ 0 & e^{\frac{\Theta^2 - r^2}{2}} \end{pmatrix} \begin{pmatrix} \Theta \\ 0 \end{pmatrix} \alpha \\ &= \Theta e^{\frac{\Theta^2 - r^2}{2}} \alpha, \end{aligned}$$

then

$$\| \Psi(\Theta, .)^* D(\Theta)^* \alpha \|_{L^2(0,\Theta;\mathbb{R}^n)} \leq \gamma \| B(.)^* \Psi(\Theta, .)^* D(\Theta)^* \alpha \|_{L^2(0,\Theta;\mathbb{R}^p)}.$$

The inequality (3.1) is then true for $\gamma = 1$, therefore (2.1)+ (2.2) is remediable on $[0, \Theta]$. We have for any $\bar{s} \in [0, \Theta]$

$$rank \begin{pmatrix} B_0(\bar{s}) & B_1(\bar{s}) \end{pmatrix} = rank \begin{pmatrix} 1 & -\bar{s} \\ 0 & 0 \end{pmatrix} = 1 < 2.$$

Then (2.5) is not controllable on $[0, \Theta]$. □

Remark 5.1 *In the case (2.5) is controllable on $[0, \Theta]$, let us define $\bar{u} \in L^2(0, \Theta; \mathbb{R}^p)$ in the following way:*

$$\bar{u}(r) = B(r)^* \Psi(\Theta, r)^* \Delta(\Theta)^{-1} (-G_\Theta K); \qquad r \in [0, \Theta], \tag{5.1}$$

and let $\bar{z} \in C^0(0, \Theta; \mathbb{R}^n)$ such that

$$\begin{cases} \dot{\bar{z}}(s) = A(s)\bar{z}(s) + B(s)\bar{u}(s) + K(s) \; ; \; 0 < s < \Theta \\ \bar{z}(0) = z_0, \end{cases} \tag{5.2}$$

the system (5.2) is augmented by the output equation

$$\bar{y}(s) = D(s)\bar{z}(s) \; ; \; \; 0 < s < \Theta. \tag{5.3}$$

Then

$$\begin{aligned} \bar{z}(\Theta) &= \Psi(\Theta, 0) z_0 + \int_0^\Theta \Psi(\Theta, r) B(r) B(r)^* \Psi(\Theta, r)^* \Delta(\Theta)^{-1} (-G_\Theta K) dr + G_\Theta K \\ &= \Psi(\Theta, 0) z_0. \end{aligned}$$

One has

$$\bar{z}(\Theta) - \Psi(\Theta, 0) z_0 = 0. \tag{5.4}$$

However, we have

$$\bar{z}(\Theta) = \Psi(\Theta, 0)z_0 + H_\Theta \bar{u} + G_\Theta K.$$

Consequently

$$D(\Theta)H_\Theta \bar{u} + D(\Theta)G_\Theta K = 0,$$

then (2.1)+ (2.2) is remediable on $[0, \Theta]$.

Numerical example

Let us take $n = 2$ with

$$A(s) = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix},$$

compute the resolvent of $\dot{z}(s) = A(s)z(s)$.
One has

$$\Psi(\Theta, s) = \begin{pmatrix} 1 & 0 \\ \frac{\Theta^2 - s^2}{2} & 1 \end{pmatrix}.$$

We have $p = 1$, $q = 2$ and

$$B(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad D(s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix},$$

with the following disturbance

$$K(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using remark 5.1, one gets where $\Theta = 10$

$$u(s) = 15\Theta^{-2} - \frac{45}{2}\Theta^{-4}(\Theta^2 - s^2).$$

The initial state is considered null $z_0 = 0$, then $y_{(0,0)} = 0$. Then

$$y_{(u,K)}(s) = \begin{pmatrix} 15\Theta^{-1}s - 15\Theta^{-3}s^3 \\ \frac{15}{2}\left(\Theta s - \frac{\Theta^{-1}s^3}{3}\right) - 6\Theta^{-3}s^5 + s^2 \end{pmatrix},$$

$$y_{(0,K)}(s) = \begin{pmatrix} 0 \\ s^2 \end{pmatrix}.$$

We acquire the subsequent numerical results that exemplify the preceding developments. In figure 1 and 2, we provide the depiction of the observations $y_{(0,K)}$ and $y_{(u,K)}$.

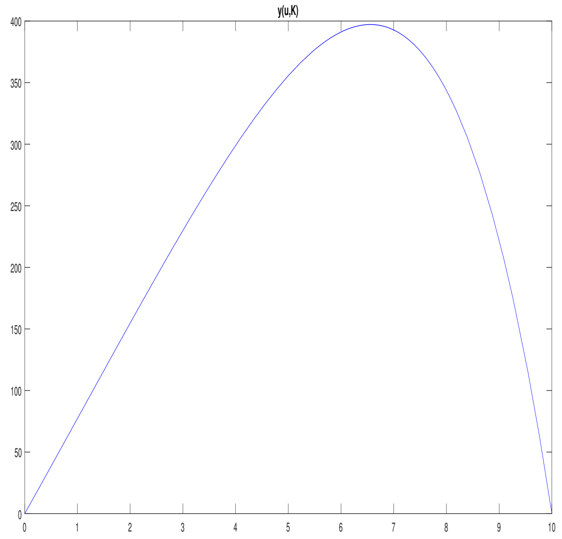

Figure 1 : Representation of $y_{(u,K)}$.

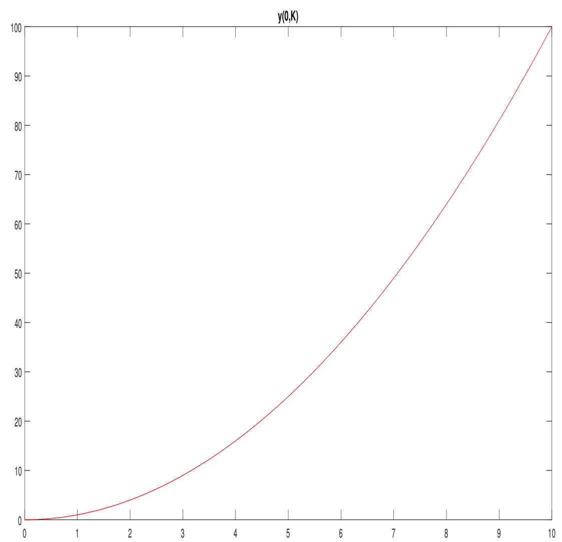

Figure 2 : Representation of $y_{(0,K)}$.

And in figure 3, we give the representation of the optimal control u.

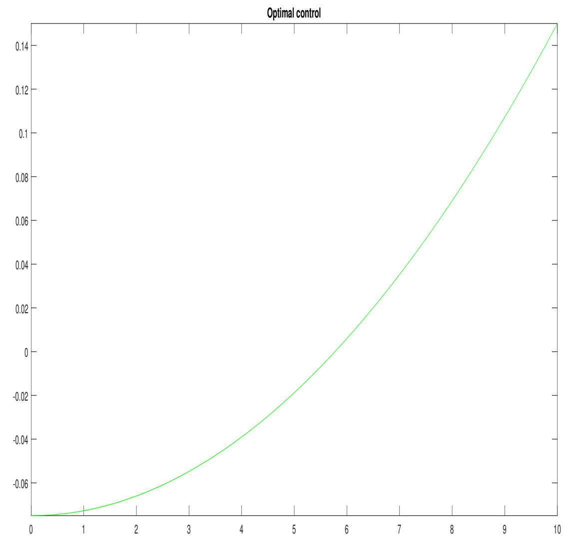

Figure 3 : Representation of control u.

6. Conclusion

Remediability in linear perturbed systems is a critical aspect of control theory. It involves designing control strategies that can effectively reject disturbances. We give necessary and sufficient conditions for the compensation, and we investigate the possibility of eliminating the disturbance effect with an appropriate choice of the control operator. We present a variety of situations and examples. In the analytic case, remediability is studied. The paper also provides a comparison between controllability and remediability. To demonstrate this in our paper, numerous examples are provided. This paper is an extension of previous works on the remediability concept; however, more general situations require further developments, such as the cases of nonlinear and stochastic systems.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

Funding: No funds, grants, or other support was received.

Conflict of interest: It is declared that none of the authors have any competing interests in this manuscript.

References

1. Amissi C., Magri E. M., Lhous M., and Afifi L., Compensation Problem in Linear Fractional Order Disturbed Systems. Mathematical Modelling and Analysis. Volume 29, Issue 3, 546-559, 2024. https://doi.org/10.3846/mma.2024.18927, (2024).

2. Chang A., An algebraic characterization of controllability. IEEE Trans. Automatic Control, AC-10, pp. 112-113, (1965).

3. Curtain R.F., and Pritchard A.J., Infinite Dimensional Linear Systems Theory. Lecture Notes in Control and Information Sciences, vol. 8, Berlin, (1978).

4. Fenga, H. , Wu, X.H., and Guo, B.Z., Dynamics Compensation in Observation of Abstract Linear Systems. arXiv:2009.01643. https://doi.org/10.48550/arXiv.2009.01643. 3 Sep (2020).

5. Hizazi H., Amissi C., Lhous M., and Magri E. M., Domination in linear fractional-order distributed systems. MATHEMATICAL MODELING AND COMPUTING, Vol. 11, No. 2, pp. 455-462, (2024).

6. Huang, Y., and Xue, W., Active disturbance rejection control: Methodology and theoretical analysis, ISA transactions, 53:4, 963–976, (2014).

7. Larrache A., Lhous M., Ben Rhila S., Rachik M., and Tridane A., An output sensitivity problem for a class of linear distributed systems with uncertain initial state. Archives of Control Sciences, 30(1): 77-93, (2020).

8. Magri E.M., Amissi C., Afifi L., and Lhous M., On the minimum energy compensation for linear time-varying disturbed systems. Archives of Control Sciences, Volume 32(LXVIII), No. 4, pages 1-22, 2022. 10.24425/acs.2022.143669, (2022).

9. Mei, Q., She, J., Liu, Z. et al. Estimation and compensation of periodic disturbance using internal-model-based equivalent-input-disturbance approach. Sci. China Inf. Sci. 65, 182205, https://doi.org/10.1007/s11432-020-3192-5, (2022).

10. Rachik M., and Lhous M., An Observer-based control of linear systems with uncertain parameters, Archives of Control Sciences, Volume 26(LXII), No. 4, pp. 565-576, (2016).

11. Silverman L. M., and Meadows H. E., *Controllability and observability in time-variable linear systems.* SIAM J. Control 5, p. 64-73, (1965).

12. Souhail S., and Afifi L., Cheap controls for disturbances compensation in hyperbolic delayed systems. International Journal of Dynamical Systems and Differential Equations, 10:6, 511-536, (2020).

13. Yu, P., Wu, M., She, and J.H., Robust tracking and disturbance rejection for linear uncertain system with unknown state delay and disturbance. IEEE/ASME Trans Mechatron, 23: 1445–1455, (2018).

14. Zattoni, E., and Marro, G., Disturbance compensation in discrete-time switching linear systems subject to a dwell-time constraint. 54th IEEE Conference on Decision and Control (CDC), Osaka, Japan, 2307-2312, doi: 10.1109/CDC.2015.7402551. (2015).

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