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On Conformable Fractional Differential Equations with Nonlocal-Impulsive Conditions

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ABSTRACT: This paper addresses the existence and controllability of the integral solution to a nondense conformable fractional differential equation with a nonlocal-impulsive condition. The main findings are derived using fixed-point theorems in conjunction with the theory of integrated semigroups.

Key Words: Fractional ordinary differential equations, fixed-point theorems, nonlocal-impulsive condition, conformable fractional derivative.

Contents

1. Introduction

In the work [1], researchers first defined the conformable fractional derivative. This innovation prompted widespread investigation into such derivatives across numerous disciplines [2,3]. Fractional differential equations involving these derivatives have become a major focus due to their extensive utility in scientific and engineering contexts [4-7].

This article examines the solvability and control properties of integral solutions for impulsive conformable fractional differential equations under nondense conditions. Specifically, we analyze the following initial value problem:

$$\begin{cases} \frac{d^\gamma x(t)}{dt^\gamma} = Ax(t) + f(t, x(t)), \quad t \in [0, \tau], \ t \neq t_1, t_2, \ldots, t_n, \ 0 < \gamma < 1, \\ x(0) = x_0 + g(x), \\ x(t_i^+) = x(t_i^-) + h_i(x(t_i)), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.1}$$

where $\frac{d^\gamma(\cdot)}{dt^\gamma}$ is the conformable fractional derivative [1]. The relation $x(0) = x_0 + g(x)$ represents a nonlocal condition [8], which arises in many real-world scenarios [9,10]. The jumps $x(t_i^+) = x(t_i^-) + h_i(x(t_i))$ capture impulsive effects, where $x(t_i^+)$ and $x(t_i^-)$ are the right- and left-sided limits at $t = t_i$. Such conditions are vital for modeling sudden changes in dynamic systems [11-14].

The operator $(A, D(A))$ is closed and satisfies the Hille-Yosida condition in the Banach space $(X, \|\cdot\|)$. The domain $D(A)$ may not be dense in X, and $x_0 \in \overline{D(A)}$. The mappings $f : [0, \tau] \times X \to X$, $h_i : X \to \overline{D(A)}$, and $g : \mathcal{C} \to \overline{D(A)}$ adhere to certain assumptions, where $\mathcal{C}$ consists of functions $x(\cdot) : [0, \tau] \to X$ that are piecewise continuous with existing one-sided limits at each t_i. The space $\mathcal{C}$ is Banach when equipped with $\|x\|_{\mathcal{C}} = \sup_{t\in[0,\tau]} \|x(t)\|$. The norm in $\mathcal{L}(X)$ (the bounded linear operators on X) is also denoted by $\|\cdot\|$.

The remainder of this work is structured as follows: Section 2 covers essential background on conformable derivatives and integrated semigroups. Section 3 establishes the core theoretical results.

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2. Preliminaries

This section outlines fundamental concepts regarding conformable fractional calculus and the theory of integrated semigroups.

Definition 2.1 *([1]) Given $\gamma \in]0,1]$, the conformable fractional derivative of order γ for a function $x(.)$ at $t>0$ is given by*

$$\frac{d^\gamma x(t)}{dt^\gamma} = \lim_{\varepsilon \longrightarrow 0} \frac{x(t+\varepsilon t^{1-\gamma}) - x(t)}{\varepsilon}.$$

At $t=0$, we use the following convention:

$$\frac{d^\gamma x(0)}{dt^\gamma} = \lim_{t \longrightarrow 0^+} \frac{d^\gamma x(t)}{dt^\gamma}.$$

The corresponding fractional integral operator $I^\gamma(.)$ is expressed as

$$I^\gamma(x)(t) = \int_0^t s^{\gamma-1} x(s) ds.$$

Theorem 2.1 *([1]) For any continuous function $x(.)$ within the domain of $I^\gamma(.)$, the following identity holds:*

$$\frac{d^\gamma (I^\gamma(x)(t))}{dt^\gamma} = x(t).$$

Definition 2.2 *([2]) The fractional Laplace transform of degree $\gamma \in]0,1]$ applied to $x(.)$ is defined through*

$$\mathcal{L}_\gamma(x(t))(\lambda) := \int_0^{+\infty} t^{\gamma-1} e^{-\lambda \frac{t^\gamma}{\gamma}} x(t) dt, \quad \lambda > 0.$$

The next result establishes how fractional integration and Laplace transformation interact with conformable differentiation.

Proposition 2.1 *([2]) For differentiable $x(.)$, these fundamental relations apply:*

$$I^\gamma \left(\frac{d^\gamma x(.)}{dt^\gamma} \right)(t) = x(t) - x(0),$$

$$\mathcal{L}_\gamma \left(\frac{d^\gamma x(t)}{dt^\gamma} \right)(\lambda) = \lambda \mathcal{L}_\gamma(x(t))(\lambda) - x(0).$$

Building on [3], we note the following properties:

Remark 2.1 *For arbitrary functions $x(.)$ and $y(.)$, we observe:*

$$\mathcal{L}_\gamma \left(x\left(\frac{t^\gamma}{\gamma} \right) \right)(\lambda) = \mathcal{L}_1(x(t))(\lambda),$$

$$\mathcal{L}_\gamma \left(\int_0^t s^{\gamma-1} x\left(\frac{t^\gamma - s^\gamma}{\gamma} \right) y(s) ds \right)(\lambda) = \mathcal{L}_1(x(t))(\lambda) \mathcal{L}_\gamma(y(t))(\lambda).$$

We now turn to key notions in integrated semigroup theory.

Definition 2.3 *([15], [16]) An integrated semigroup comprises a family $(S(t))_{t\geq 0}$ of bounded linear operators on X characterized by:*

1. *Null initialization: $S(0)=0$,*
2. *Strong continuity: $t \mapsto S(t)$ is continuous in the strong operator topology,*

3. *Composition law:* $S(s)S(t) = \int_0^s (S(t+\tau) - S(\tau))d\tau$ *holds for* $t, s \geq 0$.

Definition 2.4 *([16]) An integrated semigroup* $(S(t))_{t\geq 0}$ *may satisfy:*

1. *Exponential boundedness:* $\exists M \geq 0, \omega \in \mathbb{R}$ *with* $\|S(t)\| \leq Me^{\omega t}$ *for* $t \geq 0$.
2. *Non-degeneracy:* $S(t)x = 0$ $\forall t \geq 0$ *implies* $x = 0$.
3. *Local Lipschitz condition: For each* $b > 0$, $\exists L$ *such that* $\|S(t) - S(s)\| \leq L|t-s|$ *when* $s, t \in [0, b]$.

Definition 2.5 *([16]) An operator* A *generates an integrated semigroup if* $\exists \omega \in \mathbb{R}$ *where* $]\omega, +\infty[\subset \rho(A)$, *and a strongly continuous, exponentially bounded family* $(S(t))_{t\geq 0}$ *exists with:*

1. $S(0) = 0$,
2. *The resolvent representation:* $(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S(t)dt$ *for* $\lambda > \omega$.

Proposition 2.2 *([15]) For* A *generating* $(S(t))_{t\geq 0}$, *these structural properties emerge:*

1. *Domain inclusion:* $\int_0^t S(s)x\,ds \in D(A)$ *when* $x \in X$, $t > 0$.
2. *Preservation of domain:* $S(t)y \in D(A)$ *for* $y \in D(A)$, $t > 0$.
3. *Generator relation:* $S(t)x = A\int_0^t S(s)x\,ds + tx$ *when* $x \in X$, $t > 0$.
4. *Commutation:* $AS(t)y = S(t)Ay$ *for* $y \in D(A)$, $t > 0$.
5. *Reconstruction formula:* $S(t)y = \int_0^t S(s)Ay\,ds + ty$ *when* $y \in D(A)$, $t > 0$.

For $x \in \overline{D(A)}$, the mapping $t \mapsto S(t)x$ becomes continuously differentiable, inducing a semigroup $\dot{S}(t)$ on $\overline{D(A)}$.

Definition 2.6 *([18]) A linear operator* A *qualifies as a Hille-Yosida operator when* $\exists \theta \geq 0$, $\omega \in \mathbb{R}$ *such that:*

1. *Resolvent condition:* $]\omega, +\infty[\subset \rho(A)$,
2. *Iterated resolvent estimate:* $\|R(\lambda, A)^n\| \leq \frac{\theta}{(\lambda-\omega)^n}$ *for* $n \in \mathbb{N}$, $\lambda > \omega$ *(where* $R(\lambda, A) := (\lambda - A)^{-1}$*)).*

Theorem 2.2 *([16]) These statements are equivalent:*

1. *A satisfies Hille-Yosida conditions.*
2. *A generates a locally Lipschitz continuous integrated semigroup.*

We conclude with an essential limit property:

Remark 2.2 *([19]) For all* $x \in \overline{D(A)}$, $\lim_{\lambda \longrightarrow +\infty} \lambda(\lambda - A)^{-1}x = x$.

3. Main results

3.1. Existence of integral solutions

We first prove the following lem.

Lemma 3.1 *Let* $t \in [0, \tau]$, *if* $\int_0^t s^{\gamma-1}x(s)ds \in D(A)$ *then by integration of equation (1.1) we obtain*

$$x(t) = x_0 + g(x) + \sum_{0<t_i<t} h_i(x(t_i)) + A\int_0^t s^{\gamma-1}x(s)ds + \int_0^t s^{\gamma-1}f(s, x(s))ds.$$

Proof: For $t \in [0, t_1]$, we have

$$x(t) = x_0 + g(x) + A\int_0^t s^{\gamma-1}x(s)ds + \int_0^t s^{\gamma-1}f(s,x(s))ds.$$

As in [13], we assume that the solution of equation (1.1) is such that at the point of discontinuity t_k, $x(t_k^-) = x(t_k)$. Then, we get

$$x(t_1^-) = x_0 + g(x) + A\int_0^{t_1} s^{\gamma-1}x(s)ds + \int_0^{t_1} s^{\gamma-1}f(s,x(s))ds.$$

For $t \in]t_1, t_2]$, we obtain

$$\begin{aligned} x(t) &= x(t_1^+) + A\int_{t_1}^t s^{\gamma-1}x(s)ds + \int_{t_1}^t s^{\gamma-1}f(s,x(s))ds \\ &= x(t_1^-) + h_1(x(t_1)) + A\int_{t_1}^t s^{\gamma-1}x(s)ds + \int_{t_1}^t s^{\gamma-1}f(s,x(s))ds. \end{aligned}$$

We replace $x(t_1^-)$ by its expression in the above equation, we get

$$\begin{aligned} x(t) = x_0 + g(x) + A\int_0^{t_1} s^{\gamma-1}x(s)ds + \int_0^{t_1} s^{\gamma-1}f(s,x(s))ds \\ + h_1(x(t_1)) + A\int_{t_1}^t s^{\gamma-1}x(s)ds + \int_{t_1}^t s^{\gamma-1}f(s,x(s))ds. \end{aligned}$$

Through computation, we find

$$x(t) = x_0 + g(x) + h_1(x(t_1)) + A\int_0^t s^{\gamma-1}x(s)ds + \int_0^t s^{\gamma-1}f(s,x(s))ds.$$

In particular, we have

$$x(t_2^-) = x_0 + g(x) + h_1(x(t_1)) + A\int_0^{t_2} s^{\gamma-1}x(s)ds + \int_0^{t_2} s^{\gamma-1}f(s,x(s))ds.$$

Likewise, for $t \in]t_2, t_3]$, we have

$$\begin{aligned} x(t) &= x(t_2^+) + A\int_{t_2}^t s^{\gamma-1}x(s)ds + \int_{t_2}^t s^{\gamma-1}f(s,x(s))ds \\ &= x(t_2^-) + h_2(x(t_2)) + A\int_{t_2}^t s^{\gamma-1}x(s)ds + \int_{t_2}^t s^{\gamma-1}f(s,x(s))ds \\ &= x_0 + g(x) + h_1(x(t_1)) + h_2(x(t_2)) + A\int_0^t s^{\gamma-1}x(s)ds + \int_0^t s^{\gamma-1}f(s,x(s))ds. \end{aligned}$$

By applying the same process repeatedly, we arrive at the proof of the lemma. □

In light of the above lemma, we now introduce the following definition.

Definition 3.1 *A function $x \in \mathcal{C}$ is an integral solution of equation (1.1) if*

$$\int_0^t s^{\gamma-1}x(s)ds \in D(A),\ t \in [0,\tau], \tag{3.1}$$

$$x(t) = x_0 + g(x) + \sum_{0<t_i<t} h_i(x(t_i)) + A\int_0^t s^{\gamma-1}x(s)ds + \int_0^t s^{\gamma-1}f(s,x(s))ds,\ t \in [0,\tau]. \tag{3.2}$$

Remark 3.1 *([3]) We have the following equality*

$$x(t) = \lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\gamma}} s^{\gamma-1} x(s) ds \in \overline{D(A)}.$$

Lemma 3.2 *The integral solution of equation (1.1) satisfied*

$$x(t) = \dot{S}(\frac{t^\gamma}{\gamma})[x_0 + g(x)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma}) h_i(x(t_i))$$
$$+ \lim_{\lambda \longrightarrow +\infty} \int_0^t s^{\gamma-1} \dot{S}(\frac{t^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, x(s)) ds,$$

where $(S(t))_{t\geq 0}$ represents the integrated semigroup generated by the operator A.

Proof: Under fractional Laplace transform and remark (2.1), equation (3.2) becomes

$$\mathcal{L}_\gamma(x(t))(\lambda) = \frac{1}{\lambda}[x_0 + g(x) + \sum_{0<t_i<t} h_i(x(t_i)) + A\mathcal{L}_\gamma(x(t))(\lambda) + \mathcal{L}_\gamma(f(t, x(t)))(\lambda)].$$

Consequently, one has

$$\mathcal{L}_\gamma(x(t))(\lambda) = (\lambda - A)^{-1}[x_0 + g(x) + \sum_{0<t_i<t} h_i(x(t_i))] + (\lambda - A)^{-1}\mathcal{L}_\gamma(f(t, x(t)))(\lambda).$$

Using the inverse fractional Laplace transform in conjunction with remark (2.1), we get

$$x(t) = \dot{S}(\frac{t^\gamma}{\gamma})[x_0 + g(x)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma}) h_i(x(t_i))$$
$$+ \frac{d^\gamma}{dt^\gamma} \int_0^t s^{\gamma-1} S(\frac{t^\gamma - s^\gamma}{\gamma}) f(s, x(s)) ds,$$

Next combining remark (2.2) and remark (3.1), we obtain the following Duhamel formula

$$x(t) = \dot{S}(\frac{t^\gamma}{\gamma})[x_0 + g(x)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma}) h_i(x(t_i))$$
$$+ \lim_{\lambda \longrightarrow +\infty} \int_0^t s^{\gamma-1} \dot{S}(\frac{t^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, x(s)) ds.$$

□

Prior to establishing the existence of integral solutions, we formulate the following fundamental hypotheses:

(A_1) The nonlinear mapping $f(t, \cdot) : X \to X$ is jointly continuous and satisfies the local growth condition: for every $r > 0$, there exists $\mu_r \in L^\infty([0, \tau], \mathbb{R}^+)$ such that

$$\sup_{\|x\|\leq r} \|f(t, x)\| \leq \mu_r(t), \quad \forall t \in [0, \tau].$$

(A_2) For each fixed $x \in X$, the temporal mapping $f(\cdot, x) : [0, \tau] \to X$ is continuous.

(A_3) The nonlocal function $g : \mathcal{C} \to X$ is Lipschitz continuous with constant $L_2 > 0$, i.e.,

$$\|g(y) - g(x)\| \leq L_2\|y - x\|_c, \quad \forall x, y \in \mathcal{C}.$$

(A_4) The impulse functions $h_i : X \to X$ satisfy the uniform Lipschitz condition with constants $C_i > 0$:

$$\|h_i(y(t_i)) - h_i(x(t_i))\| \leq C_i\|y - x\|_c, \quad \forall x, y \in \mathcal{C},\ i = 1, \ldots, n.$$

Theorem 3.1 *If $(\dot{S}(t))_{t>0}$ is compact and $(A_1) - (A_4)$ are satisfied, then equation (1.1) has at least one integral solution, provided only that*

$$(L_2 + \sum_{i=1}^{n} C_i) \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})| < 1.$$

Proof: Let $B_r = \{x \in \mathcal{C},\ |\ x\ |_c \leq r\}$, where

$$r \geq \frac{\sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[\|\ x_0\ \| + \|\ g(0)\ \| + \sum_{i=1}^{n} \|\ h_i(0)\ \| + \theta\frac{\tau^\gamma}{\gamma}\ |\ \mu_r\ |_{L^\infty([0,\tau],\mathbb{R}^+)}]}{1 - (L_2 + \sum_{i=1}^{n} C_i) \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|}.$$

For $x \in B_r$ and $t \in [0, \tau]$ define the operators Γ_1 and Γ_2 by:

$$\Gamma_1(x)(t) = \dot{S}(\frac{t^\gamma}{\gamma})[x_0 + g(x)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma})h_i(x(t_i)),$$

$$\Gamma_2(x)(t) = \lim_{\lambda\longrightarrow+\infty} \int_0^t s^{\gamma-1}\dot{S}(\frac{t^\gamma - s^\gamma}{\gamma})\lambda(\lambda - A)^{-1}f(s, x(s))ds.$$

Claim 1: Prove that Γ_1 is a contraction operator on B_r.

For $x, y \in \mathcal{C}$, we have

$$\Gamma(y)(t) - \Gamma(x)(t) = \dot{S}(\frac{t^\gamma}{\gamma})[g(y) - g(x)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma})[h_i(y(t_i)) - h_i(x(t_i))].$$

Accordingly, we obtain

$$\|\ \Gamma(y)(t) - \Gamma(x)(t)\ \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\ \|\ g(y) - g(x)\ \| + \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})| \sum_{0<t_i<t} \|\ h_i(y(t_i)) - h_i(x(t_i))\ \|.$$

Using assumptions (A_3) and (A_4), we get

$$\|\ \Gamma(y)(t) - \Gamma(x)(t)\ \| \leq (L_2 + \sum_{i=1}^{n} C_i) \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\ |\ y - x\ |_c.$$

Taking the supremum, we obtain

$$|\ \Gamma(y) - \Gamma(x)\ |_c \leq (L_2 + \sum_{i=1}^{n} C_i) \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\ |\ y - x\ |_c.$$

Hence, Γ_1 is a contraction operator on B_r.

Claim 2: Prove that Γ_2 is continuous.

Let $(x_n) \subset B_r$ such that $x_n \longrightarrow x$ in B_r. We have

$$\Gamma_2(x_n)(t) - \Gamma_2(x)(t) = \lim_{\lambda \longrightarrow +\infty} \int_0^t s^{\gamma-1} \dot{S}(\frac{t^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1}[f(s, x_n(s)) - f(s, x(s))]ds.$$

By using that $\lim_{\lambda \to +\infty} \mid \lambda(\lambda - A)^{-1} \mid \leq \theta$, we obtain

$$\mid \Gamma_2(x_n) - \Gamma_2(x) \mid_c \leq \theta \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})| \int_0^\tau s^{\gamma-1} \parallel f(s, x_n(s)) - f(s, x(s)) \parallel ds.$$

According to assumption (A_1), we get

$$\parallel s^{\gamma-1}[f(s, x_n(s)) - f(s, x(s))] \parallel \leq 2\mu_r(s)s^{\gamma-1} \text{ and } f(s, x_n(s)) \longrightarrow f(s, x(s)) \text{ as } n \longrightarrow +\infty.$$

Hence, the Lebesgue dominated convergence theorem ensures that

$$\lim_{n \longrightarrow +\infty} \mid \Gamma_2(x_n) - \Gamma_2(x) \mid_c = 0.$$

Claim 3: Prove that Γ_2 is compact.

Firstly: We prove that the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X.
To do so, for some fixed $t \in]0, \tau[$. Let $\varepsilon \in]0, t[$, $x \in B_r$ and define the operator Γ_2^ε by

$$\Gamma_2^\varepsilon(x)(t) = \lim_{\lambda \longrightarrow +\infty} \int_0^{(t^\gamma - \varepsilon^\gamma)^{\frac{1}{\gamma}}} s^{\gamma-1} \dot{S}(\frac{t^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, x(s))ds.$$

We can writ Γ_2^ε as follows

$$\Gamma_2^\varepsilon(x)(t) = \dot{S}(\frac{\varepsilon^\gamma}{\gamma}) \lim_{\lambda \longrightarrow +\infty} \int_0^{(t^\gamma - \varepsilon^\gamma)^{\frac{1}{\gamma}}} s^{\gamma-1} \dot{S}(\frac{t^\gamma - s^\gamma - \varepsilon^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, x(s))ds.$$

Since, the compactness of $(\dot{S}(t))_{t>0}$ proves that the set $\{\Gamma_2^\varepsilon(x)(t),\ x \in B_r\}$ is relatively compact in X. By using a computation combined with assumption (A_5), we get

$$\parallel \Gamma_2^\varepsilon(x)(t) - \Gamma_2(x)(t) \parallel \leq \theta \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)} \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})| \frac{\varepsilon^\gamma}{\gamma}.$$

This proves that the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X.
For $t = 0$, the set $\{\Gamma_2(x)(0),\ x \in B_r\}$ is compact. Hence, the set $\{\Gamma_2(x)(t),\ x \in B_r\}$ is relatively compact in X for all $t \in [0, \tau]$.
Secondly: We prove that $\Gamma_2(B_r)$ is equicontinuous.
For $t_1, t_2 \in]0, \tau]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) &= \lim_{\lambda \longrightarrow +\infty} \int_0^{t_1} s^{\gamma-1}[\dot{S}(\frac{t_2^\gamma - s^\gamma}{\gamma}) - \dot{S}(\frac{t_1^\gamma - s^\gamma}{\gamma})] \lambda(\lambda - A)^{-1} f(s, x(s))ds \\
&+ \lim_{\lambda \longrightarrow +\infty} \int_{t_1}^{t_2} s^{\gamma-1} \dot{S}(\frac{t_2^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, x(s))ds \\
&= [\dot{S}(\frac{t_2^\gamma - t_1^\gamma}{\gamma}) - I)] \lim_{\lambda \longrightarrow +\infty} \int_0^{t_1} s^{\gamma-1} \dot{S}(\frac{t_1^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, x(s))ds \\
&+ \lim_{\lambda \longrightarrow +\infty} \int_{t_1}^{t_2} s^{\gamma-1} \dot{S}(\frac{t_2^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, x(s))ds.
\end{aligned}$$

Using a computation and assumption (A_1), we get

$$\| \Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) \| \leq \frac{\theta \sup\limits_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})| \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)}}{\gamma}[(t_2^\gamma - t_1^\gamma) + \tau^\gamma \mid \dot{S}(\frac{t_2^\gamma - t_1^\gamma}{\gamma}) - I \mid].$$

This implies that $\Gamma_2(x)$, $x \in B_r$ are equicontinuous at $t \in [0,\tau]$. Consequently, the ArzelÃ -Ascoli theorem establishes that the operator Γ_2 is compact. Finally, by applying the Krasnoselskii fixed-point theorem, we infer that the operator $\Gamma_1 + \Gamma_2$ has at least one fixed point in $\mathcal{C}$, which is the integral solution of equation (1.1). □

3.2. Controllability of integral solutions

Here, we will examine the controllability of the integral solution of the equation. (1.1). Precisely, we shall be concerned with the controllability of integral solution of the following Cauchy problem

$$\begin{cases} \frac{d^\gamma y(t)}{dt^\gamma} = Ay(t) + f(t, y(t)) + Bu(t), \quad t \in [0,\tau], \quad t \neq t_1, t_2, ..., t_n, \quad 0 < \gamma < 1, \\ y(0) = y_0 + g(y), \\ y(t_i^+) = y(t_i^-) + h_i(y(t_i)), \quad i = 1, 2, ..., n, \end{cases} \tag{3.3}$$

where B is a bounded linear operator from U into X with U is a Banach space, the control function u is an element of $L^2([0,\tau], U)$.

Lemma 3.3 *The integral solution of equation (3.3) satisfied the following Duhamel formula*

$$\begin{aligned} y(t) = \dot{S}(\frac{t^\gamma}{\gamma})[y_0 + g(y)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma})h_i(y(t_i)) \\ + \lim_{\lambda \longrightarrow +\infty} \int_0^t s^{\gamma-1}\dot{S}(\frac{t^\gamma - s^\gamma}{\gamma})\lambda(\lambda - A)^{-1}(f(s, y(s)) + Bu(t))ds. \end{aligned}$$

Definition 3.2 *The Cauchy problem (3.3) is said to be controllable on $[0,\tau]$, if for every $y_1 \in X$ there is exists a control $u \in L^2([0,\tau], U)$ such that the integral solution $y(.)$ of (3.3) satisfies $y(\tau) = y_1$.*

In the sequel, we will employ the following fundamental assumptions:

(B_1) The nonlinear mapping $f : [0,\tau] \times X \to X$ is continuous and satisfies:

- Linear growth condition: $\exists L > 0$ such that $\|f(t, y)\| \leq L\|y\|$ for all $t \in [0,\tau]$, $y \in X$.
- Uniform Lipschitz condition: $\exists K > 0$ such that $\|f(t, y) - f(t, x)\| \leq K\|y - x\|$ for all $t \in [0,\tau]$, $x, y \in X$.

(B_2) For each fixed $y \in X$, the temporal mapping $f(\cdot, y) : [0,\tau] \to X$ is continuous.

(B_3) The nonlocal operator $g : \mathcal{C}([0,\tau]; X) \to X$ is continuous.

(B_4) The functional g satisfies:

$$\|g(y)\| \leq M\|y\|_{\mathcal{C}},$$
$$\|g(y) - g(x)\| \leq N\|y - x\|_{\mathcal{C}},$$

for some constants $M, N > 0$ and all $x, y \in \mathcal{C}([0,\tau]; X)$.

(B_5) For each $i \in \{1, \dots, n\}$, the impulse functions $h_i : X \to X$ satisfies:

$$\|h_i(y(t_i))\| \le d_i \|y\|_{\mathcal{C}},$$
$$\|h_i(y(t_i)) - h_i(x(t_i))\| \le c_i \|y - x\|_{\mathcal{C}},$$

with constants $d_i, c_i > 0$ and all $x, y \in \mathcal{C}([0, \tau]; X)$.

(B_6) The linear controllability operator $W : L^2([0, \tau]; U) \to X$ defined by

$$Wu := \lim_{\lambda \to \infty} \int_0^\tau s^{\gamma-1} \dot{S}\left(\tfrac{\tau^\gamma - s^\gamma}{\gamma}\right) \lambda(\lambda - A)^{-1} Bu(s)\, ds$$

has a bounded pseudo-inverse $\widetilde{W}^{-1} : X \to L^2([0, \tau]; U)/\ker W$ with:

- $\|B\|_{\mathcal{L}(U,X)} \le R_1$,
- $\|\widetilde{W}^{-1}\|_{\mathcal{L}(X, L^2/\ker W)} \le R_2$,

for positive constants $R_1, R_2 > 0$.

Theorem 3.2 *Assume that $(B_1)-(B_6)$ hold, then Cauchy problem (3.3) is controllable on $[0, \tau]$, provided that*

$$\sup_{t \in [0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(1 + R_1 R_2 \theta \frac{\tau^\gamma}{\gamma} \sup_{t \in [0,\tau]} |\dot{S}(\frac{\tau^\gamma}{\gamma})|) \max(M + \sum_{i=1}^n d_i + \theta \frac{\tau^\gamma}{\gamma} L, N + \sum_{i=1}^n c_i + \theta \frac{\tau^\gamma}{\gamma} K) < 1.$$

Proof: By using hypothesis (B_5) for an arbitrary function $y(.)$, we can define a control $u_y(.)$ as follows

$$u_y(.) = \tilde{W}^{-1}\Big(y_1 - \dot{S}(\frac{\tau^\gamma}{\gamma})[y_0 + g(y)] - \sum_{0<t_i<\tau} \dot{S}(\frac{\tau^\gamma - t_i^\gamma}{\gamma}) h_i(y(t_i))$$
$$- \lim_{\lambda \longrightarrow +\infty} \int_0^\tau s^{\gamma-1} \dot{S}(\frac{\tau^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} f(s, y(s)) ds\Big)(.). \tag{3.4}$$

For this control, we define the operator $\Psi : \mathcal{C} \longrightarrow \mathcal{C}$ by

$$\Psi(y)(t) = \dot{S}(\frac{t^\gamma}{\gamma})[y_0 + g(y)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma}) h_i(y(t_i))$$
$$+ \lim_{\lambda \longrightarrow +\infty} \int_0^t s^{\gamma-1} \dot{S}(\frac{t^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} \Big(f(s, y(s)) + Bu_y(s)\Big) ds.$$

We also introduce for a radius $r > 0$ the ball $B_r := \{y \in \mathcal{C}, \;\; \| y \|_c \le r\}$ and we denote by $|.|$ the norm in the space $\mathcal{L}(X)$ of bounded operators defined from X into itself.
We will demonstrate that the operator Ψ has a fixed point, which is a integral solution of the control problem (3.3). To do so, we will give the proof in two steps.

Step 1: Prove that there exists a radius $\delta > 0$ such that $\Psi : B_\delta \longrightarrow B_\delta$.

For $x \in \mathcal{C}$ and $t \in [0, \tau]$, we have

$$\Psi(y)(t) = \dot{S}(\frac{t^\gamma}{\gamma})[y_0 + g(y)] + \sum_{0<t_i<t} \dot{S}(\frac{t^\gamma - t_i^\gamma}{\gamma}) h_i(y(t_i))$$
$$+ \lim_{\lambda \longrightarrow +\infty} \int_0^t s^{\gamma-1} \dot{S}(\frac{t^\gamma - s^\gamma}{\gamma}) \lambda(\lambda - A)^{-1} \Big(f(s, y(s)) + Bu_y(s)\Big) ds.$$

Then one has

$$\| \Psi(y)(t) \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[\| y_0 + g(y) + \sum_{0<t_i<t} h_i(y(t_i)) \|$$
$$+ \lim_{\lambda\longrightarrow+\infty} | \lambda(\lambda-A)^{-1} | \int_0^t s^{\gamma-1} \| f(s,y(s)) + Bu_y(s) \| ds].$$

By using hypothesis (B_1), (B_5) and (B_6),and using the fact that $\lim_{\lambda\to+\infty} | \lambda(\lambda-A)^{-1} | \leq \theta$ we obtain

$$\| \Psi(y)(t) \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[\| y_0 \| + M \| y \|_c + \sum_{i=1}^n d_i \| y \|_c + \theta(L \| y \|_c + R_1 \| u_y \|_2) \int_0^\tau s^{\gamma-1} ds]$$
$$\leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[\| y_0 \| + M \| y \|_c + \sum_{i=1}^n d_i \| y \|_c + \theta(L \| y \|_c + R_1 \| u_y \|_2) \frac{\tau^\gamma}{\gamma}]. \quad (*)$$

On the other hand, we known that

$$u_y = \tilde{W}^{-1}\Big(y_1 - \dot{S}(\frac{\tau^\gamma}{\gamma})[y_0 + g(y)] - \sum_{0<t_i<\tau} \dot{S}(\frac{\tau^\gamma - t_i^\gamma}{\gamma}) h_i(y(t_i))$$
$$- \lim_{\lambda\longrightarrow+\infty} \int_0^\tau s^{\gamma-1} \dot{S}(\frac{\tau^\gamma - s^\gamma}{\gamma}) \lambda(\lambda-A)^{-1} f(s,y(s)) ds\Big).$$

In view of assumptions (B_1), (B_5) and (B_6), we obtain

$$\| u_y \|_2 \leq R_2 \| y_1 - \dot{S}(\frac{\tau^\gamma}{\gamma})[y_0 + g(y)] - \sum_{0<t_i<\tau} \dot{S}(\frac{\tau^\gamma - t_i^\gamma}{\gamma}) h_i(y(t_i))$$
$$- \lim_{\lambda\longrightarrow+\infty} \int_0^\tau s^{\gamma-1} \dot{S}(\frac{\tau^\gamma - s^\gamma}{\gamma}) \lambda(\lambda-A)^{-1} f(s,y(s)) ds \|$$
$$\leq R_2[\| y_1 \| + \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(\| y_0 + g(y) + \sum_{0<t_i<t} h_i(y(t_i)) \|$$
$$+ \lim_{\lambda\to+\infty} | \lambda(\lambda-A)^{-1} | \int_0^\tau s^{\gamma-1} \| f(s,y(s)) \| ds)]$$
$$\leq R_2[\| y_1 \| + \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(\| y_0 \| + M \| y \|_c + \sum_{i=1}^n d_i \| y \|_c + \theta L \| y \|_c \int_0^\tau s^{\gamma-1} ds]$$
$$\leq R_2[\| y_1 \| + \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(\| y_0 \| + M \| y \|_c + \sum_{i=1}^n d_i \| y \|_c + \theta L \| y \|_c \frac{\tau^\gamma}{\gamma}]$$
$$\leq R_2[\| y_1 \| + \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(\| y_0 \| + (M + \sum_{i=1}^n d_i + \theta L \frac{\tau^\gamma}{\gamma}) \| y \|_c].$$

By replacing this estimate in $(*)$, we get

$$\| \Psi(y)(t) \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[\| y_0 \| + M \| y \|_c + \sum_{i=1}^n d_i \| y \|_c + \theta(L \| y \|_c + R_1 R_2[\| y_1 \|$$
$$+ \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(\| y_0 \| + (M + \sum_{i=1}^n d_i + \theta L \frac{\tau^\gamma}{\gamma}) \| y \|_c]) \frac{\tau^\gamma}{\gamma}].$$

Separating the terms that contain the expression $\| y \|_c$, one has

$$\| \Psi(y)(t) \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\big[M + \sum_{i=1}^n d_i + \theta L \frac{\tau^\gamma}{\gamma} + \theta R_1 R_2 \frac{\tau^\gamma}{\gamma} \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(M + \sum_{i=1}^n d_i + \theta L \frac{\tau^\gamma}{\gamma})\big] \| y \|_c$$
$$+ \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\big[\| y_0 \| + \theta \frac{\tau^\gamma}{\gamma} R_1 R_2 \| y_1 \| + \theta \frac{\tau^\gamma}{\gamma} R_1 R_2 \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})| \| y_0 \| \big].$$

By applying a straightforward factorization, we obtain

$$\| \Psi(y)(t) \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(M+\sum_{i=1}^n d_i + \theta L\frac{\tau^\gamma}{\gamma})\big[1+\theta R_1R_2\frac{\tau^\gamma}{\gamma}\sup_{t\in[0,\tau]}|\dot{S}(\frac{t^\gamma}{\gamma})|\big] \| y \|_c$$
$$+ \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\big[(1+\theta\frac{\tau^\gamma}{\gamma}R_1R_2\sup_{t\in[0,\tau]}|\dot{S}(\frac{t^\gamma}{\gamma})|) \| y_0 \| + \theta\frac{\tau^\gamma}{\gamma}R_1R_2 \| y_1 \| \big].$$

Hence, it suffices to consider δ as a solution in r of the following inequality

$$\sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(M+\sum_{i=1}^n d_i + \theta L\frac{\tau^\gamma}{\gamma})\big[1+\theta R_1R_2\frac{\tau^\gamma}{\gamma}\sup_{t\in[0,\tau]}|\dot{S}(\frac{t^\gamma}{\gamma})|\big] r$$
$$+ \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\big[(1+\theta\frac{\tau^\gamma}{\gamma}R_1R_2\sup_{t\in[0,\tau]}|\dot{S}(\frac{t^\gamma}{\gamma})|) \| y_0 \| + \theta\frac{\tau^\gamma}{\gamma}R_1R_2 \| y_1 \| \big] \leq r.$$

Precisely, we can choose δ such that

$$\delta \geq \frac{\sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|\big[(1+\theta\frac{\tau^\gamma}{\gamma}R_1R_2\sup_{t\in[0,\tau]}|\dot{S}(\frac{t^\gamma}{\gamma})|) \| y_0 \| + \theta\frac{\tau^\gamma}{\gamma}R_1R_2 \| y_1 \| \big]}{1-\sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(M+\sum_{i=1}^n d_i + \theta L\frac{\tau^\gamma}{\gamma})\big[1+\theta R_1R_2\frac{\tau^\gamma}{\gamma}\sup_{t\in[0,\tau]}|\dot{S}(\frac{t^\gamma}{\gamma})|\big]}.$$

Step 2: We show that Ψ is a contraction operator on B_δ.
For y, $x \in \mathcal{C}$, we have

$$\Psi(y)(t)-\Psi(x)(t) = \dot{S}(\frac{t^\gamma}{\gamma})[g(y)-g(x)] + \sum_{0<t_i<t}\dot{S}(\frac{t^\gamma-t_i^\gamma}{\gamma})[h_i(y(t_i))-h_i(x(t_i))]$$
$$+\lim_{\lambda\longrightarrow+\infty}\int_0^t s^{\gamma-1}\dot{S}(\frac{t^\gamma-s^\gamma}{\gamma})\lambda(\lambda-A)^{-1}\Big(f(s,y(s))-f(s,x(s))+B(u_y-u_x)(s)\Big)ds.$$

According to (B_1), (B_5) and (B_6), we obtain

$$\| \Psi(y)(t)-\Psi(x)(t) \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[\| g(y)-g(x) \| + \sum_{0<t_i<t} \| h_i(y(t_i))-h_i(x(t_i)) \|$$
$$+\theta\int_0^t s^{\gamma-1} \| f(s,y(s))-f(s,x(s))+B(u_y-u_x)(s) \| ds]$$
$$\leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[N \| y-x \|_c + \sum_{i=1}^n c_i \| y-x \|_c + \theta(K \mid y-x \|_c + R_1 \| u_y-u_x \|_2)\int_0^t s^{\gamma-1}ds]$$
$$\leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[N \| y-x \|_c + \sum_{i=1}^n c_i \| y-x \|_c + \theta\frac{\tau^\gamma}{\gamma}(K \| y-x \|_c + R_1 \| u_y-u_x \|_2)]. \qquad (**)$$

On the other hand, we know that

$$u_y-u_x = \tilde{W}^{-1}\Big(-\dot{S}(\frac{\tau^\gamma}{\gamma})[g(y)-g(x)] - \sum_{0<t_i<\tau}\dot{S}(\frac{\tau^\gamma-t_i^\gamma}{\gamma})[h_i(y(t_i))-h_i(x(t_i))]$$
$$-\lim_{\lambda\longrightarrow+\infty}\int_0^\tau s^{\gamma-1}\dot{S}(\frac{\tau^\gamma-s^\gamma}{\gamma})\lambda(\lambda-A)^{-1}(f(s,y(s))-f(s,x(s)))ds\Big).$$

Then one has

$$\| u_y-u_x \|_2 \leq R_2\sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[\| g(y)-g(x) \| + \sum_{0<t_i<t} \| h_i(y(t_i))-h_i(x(t_i)) \|$$
$$+\theta\int_0^\tau s^{\gamma-1} \| f(s,y(s))-f(s,x(s))) \| ds]$$
$$\leq R_2\sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[N+\sum_{i=1}^n c_i + \theta K\frac{\tau^\gamma}{\gamma}] \| y-x \|_c .$$

By replacing this estimate in $(**)$, we obtain

$$\begin{aligned}
\| \Psi(y)(t)-\Psi(x)(t) \| &\leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[N \| y-x \|_c + \sum_{i=1}^{n} c_i \| y-x \|_c + \theta\frac{\tau^\gamma}{\gamma}(K \| y-x \|_c \\
&\quad + R_1R_2 \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(N+\sum_{i=1}^{n} c_i + \theta K\frac{\tau^\gamma}{\gamma}) \| y-x \|_c)] \\
&\leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|[N+\sum_{i=1}^{n} c_i + \theta\frac{\tau^\gamma}{\gamma}K + \theta\frac{\tau^\gamma}{\gamma}R_1R_2 \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(N+\sum_{i=1}^{n} c_i + \theta K\frac{\tau^\gamma}{\gamma})] \| y-x \|_c \\
&\leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(N+\sum_{i=1}^{n} c_i + \theta\frac{\tau^\gamma}{\gamma}K)[1+\theta\frac{\tau^\gamma}{\gamma}R_1R_2 \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|] \| y-x \|_c .
\end{aligned}$$

Taking the supremum, we get

$$\| \Psi(y)(t)-\Psi(x)(t) \| \leq \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(N+\sum_{i=1}^{n} c_i + \theta\frac{\tau^\gamma}{\gamma}K)[1+\theta\frac{\tau^\gamma}{\gamma}R_1R_2 \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|] \| y-x \|_c .$$

Since $\sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|(N+\sum_{i=1}^{n} c_i + \theta\frac{\tau^\gamma}{\gamma}K)[1+\theta\frac{\tau^\gamma}{\gamma}R_1R_2 \sup_{t\in[0,\tau]} |\dot{S}(\frac{t^\gamma}{\gamma})|] < 1$, then Ψ is a contraction operator on B_δ. Hence, there exists a unique element $y_\delta(.) \in B_\delta$ such that $\Psi(y_\delta)(t) = y_\delta(t)$ for all $t \in [0,\tau]$. It remains to show that the integral solution y_δ is controllable. To this end, we have

$$\begin{aligned}
y_\delta(\tau) &= \Psi(y_\delta)(\tau) \\
&:= \dot{S}(\frac{\tau^\gamma}{\gamma})[y_0+g(y_\delta)] + \sum_{0<t_i<\tau} \dot{S}(\frac{\tau^\gamma - t_i^\gamma}{\gamma})h_i(y_\delta(t_i)) \\
&+ \lim_{\lambda\longrightarrow+\infty} \int_0^\tau s^{\gamma-1}\dot{S}(\frac{\tau^\gamma - s^\gamma}{\gamma})\lambda(\lambda-A)^{-1}\Big(f(s,y_\delta(s)) + Bu_{y_\delta}(s)\Big)ds \\
&= \dot{S}(\frac{\tau^\gamma}{\gamma})[y_0+g(y_\delta)] + \lim_{\lambda\longrightarrow+\infty} \int_0^\tau s^{\gamma-1}\dot{S}(\frac{\tau^\gamma - s^\gamma}{\gamma})\lambda(\lambda-A)^{-1}f(s,y_\delta(s))ds + \sum_{0<t_i<\tau} \dot{S}(\frac{\tau^\gamma - t_i^\gamma}{\gamma})h_i(y_\delta(t_i)) \\
&+ \lim_{\lambda\longrightarrow+\infty} \int_0^\tau s^{\gamma-1}\dot{S}(\frac{\tau^\gamma - s^\gamma}{\gamma})\lambda(\lambda-A)^{-1}Bu_{y_\delta}(s)ds \\
&= -W(y_\delta) + y_1 + \lim_{\lambda\longrightarrow+\infty} \int_0^\tau s^{\gamma-1}\dot{S}(\frac{\tau^\gamma - s^\gamma}{\gamma})\lambda(\lambda-A)^{-1}Bu_{y_\delta}(s)ds \\
&= -W(y_\delta) + y_1 + W(y_\delta) \\
&= y_1.
\end{aligned}$$

Thus Cauchy problem (3.3) is controllable on $[0,\tau]$. □

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