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Certain Recurrence Relations of Generalized M-Series and Its Application to Statistical Distribution

Dheerandra Shanker Sachan*, Dinesh Kumar, D. L. Suthar and K. S. Nisar

ABSTRACT: This paper addresses the prominent field of research concerning recurrence relations for special functions. Recognizing the significance of this area, we present a novel contribution by introducing new recurrence relations specifically designed for generalized M-Series. Through our analysis, we uncover intriguing connections between $(p+q+1)$ and $(p+q+2)$ parametric M-Series, which emerge as a direct consequence of these newly established recurrence relations. Furthermore, we leverage these relations to derive additional recurrence relations for both the Fox-Wright function and the generalized hypergeometric function, serving as noteworthy corollaries to our main findings. In addition, we explore the application of M-series in the establishment of distribution functions, ultimately enabling us to determine the Laplace transform of the density function. By delving into these interrelated topics, our research expands the understanding and knowledge base in this field, paving the way for further exploration and advancements.

Key Words: Mittag-Leffler function, generalized hypergeometric function, Fox-Wright function, generalized M-Series, statistical distribution, density function, Laplace transform.

Contents

1. Introduction

Special functions have distinct properties and applications and are often employed in areas such as mathematical physics, signal processing, number theory, quantum mechanics and more. These functions have been extensively studied and are typically denoted by specific symbol or names, depending on the context and the field of study . The generalized M-series (G-M-series) is a novel mathematical construct recently introduced to expand the understanding and applications of special functions. The significance of it, lies in the fact that, Mittag-Leffler (M-L) function and hypergeometric function are particular cases of it, which play very important role to solve various problems related to applied sciences. The definition of G-M-series [35] is explicated as

$$
\begin{aligned}
{}_pM_q^{\alpha,\beta}(z) &= {}_pM_q^{\alpha,\beta}\left[\begin{array}{c} c_1,\ldots,c_p; \\ d_1,\ldots,d_q; \end{array} z\right] \\
&= \sum_{k=0}^{\infty} \frac{(c_1)_k\ldots(c_p)_k}{(d_1)_k\ldots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)},
\end{aligned} \tag{1.1}
$$

* Corresponding author.
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where $z, \alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$; $(c_\ell)_k$ $(\ell = \overline{1,p})$ and $(d_\tau)_k$ $(\tau = \overline{1,q})$ are familiar Pochhammer symbols defined by, (see [36, pp. 4–6]),

$$\begin{aligned}(\kappa)_n &= \begin{cases} 1 & n = 0, \kappa \neq 0 \\ \kappa(\kappa+1)\cdots(\kappa+n-1) & n \in \mathbb{N}, \kappa \in \mathbb{C} \end{cases} \\ &= \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)}, \ n \in \mathbb{N} \cup \{0\}.\end{aligned} \tag{1.2}$$

wherein familiar Gamma function, denoted as $\Gamma(z)$, finds its definition as

$$\Gamma(z) = \int_0^\infty \xi^{z-1} e^{-\xi}\, \mathrm{d}\xi \quad (\Re(z) > 0). \tag{1.3}$$

The series (1.1) is defined when no (d_τ) $(\tau = \overline{1,q})$ parameter is a negative integer or zero. If any of the c_ℓ parameters within the numerator assume a value of either a negative integer or zero, the series (1.1) reaches its conclusion by converging to a polynomial expression in z. If $p \leq q$, the series (1.1) converges for all values of z; in the specific case, where $p = q+1$, the series converges for $|z| < \delta = \alpha^\alpha$ and diverges, if $p > q+1$. In the scenario, where $p = q+1$ and $|z| = \delta$, the series (1.1) may exhibit convergence, but this outcome is subject to additional conditions that rely on the parameters inherent in the series. To access comprehensive insights into the M-series, we recommend consulting the paper referenced as [35], which contains an extensive description.

By adjusting the parameters in equation (1.1), we can derive specific instances of the G-M-series. This includes the M-series [34], the M-L function [20], the generalized M-L function [24], and the generalized hypergeometric function [25]. These functions represent particular cases within the broader framework of the G-M-series. For example,

$$\begin{aligned}{}_pM_q^\alpha(z) &= {}_pM_q^\alpha\left[\begin{array}{c} c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{array} z\right] \\ &= \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_p)_k}{(d_1)_k \ldots (d_q)_k} \frac{z^k}{\Gamma(1+k\alpha)},\end{aligned} \tag{1.4}$$

$${}_0M_0^{\alpha,\beta}\left[\begin{array}{c} -; \\ -; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+k\alpha)} = E_{\alpha,\beta}(z), \tag{1.5}$$

$$(\Re(\alpha) > 0,\ \beta,\ z \in \mathbb{C})$$

$${}_1M_1^{\alpha,\beta}\left[\begin{array}{c} \rho; \\ 1; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{(\rho)_k}{(1)_k} \frac{z^k}{\Gamma(\beta+k\alpha)} = E_{\alpha,\beta}^{\rho}(z), \tag{1.6}$$

$$(\Re(\alpha) > 0, \Re(\beta) > 0\ \rho,\ z \in \mathbb{C})$$

$$\begin{aligned}{}_pM_q^{1,1}\left[\begin{array}{c} c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{array} z\right] &= \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_p)_k}{(d_1)_k \ldots (d_q)_k} \frac{z^k}{k!} \\ &= {}_pF_q\left[\begin{array}{c} c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{array} z\right], \quad |z| < 1.\end{aligned} \tag{1.7}$$

The relationship between the G-M-series in equation (1.1) and the Fox-Wright function is evident, as demonstrated below.

$${}_pM_q^{\alpha,\beta}\left[\begin{array}{c} c_1,\, \ldots,\, c_p\,; \\ d_1,\, \ldots,\, d_q\,; \end{array} z\right] = \frac{\prod\limits_{\ell=1}^{q} \Gamma(d_\tau)}{\prod\limits_{\tau=1}^{p} \Gamma(c_\ell)}\, {}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (c_1,\, 1),\, \ldots,\, (c_p,\, 1),(1,1)\,; \\ (d_1,\, 1),\, \ldots,\, (d_q,\, 1),(\beta,\alpha)\,; \end{array} z\right] \tag{1.8}$$

The Fox-Wright function, a widely recognized mathematical construct, is precisely defined by a formulation that has garnered substantial familiarity and academic recognition (see [40,41,42]; see also [7], ([37], [p. 21])).

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,\,\mathcal{B}_1),\,\ldots,\,(\alpha_p,\,\mathcal{B}_p)\,;\\(\beta_1,\,\mathcal{C}_1),\,\ldots,\,(\beta_q,\,\mathcal{C}_q)\,;\end{matrix}\;z\right]=\sum_{k=0}^{\infty}\frac{\prod\limits_{\varsigma=1}^{p}\Gamma\left(\alpha_\varsigma+\mathcal{B}_\varsigma\,k\right)}{\prod\limits_{\varsigma=1}^{q}\Gamma\left(\beta_\varsigma+\mathcal{C}_\varsigma\,k\right)}\frac{z^k}{k!}, \tag{1.9}$$

where $z\in\mathbb{C}$, the coefficients $\mathcal{B}_\varsigma(\varsigma=\overline{1,p}),\mathcal{C}_\varsigma(\varsigma=\overline{1,q})\in\mathbb{R}$ and $\alpha_\varsigma(\varsigma=\overline{1,p}),\beta_\varsigma(\varsigma=\overline{1,q})\in\mathbb{C}$. The expression of the convergence criteria for the series presented in equation (1.9) is articulated in the following manner (see ([14], [Theorem 1.5])): Let

$$\begin{aligned}\sigma &:=\sum_{\varsigma=1}^{q}\mathcal{C}_\varsigma-\sum_{\varsigma=1}^{p}\mathcal{B}_\varsigma,\\ \omega &:=\prod_{\varsigma=1}^{p}|\mathcal{B}_\varsigma|^{-\mathcal{B}_\varsigma}\prod_{\varsigma=1}^{q}|\mathcal{C}_\varsigma|^{\mathcal{C}_\varsigma},\\ \nu &:=\sum_{\varsigma=1}^{q}\beta_\varsigma-\sum_{\varsigma=1}^{p}\alpha_\varsigma+\frac{p-q}{2}.\end{aligned}\tag{1.10}$$

Then

(i) For all $z\in\mathbb{C}$, the series (1.9) is absolutely convergent when $\sigma>-1$;

(ii) For $|z|<\omega$, the series (1.9) is absolutely convergent when $\sigma=-1$;

(iii) For $|z|=\omega$, the series (1.9) is absolutely convergent when $\sigma=-1$ and $\Re(\nu)>\frac{1}{2}$.

The mathematical properties and practical applications of (1.1) have attracted the attention of several mathematicians who have conducted extensive research in this area. For instance, Chaurasia and Kumar [3] proposed a solution for a generalized fractional kinetic equation by incorporating the use of G-M-series. Suthar et al. [39] introduced a novel form of fractional integral operators, where the I-function serves as a kernel. They derived two theorems using these operators, which resulted in an intriguing formula for G-M-series. Additionally, they studied various properties associated with G-M-series. Sachan and Jaloree [27] explored different integral transforms, including Laplace transform, Beta transform, Hankel transform, and even fractional Fourier transform. Kumar and Singh [18] found an application for a specific product of G-M-series with the $\overline{H}$-function in electric circuit theory. Through the utilization of the q-analog of G-M-series and the incorporation of k-symmetric points, Najafzadeh [22] introduced a distinctive subclass of analytic functions. In the studies conducted by Gehlot [8], [9], notable progress was made in exploring the integrals and characteristics of M-series pertaining to the field of fractional calculus. Kumar and Saxena [17] delved into a comprehensive examination of specialized fractional calculus formulas designed explicitly for G-M-series. In their research, Kumar et al. [16] derived specific integration formulas of a generalized nature, encompassing M-series, Aleph-function, and Chebyshev Hermite polynomials as integral components. These derived formulas were subsequently applied to determine a comprehensive solution for a partial differential equation associated with the phenomenon of heat conduction. In the scholarly work presented by Najafzadeh [21], a novel subclass of univalent functions was introduced, establishing a profound relationship with M-series. This newly defined subclass incorporated the utilization of q-derivatives, which added a unique dimension to the study of these functions. Jain and Bhargava [11] derived intriguing integrals that incorporated the generalized Bessel-Maitland function and M-series, uncovering notable relationships and properties within these mathematical constructs. In their work, Kıymaz et al. [15] introduced a comprehensive generalization of M-Series through the utilization of an extended beta function. Furthermore, they investigated and established several crucial properties of this G-M-series, such as integral representation, derivative formulas, fractional derivative and integral formulas, as well as transformations including Laplace, Mellin, and Beta transforms. Sachan et al. [31] formulated and proved several theorems to derive specific single and double integral formulas linked to the G-M-series. Furthermore, by employing the Hadamard product of two analytic functions, they elegantly unified their principal results within the framework of the Hadamard product of two well-known functions. For more properties and application of (1.1), also see Sharma [34], Singh [38], Jain et al. [12], Chouhan and Saraswat [5], Khan et al. [13], Sachan et al. [30] [32], Chouhan and Khan [4], Saxena [33], Bansal et al. [2] etc.

Recurrence relations are powerful tool to understand the properties and behaviour of special functions. The applications of recurrence relations demonstrate the versatility and importance in the study and analysis of special functions.

The primary focus of this paper revolves around the establishment of specific recurrence relations that are associated with G-M-series, the Fox-Wright function, and the generalized hypergeometric function. (Notably, for the latest research on the recurrence relations of the Fox-Wright function and the generalized hypergeometric function, please refer to [28], [29]). Additionally, we present a distribution function linked to the M-series and subsequently obtain the Laplace transform of the density function.

2. Recurrence relations of generalized M-series

Within this section, our focus is directed towards the examination of novel recurrence relations associated with G-M-series, unveiling previously undiscovered connections and patterns within these mathematical constructs.

Theorem 2.1 *Let $z, \alpha, \beta, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(\alpha) > 0, \Re(c_\ell) > 1, \Re(\beta - \alpha) > 0, \Re(\beta) > -1,$ and $\Re(d_\tau) > 1$, then*

$$\begin{aligned}
{}_pM_q^{\alpha,\beta}\left[\begin{array}{c} c_1,\ldots,c_p; \\ d_1,\ldots,d_q; \end{array} z\right] &= \frac{z}{\beta}\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right){}_pM_q^{\alpha,\alpha+\beta}\left[\begin{array}{c} c_1+1,\ldots,c_p+1; \\ d_1+1,\ldots,d_q+1; \end{array} z\right] \\
&+\left(\frac{\beta-1}{z\beta}\right)\left(\frac{\prod_{\tau=1}^{q}(d_\tau-1)}{\prod_{\ell=1}^{p}(c_\ell-1)}\right)\left({}_pM_q^{\alpha,\beta-\alpha}\left[\begin{array}{c} c_1-1,\ldots,c_p-1; \\ d_1-1,\ldots,d_q-1; \end{array} z\right]-\frac{1}{\Gamma(\beta-\alpha)}\right) \\
&+\frac{1}{\Gamma(\beta+1)}
\end{aligned} \tag{2.1}$$

Theorem 2.2 *Let $z, \alpha, \beta, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(\alpha) > 0, \Re(\beta) > 0,$ and $\Re(d_\tau) > 0$, then*

$$\begin{aligned}
{}_pM_q^{\alpha,\beta}\left[\begin{array}{c} c_1,\ldots,c_p; \\ d_1,\ldots,d_q; \end{array} z\right] &= \left(\frac{z\beta}{\beta-1}\right)\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right){}_pM_q^{\alpha,\alpha+\beta}\left[\begin{array}{c} c_1+1,\ldots,c_p+1; \\ d_1+1,\ldots,d_q+1; \end{array} z\right] \\
&-\left(\frac{z^2}{\beta-1}\right)\left(\frac{\prod_{\ell=1}^{p}c_\ell(c_\ell+1)}{\prod_{\tau=1}^{q}d_\tau(d_\tau+1)}\right){}_pM_q^{\alpha,2\alpha+\beta}\left[\begin{array}{c} c_1+2,\ldots,c_p+2; \\ d_1+2,\ldots,d_q+2; \end{array} z\right]
\end{aligned}$$

$$-\frac{z}{(\beta-1)\Gamma(\alpha+\beta)}\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right)+\frac{1}{\Gamma(\beta)} \tag{2.2}$$

Theorem 2.3 *Let* $z,\alpha,\beta,c_\ell,d_\tau\in\mathbb{C}\,(\ell=\overline{1,p};\tau=\overline{1,q}),\Re(\alpha)>0,\Re(\beta)>-1,$ *and* $\Re(d_\tau)>0,$ *then*

$$\begin{aligned}{}_pM_q^{\alpha,\beta}\left[\begin{array}{c}c_1,\ldots,c_p;\\ d_1,\ldots,d_q;\end{array}z\right]&=(\beta-\alpha c_1)\,{}_pM_q^{\alpha,\beta+1}\left[\begin{array}{c}c_1,\ldots,c_p;\\ d_1,\ldots,d_q;\end{array}z\right]\\&+z\alpha(c_1+1)\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right){}_pM_q^{\alpha,\alpha+\beta+1}\left[\begin{array}{r}c_1+2,c_2+1,\ldots,c_p+1;\\ d_1+1,\ldots,d_q+1;\end{array}z\right]\\&+\frac{\alpha c_1}{\Gamma(\beta+1)}\end{aligned} \tag{2.3}$$

Theorem 2.4 *Let* $z,\alpha,\beta,c_\ell,d_\tau\in\mathbb{C}\,(\ell=\overline{1,p};\tau=\overline{1,q}),\Re(\alpha)>0,\Re(\beta)>1,\Re(c_1)>1,$ *and* $\Re(d_\tau)>0,$ *then*

$$\begin{aligned}\alpha(c_1-1)\,{}_pM_q^{\alpha,\beta}\left[\begin{array}{c}c_1,\ldots,c_p;\\ d_1,\ldots,d_q;\end{array}z\right]&={}_pM_q^{\alpha,\beta-1}\left[\begin{array}{r}c_1-1,c_2,\ldots,c_p;\\ d_1,\ldots,d_q;\end{array}z\right]\\&+[1-(\alpha+\beta)+\alpha c_1]\,{}_pM_q^{\alpha,\beta}\left[\begin{array}{r}c_1-1,c_2,\ldots,c_p;\\ d_1,\ldots,d_q;\end{array}z\right]\end{aligned} \tag{2.4}$$

Theorem 2.5 *Let* $z,\alpha,\beta,c_\ell,d_\tau\in\mathbb{C}\,(\ell=\overline{1,p};\tau=\overline{1,q}),\Re(\alpha)>0,\Re(\beta)>0,$ $m\in\mathbb{N}$ *and* $\Re(d_\tau)>0,$ *then*

$$\begin{aligned}{}_pM_q^{\alpha,\beta}\left[\begin{array}{c}c_1,\ldots,c_p;\\ d_1,\ldots,d_q;\end{array}z\right]=&z^m\left(\frac{\prod_{\ell=1}^{p}(c_\ell)_m}{\prod_{\tau=1}^{q}(d_\tau)_m}\right){}_pM_q^{\alpha,\alpha m+\beta}\left[\begin{array}{c}c_1+m,\ldots,c_p+m;\\ d_1+m,\ldots,d_q+m;\end{array}z\right]\\&+\sum_{k=0}^{m-1}\left(\frac{\prod_{\ell=1}^{p}(c_\ell)_k}{\prod_{\tau=1}^{q}(d_\tau)_k}\right)\frac{z^k}{\Gamma(\beta+k\alpha)}\end{aligned} \tag{2.5}$$

If we set $m=1,2,3,\cdots,$ in (2.5), we obtain following results.

$$\begin{aligned}{}_pM_q^{\alpha,\beta}\left[\begin{array}{c}c_1,\ldots,c_p;\\ d_1,\ldots,d_q;\end{array}z\right]=&z\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right){}_pM_q^{\alpha,\alpha+\beta}\left[\begin{array}{c}c_1+1,\ldots,c_p+1;\\ d_1+1,\ldots,d_q+1;\end{array}z\right]\\&+\frac{1}{\Gamma(\beta)}.\end{aligned} \tag{2.6}$$

$$
{}_pM_q^{\alpha,\beta}\left[\begin{array}{c} c_1,\ldots,c_p; \\ d_1,\ldots,d_q; \end{array} z\right] = z^2\left(\frac{\prod_{\ell=1}^{p}(c_\ell)_2}{\prod_{\tau=1}^{q}(d_\tau)_2}\right){}_pM_q^{\alpha,2\alpha+\beta}\left[\begin{array}{c} c_1+2,\ldots,c_p+2; \\ d_1+2,\ldots,d_q+2; \end{array} z\right] + \left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right)\frac{z}{\Gamma(\beta+\alpha)} + \frac{1}{\Gamma(\beta)}. \tag{2.7}
$$

$$
{}_pM_q^{\alpha,\beta}\left[\begin{array}{c} c_1,\ldots,c_p; \\ d_1,\ldots,d_q; \end{array} z\right] = z^3\left(\frac{\prod_{\ell=1}^{p}(c_\ell)_3}{\prod_{\tau=1}^{q}(d_\tau)_3}\right){}_pM_q^{\alpha,3\alpha+\beta}\left[\begin{array}{c} c_1+3,\ldots,c_p+3; \\ d_1+3,\ldots,d_q+3; \end{array} z\right] + \left(\frac{\prod_{\ell=1}^{p}(c_\ell)_2}{\prod_{\tau=1}^{q}(d_\tau)_2}\right)\frac{z^2}{\Gamma(\beta+2\alpha)} + \left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right)\frac{z}{\Gamma(\beta+\alpha)} + \frac{1}{\Gamma(\beta)}. \tag{2.8}
$$

and so on.

Proof: We demonstrate only (2.1). Denote by $\mathcal{R}_1$ the expression on the right-hand side of (2.1). By applying (1.1), we obtain

$$
\begin{aligned}
\mathcal{R}_1 = {} & \frac{1}{\beta}\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right)\sum_{k=0}^{\infty}\frac{(c_1+1)_k\ldots(c_p+1)_k}{(d_1+1)_k\ldots(d_q+1)_k}\frac{z^{k+1}}{\Gamma(\alpha+\beta+k\alpha)} \\
& + \left(\frac{\beta-1}{\beta}\right)\left(\frac{\prod_{\tau=1}^{q}(d_\tau-1)}{\prod_{\ell=1}^{p}(c_\ell-1)}\right)\left\{\sum_{k=0}^{\infty}\frac{(c_1-1)_k\ldots(c_p-1)_k}{(d_1-1)_k\ldots(d_q-1)_k}\frac{z^{k-1}}{\Gamma(\beta-\alpha+k\alpha)} \right. \\
& \left. - \frac{1}{z\Gamma(\beta-\alpha)}\right\} + \frac{1}{\Gamma(\beta+1)},
\end{aligned} \tag{2.9}
$$

upon substituting $k = k' - 1$ in the first summation and $k = k' + 1$ in the second summation within equation (2.9), while also discarding the prime notation on k, we obtain the following outcome.

$$\begin{aligned}
\mathcal{R}_1 &= \frac{1}{\beta}\sum_{k=1}^{\infty}\frac{(c_1)_k\dots(c_p)_k}{(d_1)_k\dots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)} \\
&+\left(\frac{\beta-1}{\beta}\right)\sum_{k=-1}^{\infty}\frac{(c_1)_k\dots(c_p)_k}{(d_1)_k\dots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)} \\
&-\left(\frac{\beta-1}{\beta}\right)\left(\frac{\prod_{\tau=1}^{q}(d_\tau-1)}{\prod_{\ell=1}^{p}(c_\ell-1)}\right)\frac{1}{z\Gamma(\beta-\alpha)}+\frac{1}{\Gamma(\beta+1)}, \\
&= \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{(c_1)_k\dots(c_p)_k}{(d_1)_k\dots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)} \\
&+\left(\frac{\beta-1}{\beta}\right)\sum_{k=0}^{\infty}\frac{(c_1)_k\dots(c_p)_k}{(d_1)_k\dots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)} \\
&= \sum_{k=0}^{\infty}\frac{(c_1)_k\dots(c_p)_k}{(d_1)_k\dots(d_q)_k}\frac{z^k}{\Gamma(\beta+k\alpha)} \\
&= {}_pM_q^{\alpha,\beta}\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} z\right].
\end{aligned}$$

We have successfully concluded the proof of (2.1). □

Likewise, the remaining relations (2.2)–(2.5) can be demonstrated following a similar approach. However, for brevity, we omit the specific details.

Remark 2.1 The relationship described by equation (2.6) has been previously established and documented in the literature (see [32]).

3. Relations between $(p+q+1)$ and $(p+q+2)$ parametric M-Series

By substituting $\beta = 1$ into equations (2.1), (2.3), and (2.5), we derive a set of relations between the $(p+q+1)$ and $(p+q+2)$ parametric M-Series described in equations (1.4) and (1.1). These relationships are presented as corollaries, which are outlined below.

Corollary 3.1 *Let* $z, \alpha, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(\alpha) > 0$ *and* $\Re(d_\tau) > 0$, *then*

$$ {}_pM_q^{\alpha}\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} z\right] = z\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right){}_pM_q^{\alpha,\alpha+1}\left[\begin{array}{c} c_1+1,\dots,c_p+1; \\ d_1+1,\dots,d_q+1; \end{array} z\right]+1 \tag{3.1}$$

Corollary 3.2 *Let* $z, \alpha, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(\alpha) > 0,$ *and* $\Re(d_\tau) > 0$, *then*

$$\begin{aligned}
{}_pM_q^{\alpha}\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} z\right] &= (1-\alpha c_1)\,{}_pM_q^{\alpha,2}\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} z\right] \\
&+ z\alpha(c_1+1)\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right){}_pM_q^{\alpha,\alpha+2}\left[\begin{array}{r} c_1+2,c_2+1,\dots,c_p+1; \\ d_1+1,\dots,d_q+1; \end{array} z\right] \\
&+ \alpha c_1
\end{aligned} \tag{3.2}$$

Corollary 3.3 *Let* $z, \alpha, c_\ell, d_\tau \in \mathbb{C}\,(\ell=\overline{1,p}; \tau=\overline{1,q}), \Re(\alpha)>0,\ m\in\mathbb{N}$ *and* $\Re(d_\tau)>0,$ *then*

$$\begin{aligned}{}_pM_q^{\alpha}\left[\begin{array}{c} c_1,\ldots,c_p; \\ d_1,\ldots,d_q;\end{array} z\right] =& z^m\left(\frac{\prod\limits_{\ell=1}^{p}(c_\ell)_m}{\prod\limits_{\tau=1}^{q}(d_\tau)_m}\right){}_pM_q^{\alpha,\alpha m+1}\left[\begin{array}{c} c_1+m,\ldots,c_p+m; \\ d_1+m,\ldots,d_q+m;\end{array} z\right] \\ &+\sum_{k=0}^{m-1}\left(\frac{\prod\limits_{\ell=1}^{p}(c_\ell)_k}{\prod\limits_{\tau=1}^{q}(d_\tau)_k}\right)\frac{z^k}{\Gamma(1+k\alpha)}\end{aligned} \tag{3.3}$$

4. Recurrence relations of Fox–Wright function

Within this section, our focus revolves around deriving recurrence relations for the Fox-Wright function. By utilizing the reduction of M-Series to the Fox-Wright function ${}_p\Psi_q$ as expressed in (1.9) through the application of (1.8) in the aforementioned theorems, we deduce the following corollaries.

Corollary 4.1 *Let* $z, \beta, c_\ell, d_\tau \in \mathbb{C}\,(\ell=\overline{1,p}; \tau=\overline{1,q}), \alpha>0, \Re(c_\ell)>1, \Re(\beta-\alpha)>0, \Re(\beta)>-1$ *then*

$$\begin{aligned}&{}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (c_1,1),\ldots,(c_p,1),(1,1); \\ (d_1,1),\ldots,(d_q,1),(\beta,\alpha);\end{array} z\right] \\ &= \frac{z}{\beta}\,{}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (c_1+1,1),\ldots,(c_p+1,1),(1,1); \\ (d_1+1,1),\ldots,(d_q+1,1),(\alpha+\beta,\alpha);\end{array} z\right] \\ &+\left(\frac{\beta-1}{z\beta}\right){}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (c_1-1,1),\ldots,(c_p-1,1),(1,1); \\ (d_1-1,1),\ldots,(d_q-1,1),(\beta-\alpha,\alpha);\end{array} z\right] \\ &-\left(\frac{\beta-1}{\beta\Gamma(\beta-\alpha)}\right)\left(\frac{\prod\limits_{\ell=1}^{p}\Gamma(c_\ell-1)}{\prod\limits_{\tau=1}^{q}\Gamma(d_\tau-1)}\right)+\left(\frac{\prod\limits_{\ell=1}^{p}\Gamma(c_\ell)}{\prod\limits_{\tau=1}^{q}\Gamma(d_\tau)}\right)\frac{1}{\Gamma(\beta+1)}\end{aligned} \tag{4.1}$$

Corollary 4.2 *Let* $z, \beta, c_\ell, d_\tau \in \mathbb{C}\,(\ell=\overline{1,p}; \tau=\overline{1,q}), \alpha>0, \Re(c_\ell)>0, \Re(\beta)>0$ *then*

$$\begin{aligned}&{}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (c_1,1),\ldots,(c_p,1),(1,1); \\ (d_1,1),\ldots,(d_q,1),(\beta,\alpha);\end{array} z\right] \\ &= \left(\frac{z\beta}{\beta-1}\right){}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (c_1+1,1),\ldots,(c_p+1,1),(1,1); \\ (d_1+1,1),\ldots,(d_q+1,1),(\alpha+\beta,\alpha);\end{array} z\right] \\ &-\left(\frac{z^2}{\beta-1}\right){}_{p+1}\Psi_{q+1}\left[\begin{array}{c} (c_1+2,1),\ldots,(c_p+2,1),(1,1); \\ (d_1+2,1),\ldots,(d_q+2,1),(2\alpha+\beta,\alpha);\end{array} z\right] \\ &-\left(\frac{\prod\limits_{\ell=1}^{p}\Gamma(c_\ell+1)}{\prod\limits_{\tau=1}^{q}\Gamma(d_\tau+1)}\right)\frac{z}{(\beta-1)\Gamma(\alpha+\beta)}+\left(\frac{\prod\limits_{\ell=1}^{p}\Gamma(c_\ell)}{\prod\limits_{\tau=1}^{q}\Gamma(d_\tau)}\right)\frac{1}{\Gamma(\beta)}\end{aligned} \tag{4.2}$$

Corollary 4.3 *Let* $z,\beta,c_\ell,d_\tau\in\mathbb{C}\,(\ell=\overline{1,p};\tau=\overline{1,q}),\alpha>0,\Re(c_\ell)>0,\Re(\beta)>-1$ *then*

$$\begin{aligned}
&{}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(c_1,1),\dots,(c_p,1),(1,1);\\(d_1,1),\dots,(d_q,1),(\beta,\alpha);\end{array} z\right]\\
&=(\beta-\alpha c_1)\;{}_{p+1}\Psi_{q+1}\left[\begin{array}{r}(c_1,1),\dots,(c_p,1),(1,1);\\(d_1,1),\dots,(d_q,1),(\beta+1,\alpha);\end{array} z\right]\\
&+\alpha z\;{}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(c_1+2,1),(c_2+1,1),\dots,(c_p+1,1),(1,1);\\(d_1+1,1),\dots,(d_q+1,1),(\alpha+\beta+1,\alpha);\end{array} z\right]\\
&+\left(\frac{\prod_{\ell=1}^{p}\Gamma(c_\ell)}{\prod_{\tau=1}^{q}\Gamma(d_\tau)}\right)\frac{\alpha c_1}{\Gamma(\beta+1)}
\end{aligned}\tag{4.3}$$

Corollary 4.4 *Let* $z,\beta,c_\ell,d_\tau\in\mathbb{C}\,(\ell=\overline{1,p};\tau=\overline{1,q}),\alpha>0,\Re(c_1)>1,\Re(\beta)>1$ *then*

$$\begin{aligned}
&\alpha\;{}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(c_1,1),\dots,(c_p,1),(1,1);\\(d_1,1),\dots,(d_q,1),(\beta,\alpha);\end{array} z\right]\\
&={}_{p+1}\Psi_{q+1}\left[\begin{array}{r}(c_1-1,1),(c_2,1),\dots,(c_p,1),(1,1);\\(d_1,1),\dots,(d_q,1),(\beta-1,\alpha);\end{array} z\right]\\
&+[1-(\alpha+\beta)+\alpha c_1]\;{}_{p+1}\Psi_{q+1}\left[\begin{array}{r}(c_1-1,1),(c_2,1),\dots,(c_p,1),(1,1);\\(d_1,1),\dots,(d_q,1),(\beta,\alpha);\end{array} z\right]
\end{aligned}\tag{4.4}$$

Corollary 4.5 *Let* $z,\beta,c_\ell,d_\tau\in\mathbb{C}\,(\ell=\overline{1,p};\tau=\overline{1,q}),\alpha>0,\Re(c_\ell)>0$ *then*

$$\begin{aligned}
&{}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(c_1,1),\dots,(c_p,1),(1,1);\\(d_1,1),\dots,(d_q,1),(\beta,\alpha);\end{array} z\right]\\
&=z^m\;{}_{p+1}\Psi_{q+1}\left[\begin{array}{r}(c_1+m,1),\dots,(c_p+m,1),(1,1);\\(d_1+m,1),\dots,(d_q+m,1),(\alpha m+\beta,\alpha);\end{array} z\right]\\
&+\sum_{k=0}^{m-1}\left(\frac{\prod_{\ell=1}^{p}\Gamma(c_\ell+k)}{\prod_{\tau=1}^{q}\Gamma(d_\tau+k)}\right)\frac{z^k}{\Gamma(\beta+k\alpha)}
\end{aligned}\tag{4.5}$$

5. Recurrence relations of generalized hypergeometric function

When we assign the values $\alpha=1$ and $\beta=2$ to (2.1), (2.2), and (2.4), we attain the resulting recurrence relations concerning the generalized hypergeometric function.

Corollary 5.1 *Let* $z,c_\ell,d_\tau\in\mathbb{C}\,(\ell=\overline{1,p};\tau=\overline{1,q}),\Re(c_\ell)>1$ *and* $\Re(d_\tau)>0$*, then*

$$\begin{aligned}
&{}_{p+1}F_{q+1}\left[\begin{array}{c}c_1,\dots,c_p,1;\\d_1,\dots,d_q,2;\end{array} z\right]=\frac{z}{2}\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right){}_{p+1}F_{q+1}\left[\begin{array}{c}c_1+1,\dots,c_p+1,1;\\d_1+1,\dots,d_q+1,3;\end{array} z\right]\\
&+\frac{1}{z}\left(\frac{\prod_{\tau=1}^{q}(d_\tau-1)}{\prod_{\ell=1}^{p}(c_\ell-1)}\right)\left({}_pF_q\left[\begin{array}{c}c_1-1,\dots,c_p-1;\\d_1-1,\dots,d_q-1;\end{array} z\right]-1\right)+1
\end{aligned}\tag{5.1}$$

Corollary 5.2 *Let* $z, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(c_\ell) > 0$ *and* $\Re(d_\tau) > 0$, *then*

$$\begin{aligned}
&{}_{p+1}F_{q+1}\left[\begin{array}{c} c_1, \ldots, c_p, 1; \\ d_1, \ldots, d_q, 2; \end{array} z\right] \\
&= \frac{z}{2}\left(\frac{\prod\limits_{\ell=1}^{p}(c_\ell)}{\prod\limits_{\tau=1}^{q}(d_\tau)}\right){}_{p+1}F_{q+1}\left[\begin{array}{c} c_1+1, \ldots, c_p+1, 1; \\ d_1+1, \ldots, d_q+1, 3; \end{array} z\right] \\
&- \left(\frac{z^2}{6}\right)\left(\frac{\prod\limits_{\ell=1}^{p}c_\ell(c_\ell+1)}{\prod\limits_{\tau=1}^{q}d_\tau(d_\tau+1)}\right){}_{p+1}F_{q+1}\left[\begin{array}{c} c_1+2, \ldots, c_p+2, 1; \\ d_1+2, \ldots, d_q+2, 4; \end{array} z\right] \\
&\qquad\qquad - \frac{z}{2}\left(\frac{\prod\limits_{\ell=1}^{p}(c_\ell)}{\prod\limits_{\tau=1}^{q}(d_\tau)}\right) + 1
\end{aligned} \tag{5.2}$$

Corollary 5.3 *Let* $z, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(c_1) > 1$ *and* $\Re(d_\tau) > 0$, *then*

$$\begin{aligned}
&2{}_pF_q\left[\begin{array}{c} c_1-1, c_2, \ldots, c_p; \\ d_1, \ldots, d_q; \end{array} z\right] \\
&= (c_1-1)\,{}_{p+1}F_{q+1}\left[\begin{array}{c} c_1, \ldots, c_p, 1; \\ d_1, \ldots, d_q, 2; \end{array} z\right] \\
&- (c_1-2)\,{}_{p+1}F_{q+1}\left[\begin{array}{c} c_1-1, c_2, \ldots, c_p, 1; \\ d_1, \ldots, d_q, 2; \end{array} z\right]
\end{aligned} \tag{5.3}$$

If we set $\alpha = \beta = 1$, in (2.3) and (2.5), we obtain following interesting results.

Corollary 5.4 *Let* $z, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(c_\ell) > 0$ *and* $\Re(d_\tau) > 0$, *then*

$$\begin{aligned}
&{}_pF_q\left[\begin{array}{c} c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{array} z\right] = (1-c_1)\;{}_{p+1}F_{q+1}\left[\begin{array}{c} c_1, \ldots, c_p, 1; \\ d_1, \ldots, d_q, 2; \end{array} z\right] \\
&+ (c_1+1)\frac{z}{2}\left(\frac{\prod\limits_{\ell=1}^{p}(c_\ell)}{\prod\limits_{\tau=1}^{q}(d_\tau)}\right){}_{p+1}F_{q+1}\left[\begin{array}{c} c_1+2, c_2+1, \ldots, c_p+1, 1; \\ d_1+1, \ldots, d_q+1, 3; \end{array} z\right] + c_1
\end{aligned} \tag{5.4}$$

Corollary 5.5 *Let* $z, c_\ell, d_\tau \in \mathbb{C}\,(\ell = \overline{1,p}; \tau = \overline{1,q}), \Re(c_\ell) > 0$, $m \in \mathbb{N}$ *and* $\Re(d_\tau) > 0$, *then*

$$\begin{aligned}
&{}_pF_q\left[\begin{array}{c} c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{array} z\right] = \frac{z^m}{\Gamma(m+1)}\left(\frac{\prod\limits_{\ell=1}^{p}(c_\ell)_m}{\prod\limits_{\tau=1}^{q}(d_\tau)_m}\right){}_{p+1}F_{q+1}\left[\begin{array}{c} c_1+m, \ldots, c_p+m, 1; \\ d_1+m, \ldots, d_q+m, m+1; \end{array} z\right] \\
&\qquad + \sum_{k=0}^{m-1}\left(\frac{\prod\limits_{\ell=1}^{p}(c_\ell)_k}{\prod\limits_{\tau=1}^{q}(d_\tau)_k}\right)\frac{z^k}{k!}
\end{aligned} \tag{5.5}$$

If we set $m = 1, 2, 3, \cdots$, in (5.5), we obtain following results.

$$ {}_pF_q\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} z\right] = z\left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right) {}_{p+1}F_{q+1}\left[\begin{array}{c} c_1+1,\dots,c_p+1,1; \\ d_1+1,\dots,d_q+1,2; \end{array} z\right] + 1 \tag{5.6} $$

$$ \begin{aligned} {}_pF_q\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} z\right] &= \frac{z^2}{2}\left(\frac{\prod_{\ell=1}^{p}(c_\ell)_2}{\prod_{\tau=1}^{q}(d_\tau)_2}\right) {}_{p+1}F_{q+1}\left[\begin{array}{c} c_1+2,\dots,c_p+2,1; \\ d_1+2,\dots,d_q+2,3; \end{array} z\right] \\ &+ \left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right) z + 1 \end{aligned} \tag{5.7} $$

$$ \begin{aligned} {}_pF_q\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} z\right] &= \frac{z^3}{3!}\left(\frac{\prod_{\ell=1}^{p}(c_\ell)_3}{\prod_{\tau=1}^{q}(d_\tau)_3}\right) {}_{p+1}F_{q+1}\left[\begin{array}{c} c_1+3,\dots,c_p+3,1; \\ d_1+3,\dots,d_q+3,4; \end{array} z\right] \\ &+ \left(\frac{\prod_{\ell=1}^{p}(c_\ell)_2}{\prod_{\tau=1}^{q}(d_\tau)_2}\right) \frac{z^2}{2!} + \left(\frac{\prod_{\ell=1}^{p}(c_\ell)}{\prod_{\tau=1}^{q}(d_\tau)}\right) z + 1 \end{aligned} \tag{5.8} $$

and so on.

6. Application of M-series to the statistical distribution

Statistical distributions describe the likelihood of different outcomes or events in statistical dataset and provide a powerful framework for understanding and making inferences about data in various fields of study.

Many authors have studied distribution function (for definition, see [19]) using M-L type functions, see [23,6,10,1,26]. In this section, we give distribution function involving M-series. Further, we obtained Laplace transform of density function.

M-Series statistical distribution: The distribution function provides a definition for a statistical distribution that is connected to the M-Series in (1.4) as follows:

$$ \begin{aligned} F_y(y) &= 1 - {}_pM_q^{\alpha}\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} -y^{\alpha}\right], \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(c_1)_k\cdots(c_p)_k}{(d_1)_k\cdots(d_q)_k} \frac{y^{\alpha k}}{\Gamma(1+k\alpha)}, \quad 0<\alpha\le 1,\ y>0 \end{aligned} \tag{6.1} $$

and $F_y(y) = 0$ for $y \le 0$.

Theorem 6.1 *Let $y, \alpha \in \mathbb{R}^+$ with $0<\alpha\le 1$, $y>0$ and $\Re(c_\ell) > 0\,(\ell = \overline{1,p}), \Re(d_\tau) > 0\,(\tau = \overline{1,q})$. If*

$$ F_y(y) = 1 - {}_pM_q^{\alpha}\left[\begin{array}{c} c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} -y^{\alpha}\right], $$

then the density function $f(y)$ of distribution function $F_y(y)$ is given by

$$f(y) = y^{\alpha-1} \left(\frac{\prod_{\ell=1}^{p} (c_\ell)}{\prod_{\tau=1}^{q} (d_\tau)} \right) {}_pM_q^{\alpha,\alpha} \left[\begin{array}{c} c_1+1,\dots,c_p+1; \\ d_1+1,\dots,d_q+1; \end{array} -y^\alpha \right]. \tag{6.2}$$

Proof: By taking the derivative of equation (6.1) with respect to y, we obtain the expression for the density function $f(y)$ as follows:

$$\begin{aligned} f(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_y(y) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{\alpha k y^{\alpha k-1}}{\Gamma(1+k\alpha)} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (c_1)_k \cdots (c_p)_k}{(d_1)_k \cdots (d_q)_k} \frac{y^{\alpha k-1}}{\Gamma(k\alpha)}, \end{aligned}$$

replacing k by $k+1$ and using formula $(c)_{k+1} = c(c+1)_k$, we easily obtain

$$f(y) = y^{\alpha-1} \left(\frac{\prod_{\ell=1}^{p} (c_\ell)}{\prod_{\tau=1}^{q} (d_\tau)} \right) {}_pM_q^{\alpha,\alpha} \left[\begin{array}{c} c_1+1,\dots,c_p+1; \\ d_1+1,\dots,d_q+1; \end{array} -y^\alpha \right],$$

where ${}_pM_q^{\alpha,\beta}(z)$ is G-M-series defined by (1.1). Thus, the proof of Theorem 6.1 is concluded. □

Theorem 6.2 *Let $\alpha \in \mathbb{R}^+$ with $0 < \alpha \le 1$ and $\Re(c_\ell) > 0\,(\ell = \overline{1,p}), \Re(d_\tau) > 0\,(\tau = \overline{1,q})$. Then the Laplace transform of the density function represented by equation (6.2) is given by*

$$\mathscr{L}\{f(x)\} = 1 - {}_{p+1}F_q \left[\begin{array}{c} 1, c_1,\dots,c_p; \\ d_1,\dots,d_q; \end{array} -t^{-\alpha} \right], \quad |-t^{-\alpha}| < 1. \tag{6.3}$$

Proof: In order to establish the validity of equation (6.3), we initiate the proof by considering the Laplace transform formula.

$$\mathscr{L}\{f(x)\} = f(t) = \int_0^\infty e^{-tx} f(x)\mathrm{d}x, \tag{6.4}$$

using (6.2) in (6.4) , we have

$$\mathscr{L}\{f(x)\} = \int_0^\infty e^{-tx} x^{\alpha-1} \left(\frac{\prod_{\ell=1}^{p} (c_\ell)}{\prod_{\tau=1}^{q} (d_\tau)} \right) {}_pM_q^{\alpha,\alpha} \left[\begin{array}{c} c_1+1,\dots,c_p+1; \\ d_1+1,\dots,d_q+1; \end{array} -x^\alpha \right] \mathrm{d}x,$$

by rewriting the M-series in its series form and exchanging the order of integration and summation, we obtain the following expression.

$$\mathscr{L}\{f(x)\} = \left(\frac{\prod_{\ell=1}^{p} (c_\ell)}{\prod_{\tau=1}^{q} (d_\tau)} \right) \sum_{k=0}^{\infty} \frac{(-1)^k (c_1+1)_k \cdots (c_p+1)_k}{(d_1+1)_k \cdots (d_q+1)_k \Gamma(\alpha+k\alpha)} \int_0^\infty e^{-tx} x^{\alpha k+\alpha-1} \mathrm{d}x,$$

setting $tx = s$, we have

$$\mathscr{L}\{f(x)\} = \left(\frac{\prod_{\ell=1}^{p} (c_\ell)}{\prod_{\tau=1}^{q} (d_\tau)} \right) \sum_{k=0}^{\infty} \frac{(-1)^k (c_1+1)_k \cdots (c_p+1)_k \, t^{-\alpha k - \alpha}}{(d_1+1)_k \cdots (d_q+1)_k \, \Gamma(\alpha + k\alpha)} \int_0^\infty e^{-s} s^{\alpha k + \alpha - 1} \mathrm{d}s,$$

using (1.3) and replacing k by $k-1$, we obtain

$$\begin{aligned}\mathscr{L}\{f(x)\} &= -\sum_{k=1}^{\infty} \frac{(c_1)_k \cdots (c_p)_k \, (-t^{-\alpha})^k}{(d_1)_k \cdots (d_q)_k} \\ &= 1 - {}_{p+1}F_q \left[\begin{matrix} 1, c_1, \ldots, c_p; \\ d_1, \ldots, d_q; \end{matrix} \; -t^{-\alpha} \right],\end{aligned}$$

with that, the proof of Theorem 6.2 is concluded. □

7. Conclusion

This research paper focuses on the development of new recurrence relations for G-M-series. By deriving these recurrence relations, the authors uncover interesting relationships between (p + q + 1) and (p + q + 2) parametric M-Series. These relationships emerge as a direct consequence of the derived recurrence relations. Furthermore, the authors extend the application of these recurrence relations to obtain additional recurrence relations for the Fox-Wright function and the generalized hypergeometric function. These functions are also important special functions encountered in various mathematical analyses.

The paper also addresses the establishment of a distribution function associated with the M-Series. Additionally, the authors derive the Laplace transform of the density function, which provides valuable insights into the behavior of the distribution function. Overall, the main objective of this research paper is to contribute to the understanding of recurrence relations concerning G-M-series, the Fox-Wright function, and the generalized hypergeometric function. The findings in this paper provide new insights into the relationships and properties of these special functions. The establishment of the distribution function and Laplace transform further enriches the analysis and application of the M-Series.

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Dheerandra Shanker Sachan,
Department of Mathematics,
St. Mary's PG College, Vidisha-464001, (M.P.),
India.
E-mail address: sachan.dheerandra17@gmail.com

and

D. Kumar,
Department of Applied Sciences,
College of Agriculture-Jodhpur,
Agriculture University Jodhpur, Jodhpur–342304 (Raj.),
India
E-mail address: dinesh_dino03@yahoo.com

and

D.L. Suthar,
Department of Mathematics,
Wollo University, P.O. Box 1145, Dessie,
Ethiopia.
E-mail address: dlsuthar@gmail.com

and

K.S. Nisar,
Department of Mathematics,
College of Science and Humanities,
Prince Sattam Bin Abdulaziz University, Alkharj, 11942,
Saudi Arabia.
E-mail address: ksnisar1@gmail.com