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A New Analytical Study of Measurable Functions via Approach Structure

Ali A. Hammood, Arkan J. Mohammed and Boushra Y. Hussein*

ABSTRACT: This study introduces approach measures and semi approach measure, a new form of measure based on approach structure. Then we presented a new definition of measurable functions and studied their most important properties and the relationship between this new concept and open function.

Key Words: Approach space, approach measure, measurable function, approach measurable function, open function.

Contents

1. Introduction

In 2022 Abbas and Hussein [2], studied results of space of normed approach, in 2023, Abbas and Hussein [10] announced new results for completion normed approach space, in 2022 [1], they gave a new kind of topological vector space, in 2014 [6] Ash, real analysis and probability, Abd and Hussein [3] used proximit structre to find new topological vector space with new results in this manner, in 1994, Baekeland and Lowen [7] explained the measures of Lindelof and separability in approach spaces, In 2011, Hussein [12] discovered an analogous (super/sub) martingale measure on L^p-space, whereas Hussein [13] offered one in 2010, In 2007, there was a missing link in the topology - uniformity-metric triad. Hussein [8], In 2020, Hussein [9] investigated the equivalent locally Martingale measure for the deflator process in ordered Banach algebra, in 2023 Saeed and Hussein [19], they studied further new Results of Normed Approach Space, and in 2023 Hussein and Wshayeh [11] found new results about state of measurable function space in symmetric δ-Banach algebra, in 2023 [16] Neamah and Hussein study δ -character in symmetric t^ω-δ- Banach - algebra, in [4] Abed Ali and Hussein study New kind of topological vector space Via Proximit structure, In 2022, Neamah and Hussein [17] published new results on Completion t^ω Normed Approach Space, in [15] they discuss results of a new type topological vector space via t^ω - approach structure, Lowen [14] presented approach spaces as a top super category, Taylor [20] proposed measure and integration in 1973. This paper consists of four sections: In section 1, we provided an introduction for the reader. In section 2, we gave the key definitions and concepts. In section 3, we offer additional definitions, examples, and theorems for our study on approach measure and semi-approach measure. In Section 4, we introduced new concepts (approach measurable functions) and study properties of this concept.

* Corresponding author

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2. Definitions and Preliminary

Definition 2.1 *[4,5] If $(\Omega, F), (\Omega', F')$ are a measurable spaces, a function $f : \Omega \to \Omega'$ is called measurable function when $f^{-1}(\mathcal{A}) \in F, \forall \mathcal{A} \in F'$.*

Definition 2.2 *[5] If (Ω, F) is measurable space and $\breve{A}$ is collection of sub sets of Ω, we call F generated by $\breve{A}$ if F is the smallest σ- filed containing $\breve{A}$.*

Definition 2.3 *[5] Let $(¥, \mathcal{T})$ be topological space, the σ-algebra of subsets of ¥ which is generated by $\mathcal{T}$ named Boral σ- filed of ¥ denoted by $\beta(¥)$.*

Definition 2.4 *[8] if $\Omega' = \mathfrak{R}^n$, $F' = \beta(\mathfrak{R}^n)$, where $\beta(\mathfrak{R}^n)$ Boral σ- filed of $\mathfrak{R}^n$ the function $f : \Omega \to \mathfrak{R}^n$ is said to be Boral measurable function on (Ω, F) if f is measurable function with respect to F and β.*

Definition 2.5 *[21] Let $(¥_1, \mathcal{T}_1), (¥_2, \mathcal{T}_2)$ be two topological spaces, a function $f : (¥_1, \mathcal{T}_1) \to (¥_2, \mathcal{T}_2)$ is open function if $f(\mathcal{T})$ is open sub set of $¥_2$ for each $\mathcal{T}$ open sub sets of $¥_1$.*

Proposition 2.1 *[5] If $\breve{G}$ is a non-empty set and F is σ –field on $\breve{G}$ then $F \cap \mathcal{A}$ is σ – field on $\mathcal{A}$ for each $\mathcal{A} \subseteq \breve{G}$.*

Proposition 2.2 *[18] Let R be a set of all real numbers, the Boral σ- field on R is generated by the following of collection of intervals: $\{(-\infty, c') : c' \in R\}, \{(-\infty, c'] : c' \in R\}, \{\lceil c', \infty\rceil : c' \in R\}, \{[c', \infty) : c' \in R\}$*

3. Approach Measure and Semi Approach Measure

Definition 3.1 *If ℵ a non-empty set and F a σ-filed. The jointly set function $\mu_\delta : F \times 2^{\aleph} \to [0, \infty]$ is called approach measure on ℵ if it is satisfies the following conditions:*

1. *$\mu_\delta(\emptyset, \mathcal{B}) = 0$ $\mathcal{B} \in 2^{\aleph}$.*
2. *$\mu_\delta(\mathcal{A}, \mathcal{B}) = \infty$ if $\mathcal{A} = \mathcal{B}^c \mathcal{A} \in F, \mathcal{B} \in 2^{\aleph}$ $\emptyset \neq \mathcal{A}$*
3. *$\mu_\delta\ (\bigcup_{n=1}^{\infty} \mathcal{A}_n\ ,\ \mathcal{B}) = \sum_{n-1}^{\infty} \mu_\delta(\mathcal{A}_n, \mathcal{B})$ where $\mathcal{A}_n$ is a sequence of disjoint sets in F.*

Then the triple $(\aleph, F, \mu_\delta)$ is said to be approach measure space and denoted by AMS

Definition 3.2 *Let ℵ be a non-empty set and F be σ-filed. The jointly set function $\mu_\delta : F \times 2^{\aleph} \to [0, \infty]$ is called semi approach measure on ℵ if it is satisfies the following conditions:*

1. *$\mu_\delta(\emptyset, \mathcal{B}) = 0$ $B \in 2^{\aleph}$*
2. *$\mu_\delta(\mathcal{A}, \mathcal{B}) = \infty$ if $\mathcal{A} \subseteq \mathcal{B}^c$ $\mathcal{A} \in F, \mathcal{B} \in 2^{\aleph}$ $\emptyset \neq \mathcal{A}$*
3. *$\mu_\delta(\bigcup_{n=1}^{\infty} \mathcal{A}_n, \mathcal{B}) = \sum_{n=1}^{\infty} \mu_S(\mathcal{A}_n, \mathcal{B})$ for each sequence of disjoint sets $\{\mathcal{A}_n\}_{n=1}^{\infty}$ inF $, \mathcal{B} \in 2^{\aleph}$. The triple $(\aleph, F, \mu_\delta)$ said to be semi approach measure space and denoted by SAMS*

Example 3.1 Let ℵ be a non-empty set and F be σ-filed , a function $\mu_\delta : F \times 2^{\aleph} \to [0, \infty]$ is defined by:

$$\mu_\delta(\mathcal{A}, \mathcal{B}) = \begin{cases} 0 & \mathcal{A} = \emptyset \\ \infty & \mathcal{A} \neq \emptyset \end{cases}$$

Then μ_δ is approach measure We'll call it is standard approach measure and $(\aleph, F, \mu_\delta)$ is standard AMS.

Solution: True proof μ_δ is approach measure:

1. Let $\mathcal{B} \in 2^{\aleph}, \mu_\delta(\emptyset, \mathcal{B}) = 0$ (by definition of (μ_δ)).
2. Let $\mathcal{A}, \mathcal{B} \in 2^{\aleph}$ where $\mathcal{A} \neq \emptyset$ such that $\mathcal{A} = \mathcal{B}^c$

   By definition of (μ_δ) we get $\mu_\delta(\mathcal{A}, \mathcal{B}) = \infty$

3. If $\{\mathcal{A}_n\}_{n=1}^{\infty}$ a sequence of disjoint sets in F and $\bigcup_{n=1}^{\infty} \mathcal{A}_n = \emptyset$

Then, $\mu_\delta(\bigcup_{n=1}^{\infty} \mathcal{A}_n, \mathcal{B}) = \mu_\delta(\emptyset, \mathcal{B}) = 0$ (by definition of the function (μ_δ))
Since $\bigcup_{n=1}^{\infty} \mathcal{A}_n = \emptyset, then \mathcal{A}_n = \emptyset$ for $n = 1, 2, \dots$
That is $\sum_{n=1}^{\infty} \mu_\delta(\mathcal{A}_n, \mathcal{B}) = 0 + 0 + 0 + \dots$

$$\sum_{n=1}^{\infty} \mu_\delta(\mathcal{A}_n, \mathcal{B}) = 0$$

Then,

$$\mu_\delta(\bigcup_{n=1}^{\infty} \mathcal{A}_n,\ B) = \sum_{n=1}^{\infty} \mu_\delta(\mathcal{A}_n, \mathcal{B})$$

If $\bigcup_{n=1}^{\infty} \mathcal{A}_n \neq \emptyset$ that is $\mu_\delta(\bigcup_{n=1}^{\infty} \mathcal{A}_n, \mathcal{B}) = \infty$ (by definition of the function (μ_δ))

Since $\bigcup_{n=1}^{\infty} \mathcal{A}_n \neq \emptyset$, then there exist at least one set say $\mathcal{A}_t \in \{\mathcal{A}_n\}_{n=1}^{\infty}$ such that $\mathcal{A}_t \neq \emptyset$ $(1 \leq t < \infty)$ so that $\mu_\delta(A_t, \mathcal{B}) = \infty$ and we have

$$\sum_{n=1}^{\infty} \mu_\delta(\mathcal{A}_n, \mathcal{B}) = \mu_\delta\ (\mathcal{A}_1, \mathcal{B}\ ) + \mu_\delta(\mathcal{A}_2, \mathcal{B}) + \dots + \mu_\delta(\mathcal{A}_t, \mathcal{B}) + \dots$$

Hence,

$$\sum_{n=1}^{\infty} \mu_\delta(\mathcal{A}_n, B) = \mu_\delta\ (\mathcal{A}_1, \mathcal{B}) + \mu_\delta\ (\mathcal{A}_2, \mathcal{B}) + \dots + \infty + \dots,$$

that is

$$\sum_{n=1}^{\infty} \mu_\delta(\mathcal{A}_n, \mathcal{B}) = \infty \text{ Then } \mu_\delta(\bigcup_{n=1}^{\infty} \mathcal{A}_n, \mathcal{B}) = \sum_{n=1}^{\infty} \mu_\delta(\mathcal{A}_n, \mathcal{B})$$

Therefore, μ_δ is approach measure and $(\aleph, F,\ \mu_\delta)$ approach measure space. □

4. Approach Measurable Function

Definition 4.1 *Let* $(\aleph_1, F_1,\ \mu_{\delta 1}), (\aleph_2, F_2,\ \mu_{\delta 2})$ *be two AMS . A function* $\mathcal{Q} : (\aleph_1, F_1, \mu_{\delta 1}) \to (\aleph_2, F_2, \mu_{\delta 2})$ *is called approach measurable function if it satisfies the following conditions :*

1. $\mathcal{Q}(\mathcal{A}) \in F_2$ *for all* $\mathcal{A} \in F_1$.

2. $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ *for each* $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$

Example 4.1

1. Let $(\aleph, F, \mu_\delta)$ be AMS and $\mathcal{Q} : (\aleph, F, \mu_\delta) \to (\aleph, F, \mu_\delta)$ the identity function then $\mathcal{Q}$ is approach measurable function.

2. Let $(\aleph_1, F_1, \mu_{\delta 1}), (\aleph_2, 2^{\aleph_2},\ \mu_{\delta 2})$ be two standard AMS , then any constant function $\mathcal{Q} : (\aleph_1, F_1, \mu_{\delta 1}) \to (\aleph_2, 2^{\aleph_2}, \mu_{\delta 2})$ is approach measurable function.

3. Let $(\aleph_1, F_1,\ \mu_{\delta 1})$, $(\mathfrak{R}, 2^{\mathfrak{R}}, \mu_{\delta 2})$ be two standard AMS, let $\breve{E}$ be a sub set of $\aleph_1$, a function $\mathcal{Q}\breve{E}$: $(\aleph_1, F_1, \mu_{\delta 1}) \to (\mathfrak{R}, 2^{\mathfrak{R}}, \mu_{\delta 2})$ is defined as follows :

$$\mathcal{Q}\breve{E}(x) = \begin{cases} 1 & x \in \breve{E} \\ 0 & x \notin \breve{E} \end{cases}$$

then $\mathcal{Q}\breve{E}$ is approach measurable function.

Proof: true prove (1):

i. Let $\mathcal{A} \in F, \mathcal{Q}(\mathcal{A}) = \mathcal{A}$ since $\mathcal{Q}$ is identity, then $\mathcal{Q}(\mathcal{A}) \in F$

ii. Let $\mathcal{A} \in F, \mathcal{B} \in 2^{\aleph}$ $\mu_\delta(\mathcal{A}, \mathcal{B}) = \mu_\delta(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$ since $\mathcal{Q}$ is identity

Therefor $\mu_\delta(\mathcal{A}, \mathcal{B}) \leq \mu_\delta(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$

Then $\mathcal{Q}$ is approach measurable function. By (i) , (ii)

Now true prove (2)

i. Let $\mathcal{A} \in F_1, \mathcal{Q}(\mathcal{A}) = \{c\}$ Since $\{c\} \in 2^{\aleph_2}$ then $\mathcal{Q}(\mathcal{A}) \in 2^{\aleph_2}$

ii. Let $\mathcal{A} \in F, \mathcal{B} \in 2^{\aleph}$
If $\mathcal{A} = \emptyset$ then $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) = 0$ since $\mu_{\delta 1}$ standard approach measure
either $\mathcal{Q}(\mathcal{A}) = \emptyset$ then $\mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B})) = 0$ since $\mu_{\delta 2}$ standard approach measure
Or $\mathcal{Q}(\mathcal{A}) = \{c\}$ then $\mu_{\delta 2}(\{c\}, \mathcal{Q}(\mathcal{B})) = \infty$
Therefor $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$
If $\mathcal{A} \neq \emptyset$, then $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) = \infty$ since $\mu_{\delta 1}$ standard approach measure
$\mathcal{Q}(\mathcal{A}) = \{c\}$ then $\mu_{\delta 2}(\{c\}\, \mathcal{Q}(\mathcal{B})) = \infty$ since $\mu_{\delta 2}$ standard approach measure
therefor $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$, for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$
then $\mathcal{Q}$ is approach measurable function. By (i) , (ii) □

Now true prove (3):

i. Let $\mathcal{A} \in F_1$.
Either $\mathcal{A} \cap \breve{E} = \emptyset$.
Let $x \in \mathcal{A}$ *implies* $x \notin \breve{E}$.
Then $\mathcal{Q}\breve{E}(\mathcal{A}) = \{0\}$
Since $\{0\} \in 2^{\mathfrak{R}}$ then $\mathcal{Q}\breve{E}(\mathcal{A}) \in 2^{\mathfrak{R}}$
Or $\mathcal{A} \cap \breve{E} \neq \emptyset$
Let $x \in A$ implies
Either $x \in \breve{E}$.
Then $\mathcal{Q}\breve{E}(\mathcal{A}) = \{1\}$
Since $\{1\} \in 2^{\mathfrak{R}}$ then $\mathcal{Q}\breve{E}(\mathcal{A}) \in 2^{\mathfrak{R}}$
Or $x \notin \breve{E}$ Then $\mathcal{Q}\breve{E}(\mathcal{A}) = \{0\}$
Since $\{0\} \in 2^{\mathfrak{R}}$ then $\mathcal{Q}\breve{E}(\mathcal{A}) \in 2^{\mathfrak{R}}$

ii. Let $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}_1$
If $\mathcal{A} = \emptyset$ then $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) = 0$, since $\mu_{\delta 1}$ standard approach measure
Either $\mathcal{Q}(\mathcal{A}) = \emptyset$ then $\mu_{\delta 2}(\mathcal{Q}(\mathcal{A})\mathcal{Q}(\mathcal{B})) = 0$, since $\mu_{\delta 2}$ standard approach measure
Or $\mathcal{Q}(\mathcal{A}) \neq \emptyset$ then $\mu_{\delta 2}(\mathcal{Q}(\mathcal{A})\mathcal{Q}(\mathcal{B})) = \infty$
Therefor $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}\breve{E}(\mathcal{A}), \mathcal{Q}\breve{E}(\mathcal{B}))$
If $A \neq \emptyset$ then $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) = \infty$, since $\mu_{\delta 1}$ standard approach measure.
Either $\mathcal{A} \cap \breve{E} = \emptyset$ or $\mathcal{A} \cap \breve{E} \neq \emptyset$
$\mathcal{Q}\breve{E}(\mathcal{A}) = \{0\}$ or $\{1\}$ so that $\mathcal{Q}\breve{E}(\mathcal{A}) \neq \emptyset$
then $\mu_{\delta 2}(\mathcal{Q}\breve{E}(\mathcal{A}), \mathcal{Q}\breve{E}(\mathcal{B})) = \infty$, since $\mu_{\delta 1}$ standard approach measure
$\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}\breve{E}(\mathcal{A}), \mathcal{Q}\breve{E}(B))$, for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$
then $\mathcal{Q}\breve{E}$ is approach measurable function. By (i), (ii)

□

Proposition 4.1 *Let $\mathcal{Q}_1 : (\aleph_1, F_1,\ \mu_{\delta 1}) \to (\aleph_2, F_2,\ \mu_{\delta 2})$ and $\mathcal{Q}_2 : (\aleph_2, F_2,\ \mu_{\delta 2}) \to (\aleph_3, F_3,\ \mu_{\delta 3})$ be two approach measurable functions then $\mathcal{Q}_2 \circ \mathcal{Q}_1 : (\aleph_1, F_1, \mu_{\delta 1}) \to (\aleph_3, F_3, \mu_{\delta 3})$ is also approach measurable function.*

Proof: Let $A \in F_1$.
We must proof $\mathcal{Q}_2 \circ \mathcal{Q}_1(\mathcal{A}) \in F_3$

1. Since $\mathcal{A} \in F_1$ and Q_1 is approach measurable function then $\mathcal{Q}_1(\mathcal{A}) \in F_2$. Since $\mathcal{Q}_1(\mathcal{A}) \in F_2$ and Q_2 is approach measurable function then $\mathcal{Q}_2(\mathcal{Q}_1(\mathcal{A})) \in F_3$.

   Therefor $\mathcal{Q}_2 \circ \mathcal{Q}_1(\mathcal{A}) \in F_3$

2. Let $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph_1}$

   since Q_1 is approach measurable function

$$\mu_{\delta 1}\,(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}_1(\mathcal{A}), \mathcal{Q}_1(\mathcal{B})) \tag{4.1}$$

   Since $\mathcal{A} \in F_1$ then $\mathcal{Q}_1(\mathcal{A}) \in F_2$. By $\mathcal{Q}_1$ is approach measurable function.

   since $\mathcal{Q}_2$ is approach measurable function then

$$\mu_{\delta 2}\mathcal{Q}_1(\mathcal{A}), \mathcal{Q}_1(\mathcal{B})) \leq \mu_{\delta 3}(\mathcal{Q}_2(\mathcal{Q}_1(\mathcal{A})), \mathcal{Q}_2(\mathcal{Q}_1(\mathcal{B})) \tag{4.2}$$

   By (4.1), (4.2) we get: $\mu_{\delta 1}\,(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 3}(\mathcal{Q}_2 \circ \mathcal{Q}_1(\mathcal{A})), \mathcal{Q}_2 \circ \mathcal{Q}_1(\mathcal{B}))$ for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$

Therefor $\mathcal{Q}_2 \circ \mathcal{Q}_1$ is approach measurable function. □

Proposition 4.2 *Let $(\aleph_1, 2^{\aleph_1}, \mu_{\delta 1}), (\aleph_2, F_2,\ \mu_{\delta 2})$ be two AMS , if $\mathcal{Q} : \aleph_1 \to \aleph_2$ is approach measurable function and $\breve{K} \subseteq \aleph_1$ then $\mathcal{Q}_{\breve{K}} : (\breve{K}, \breve{K} \cap 2^{\aleph_1}, \mu_{\delta 1}|_{k \cap 2^{\aleph}}) \to (\aleph_2, F_2, \mu_{\delta 2}) \to (\aleph_2, F_2, \mu_{\delta 2})$ is also approach measurable function such that $\mathcal{Q}_{\breve{K}}(\breve{c}) = \mathcal{Q}(\breve{c})$ for each $\breve{c} \in \breve{K}$.*

Proof: By Proposition 2.1 we get $(\breve{K}, \breve{K} \cap 2^{\aleph_1},\ \mu_{\delta 1}|_{k \cap 2^{\aleph}})$ is approach measure space.
We must proof $\mathcal{Q}_{\breve{K}}$ approach measurable function:

1. Let $A \epsilon \breve{K} \cap 2^{\aleph_1}$

   Then $\mathcal{A} \subseteq \breve{K}$

   $\mathcal{Q}_{\breve{K}}(\mathcal{A}) = \mathcal{Q}(\mathcal{A}) \epsilon F2$ since $\mathcal{Q}$ is approach measurable function.

2. Let $\mathcal{A} \in \breve{K} \cap 2^{\aleph_1}, \mathcal{B} \in 2^{K}$

   But $\mathcal{A} \in 2^{\aleph_1}$ and $\mathcal{B} \in 2^{\aleph_1}$

   So $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ since $\mathcal{Q}$ is approach measurable function

   Since $\mathcal{A} \in \breve{K} \cap 2^{\aleph_1}$ then $\mathcal{A} \subseteq \breve{K}$

   Since $\mathcal{B} \in 2^{\breve{K}}$ then $\mathcal{B} \subseteq \breve{K}$

   There for $\mu_{\delta 1}|_{k \cap 2^{\aleph}}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}_{\breve{K}}(\mathcal{A}), \mathcal{Q}_{\breve{K}}(\mathcal{B}))$, since $\mathcal{Q}_{\breve{K}}(\mathcal{A}) = \mathcal{Q}(\mathcal{A})$ for each $\mathcal{A} \subseteq \breve{K}$.

   By (1), (2) $\mathcal{Q}_{\breve{K}}$ is approach measurable function.

□

Proposition 4.3 *Let $(\ \aleph_1, F_1,\ \mu_{\delta 1}), (\aleph_2, F_2,\ \mu_{\delta 2})$ be two AMS, If F_1 is σ – field generated by ($\mathcal{D}$) where $\mathcal{D}$ is the collection of sub sets of $\aleph_1$ then a function $\mathcal{Q} : \aleph_1 \to \aleph_2$ such that $\mathcal{Q}$ onto is approach measurable function iff it satisfies :*

1. $\mathcal{Q}(\mathcal{A}) \in F_2\ \mathcal{A} \in \mathcal{D}$.

2.

$$\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B})),\ \textit{for each } \mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph_1} \tag{4.3}$$

Proof: Suppose $\mathcal{Q}: \aleph_1 \to \aleph_2$ is approach measurable function
Let $\mathcal{A} \in \mathcal{D}$ then $\mathcal{A} \in F_1$, since $\mathcal{D} \subseteq F_1$
Therefor $\mathcal{Q}(\mathcal{A}) \in F_2$, for each $\mathcal{A} \in \mathcal{D}$
Since $\mathcal{Q}$ is approach measurable function then
$\mu_{\delta_1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$
Now suppose $\mathcal{Q}: \aleph_1 \to \aleph_2$ satisfies:

1. $\mathcal{Q}(\mathcal{A}) \in F_2\ \mathcal{A} \in \mathcal{D}$.
2. $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$

We must proof $\mathcal{Q}$ is approach measurable function
Let $T = \{\mathcal{C} \subseteq \aleph_1 : \mathcal{Q}(\mathcal{C}) \in F_2\}$
We will proof $\mathcal{T}$ is $\sigma-$ filed.

a. Since $\aleph_1 \subseteq \aleph_1, \mathcal{Q}(\aleph_1) = \aleph_2 \in F_2$ since $\mathcal{Q}$ onto and F_2 is $\sigma-$ field.
Then $\aleph_1 \in \mathcal{T}$

b. Let $\mathcal{A} \in \mathcal{T}$, we must proof $A^{\mathsf{C}} \in \mathcal{T}$.
Since $A \in \mathcal{T}$ then $Q(A) \in F_2$ and $(Q(A))^{\mathsf{C}} \in F_2$, since F_2 is $\sigma-$ field.
Since $\mathcal{Q}$ onto then $(\mathcal{Q}(\mathcal{A}))^{\mathsf{C}} = \mathcal{Q}(\mathcal{A}^{\mathsf{C}})$, for each $\mathcal{A} \subseteq \aleph_1$
Then $\mathcal{Q}(\mathcal{A}^{\mathsf{C}}) \in F_2$ for each $\mathcal{A} \in \mathcal{T}$
Therefor $\mathcal{A}^{\mathsf{C}} \in \mathcal{T}$.
Now by (a), (b), we can proof $\emptyset \in \mathcal{T}$
By (a) $\aleph_1 \in \mathcal{T}$
By (b) ${\aleph_1}^{\mathsf{C}} \in \mathcal{T}$
Then $\emptyset = {\aleph_1}^{\mathsf{C}} \in \mathcal{T}$.

c. Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a family of sets in $\mathcal{T}$
then $\mathcal{Q}(\mathcal{A}_n) \in F_2 \qquad n = 1, 2, \ldots,$
therefor $\bigcup_{n=1}^{\infty} \mathcal{Q}(\mathcal{A}_n) \in F_2$ since F_2 is $\sigma-$ field
since $\bigcup_{n=1}^{\infty} \mathcal{Q}(\mathcal{A}_n) = \mathcal{Q}(\bigcup_{n=1}^{\infty} \mathcal{A}_n)$
then $\mathcal{Q}(\bigcup_{n=1}^{\infty} \mathcal{A}_n) \in F_2$
therefor $\bigcup_{n=1}^{\infty} \mathcal{A}_n \in \mathcal{T}$
we get $\mathcal{T}$ is $\sigma-$ field
Since $\mathcal{Q}(\mathcal{A}) \in F_2\ \mathcal{A} \in \mathcal{D}$
Then $\mathcal{D} \subseteq \mathcal{T}$.
Since F_1 is $\sigma -$ field generated by $(\mathcal{D})$ (so that F_1 is the smallest σ- filed containing $\mathcal{D}$)
Implies $F_1 \subseteq \mathcal{T}$
Therefor

$$\mathcal{Q}(\mathcal{A}) \in F_2 \text{ for each } \mathcal{A} \in F_1 \tag{4.4}$$

By (4.3), (4.4), we get $\mathcal{Q}$ is approach measurable function.

□

Proposition 4.4 *Let* $(\mathfrak{R}, \beta(\mathfrak{R}), \mu_{\delta 1}), (\aleph_2, F_2,\ \mu_{\delta 2})$ *be two AMS , If the function* $\mathcal{Q}: \mathfrak{R} \to \aleph_2$, *such that* $\mathcal{Q}$ *onto is approach measurable function iff it satisfies:*

1. $\left\{\mathcal{X} \in \aleph_2 :\ \mathcal{Q}^{-1}(\mathcal{X}) < t\right\} \in F_2$ *for each* $t \in \mathfrak{R}$
2. $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ *for each* $A \in \beta(\mathfrak{R}), \mathcal{B} \in 2^{\mathfrak{R}}$

Proof: suppose $\mathcal{Q}: \mathfrak{R} \to \aleph_2$ is approach measurable function

$$\mathcal{Q}(-\infty, t) = \left\{\mathcal{X} \in \aleph_2 :\ \mathcal{Q}^{-1}\ (\mathcal{X}) < t\ \right\}$$

Since $\beta(\mathfrak{R})$ is Boral σ- filed of $(\mathfrak{R})$ then $(-\infty, t) \in \beta(\mathfrak{R})$ by Proposition 2.2.
Since $\mathcal{Q}$ is approach measurable function.
Then $\mathcal{Q}(-\infty, t) \in F_2$
Therefor $\left\{\mathcal{X} \in \aleph_1 :\ \mathcal{Q}^{-1}\ (\mathcal{X}) < t\ \right\} \in F_2$
Since $\mathcal{Q}$ is approach measurable function then
$\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ for each $\mathcal{A} \in \beta(\mathfrak{R}), \mathcal{B} \in 2^R$
Now suppose $\mathcal{Q}: \mathfrak{R} \to \aleph_2$ satisfies:

1. $\left\{\mathcal{X} \in \aleph_2 : \mathcal{Q}^{-1}\ (\mathcal{X}) < t\ \right\} \in F_2$ for each t.
2. $$\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(B)) \text{ for each } \mathcal{A} \in \beta(\mathfrak{R}), \mathcal{B} \in 2^{\mathfrak{R}} \tag{4.5}$$

   We must proof $\mathcal{Q}: \mathfrak{R} \to \aleph_2$ is approach measurable function.

Define $\mathcal{T} = \{\mathcal{C} \subseteq \mathfrak{R} : \mathcal{Q}(\mathcal{C}) \in F_2\}$.
We will proof $\mathcal{T}$ is $\sigma-$ filed.

a. Since $\mathfrak{R} \subseteq \mathfrak{R}, \mathcal{Q}(\mathfrak{R}) = \aleph_2 \in F_2$, by Q onto and F_2 is $\sigma-$ field.

   Then $\mathfrak{R} \in \mathcal{T}$.

b. Let $\mathcal{A} \in\ \mathcal{T}$, we must proof $\mathcal{A}^{\complement} \in \mathcal{T}$.

   Since $\mathcal{A} \in\ \mathcal{T}$ then $\mathcal{Q}(\mathcal{A}) \in F_2$ and $(\mathcal{Q}(\mathcal{A}))^{\complement} \in F_2$ since F_2 is $\sigma-$ field

   Since $\mathcal{Q}$ onto $(\mathcal{Q}(\mathcal{A}))^{\complement} = \mathcal{Q}(\mathcal{A}^{\complement})$

   $Q(A^{\complement}) \in F_2$, for each $A \in \mathcal{T}$

   Therefor $\mathcal{A}^{\complement} \in \mathcal{T}$.

   Now by (a), (b) we can proof $\emptyset \in \mathcal{T}$

   By (a) $\aleph_1 \in \mathcal{T}$.

   By (b) ${\aleph_1}^{\complement} \in \mathcal{T}$.

   Then $\emptyset = {\aleph_1}^{\complement} \in \mathcal{T}$.

c. Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a family of sets in $\mathcal{T}$

   then $\mathcal{Q}(\mathcal{A}_n) \in F_2\ n = 1, 2, \ldots,$

   therefor $\bigcup_{n=1}^{\infty} \mathcal{Q}(\mathcal{A}_n) \in F_2$, since F_2 is $\sigma-$ field .

   since $\bigcup_{n=1}^{\infty} \mathcal{Q}(\mathcal{A}_n) = \mathcal{Q}(\bigcup_{n=1}^{\infty} \mathcal{A}_n)$

   then $\mathcal{Q}(\bigcup_{n=1}^{\infty} \mathcal{A}_n) \in F_2$.

   therefor $\bigcup_{n=1}^{\infty} \mathcal{A}_n \in \mathcal{T}$

   Since $\left\{\mathcal{X} \in \aleph_2 : \mathcal{Q}^{-1}\ (\mathcal{X}) < t\ \right\} \in F_2$ for each t.

Then $\mathcal{Q}(-\infty, t) \in F_2$.

$(-\infty, t) \in \mathcal{T}$ for each t

Since $\beta(\mathfrak{R})$ is the smallest $\sigma -$ field which contained intervals has form $(-\infty, t)$

Then $\beta(\mathfrak{R}) \subseteq \mathcal{T}$

Therefor

$$\mathcal{Q}(\mathcal{A}) \in F_2 \text{ for each } \mathcal{A} \in \beta(\mathfrak{R}) \tag{4.6}$$

By (4.5), (4.6), we get $\mathcal{Q}$ is approach measurable function.

□

Proposition 4.5 *Let $(\mathfrak{R}, \beta(\mathfrak{R}), \mu_{\delta 1}), (\aleph_2, F_2, \ \mu_{\delta 2})$ be two AMS , If the function $\mathcal{Q} : \mathfrak{R} \to \aleph_2$ is approach measurable function then:*

1. $\left\{X \in \aleph_2 : \mathcal{Q}^{-1} \ (\mathcal{X}) < \acute{c} \ \right\} \in F_2$ *for each* $\acute{c} \in R$.
2. $\left\{X \in \aleph_2 : \mathcal{Q}^{-1} \ (\mathcal{X}) \leq \acute{c} \ \right\} \in F_2$ *for each* $\acute{c} \in R$.
3. $\left\{X \in \aleph_2 : \mathcal{Q}^{-1} \ (\mathcal{X}) \geq \acute{c} \ \right\} \in F_2$ *for each* $\acute{c} \in R$.
4. $\left\{X \in \aleph_2 : \mathcal{Q}^{-1} \ (\mathcal{X}) > \acute{c} \ \right\} \in F_2$ *for each* $\acute{c} \in R$.

Proof: since $\mathcal{Q} : \mathfrak{R} \to \aleph_2$ is approach measurable function then
$\mathcal{Q}(\mathcal{A}) \in F_2$ for each $\mathcal{A} \in \beta(\mathfrak{R})$
by Proposition 2.2 we have: $\beta(\mathfrak{R})$ generated by $\{(-\infty, \acute{c}) : \acute{c} \in R\}$, $\{ (-\infty, \acute{c}] : \acute{c} \in R \ \}$, $\{\lceil \acute{c}, \infty\rceil : \acute{c} \in R\}$, $\{[\acute{c}, \infty) \ : \acute{c} \in R \ \}$
Then
$\mathcal{Q}((-\infty, \acute{c})) \in F_2 \ \acute{c} \in R$ by Proposition 4.4
$\mathcal{Q}((-\infty, \acute{c}]) \in F_2 \ \acute{c} \in R$
$\mathcal{Q}(\lceil \acute{c}, \infty\rceil) \in F_2 \ \acute{c} \in R$
$\mathcal{Q}([\acute{c}, \infty)) \in F_2 \ \acute{c} \in R$
Since $\mathcal{Q}(-\infty, \acute{c}) = \left\{\mathcal{X} \in \aleph_1 : \mathcal{Q}^{-1} \ (\mathcal{X}) < \acute{c}\right\}$
Therefore $\left\{\mathcal{X} \in \aleph_2 : \mathcal{Q}^{-1} \ (\mathcal{X}) < \ \acute{c} \ \right\} \in F_2$ for each $\acute{c} \in R$
Since $\mathcal{Q} \ (-\infty, \acute{c}] \ = \left\{\mathcal{X} \in \aleph_1 : \mathcal{Q}^{-1} (\mathcal{X}) \leq \acute{c}\right\}$
Therefore $\left\{\mathcal{X} \in \aleph_2 : \mathcal{Q}^{-1} (\mathcal{X}) \leq \acute{c}\right\} \in F_2$ for each $\acute{c} \in R$
Since $\mathcal{Q} \lceil \acute{c}, \infty\rceil = \left\{\mathcal{X} \in \aleph_1 : \mathcal{Q}^{-1} (\mathcal{X}) \geq \acute{c}\right\}$
Therefor $\left\{X \in \aleph_1 : \mathcal{Q}^{-1} \ (\mathcal{X}) \geq \acute{c}\right\} \in F_2$ for each $\acute{c} \in R$
Since $\mathcal{Q}([\acute{c}, \ \infty)) = \left\{\mathcal{X} \in \aleph_1 : \ \mathcal{Q}^{-1} \ (\mathcal{X}) > \acute{c} \ \right\}$
Then $\left\{\mathcal{X} \in \aleph_1 : \mathcal{Q}^{-1} (\mathcal{X}) > \acute{c}\right\} \in F_2$ for each $\acute{c} \in R$ □

Theorem 4.1 *Let $(\aleph_1, F_1, \mu_{\delta 1}), (\mathfrak{R}, \beta(\mathfrak{R}), \mu_{\delta 2})$ be AMS, if F_1 is Boral σ – field generated by $(\mathcal{D})$ where D is the collection of all open sub sets of $\aleph_1$ and $\mathcal{Q} : \mathfrak{R} \to \aleph_1$ such that $\mathcal{Q}$ onto is approach measurable function if satisfies:*

1. $\mathcal{Q} : (\mathfrak{R}, \beta(\mathfrak{R}), \mu_{\delta 2}) \to (\aleph_1, F_1, \mu_{\delta 1})$ *is open function.*
2. $$\mu_{\delta 1} (\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B})) \text{ for each } \mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph_1} \tag{4.7}$$

Proof: to proof $\mathcal{Q}$ is approach measurable function
Let $\mathcal{A}$ be open sub set of $\mathfrak{R}$
Since $\mathcal{Q}$ is open function then $\mathcal{Q}(\mathcal{A})$ is open sub set of $\aleph_1$
Then $\mathcal{Q}(\mathcal{A}) \ \in \mathcal{D}$

since F_1 is Boral $\sigma-$ field generated by $(\mathcal{D})$
Then

$$\mathcal{Q}(\mathcal{A}) \in F_1 \text{ for each } \mathcal{A} \text{ open sub set of} \mathfrak{R} \tag{4.8}$$

Since $\beta(\mathfrak{R})$ generated by open set by Proposition 2.2.
Now by Proposition 4.3, It's enough for us proof $\mathcal{Q}(\mathcal{A}) \in F_1$ for each open sub set of $\mathfrak{R}$.
By (4.7), (4.8) we get $\mathcal{Q}$ is approach measurable function. □

Theorem 4.2 *Let* $(\aleph_1, F_1, \mu_{\delta 1}), (\aleph_2, F_2, \ \mu_{\delta 2})$ *be two standard AMS , then* $\mathcal{Q}: \ \aleph_1 \to \aleph_2$ *is approach measurable function iff* $\mathcal{Q}(\mathcal{A}) \in F_2$ *for each* $\mathcal{A} \in F_1$*.*

Proof: suppose $\mathcal{Q}: \ \aleph_1 \to \aleph_2$ is approach measurable function.
It is clear that $\mathcal{Q}(\mathcal{A}) \in F_2$ for each $\mathcal{A} \in F_1$, since $\mathcal{Q}$ is approach measurable function.
Now suppose $\mathcal{Q}(\mathcal{A}) \in F_2$ for each $\mathcal{A} \in F_1$
To proof $\mathcal{Q}$ is approach measurable function
So that we must proof:

1. $\mathcal{Q}(\mathcal{A}) \in F_2$ for each $\mathcal{A} \in F_1$.
2. $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$

It is clear that $\mathcal{Q}$ satisfies (1) by hypothesis.
Now to prove $\mathcal{Q}$ satisfies (2):
If $\mathcal{A} = \emptyset$, then $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) = 0$, since $\mu_{\delta 1}$ standard approach measure
Either $\mathcal{Q}(\mathcal{A}) = \emptyset$, then $\mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B})) = 0$, since $\mu_{\delta 2}$ standard approach measure
Or $\mathcal{Q}(\mathcal{A}) \ \neq \emptyset$ then $\mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B})) = \infty$
Therefor $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$
Or $\mathcal{A} \ \neq \emptyset$ then $\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) = \infty$, since $\mu_{\delta 1}$ standard approach measure
$\mathcal{Q}(\mathcal{A}) \neq \emptyset$, then $\mu_{\delta 2}(Q(A)\mathcal{Q}(\mathcal{B})) = \infty$, since $\mu_{\delta 1}$ standard approach measure.
$\mu_{\delta 1}(\mathcal{A}, \mathcal{B}) \leq \mu_{\delta 2}(\mathcal{Q}(\mathcal{A}), \mathcal{Q}(\mathcal{B}))$ for each $\mathcal{A} \in F_1, \mathcal{B} \in 2^{\aleph}$
Then $\mathcal{Q}$ satisfies (2).
Hence $\mathcal{Q}$ is approach measurable function. □

Theorem 4.3 *Let* $(\ \aleph_1, F_1, \ \mu_{\delta 1}), (\aleph_2, F_2, \ \mu_{\delta 2})$ *be two standard AMS , if* $\mathcal{Q}_1 : (\aleph_1, F_1, \ \mu_{\delta 1}) \to (\aleph_2, F_2, \ \mu_{\delta 2})$ *and* $\mathcal{Q}_2 : (\aleph_1, F_1, \ \mu_{\delta 1}) \to (\aleph_2, F_2, \ \mu_{\delta 2})$ *is two approach measurable functions then* $\mathcal{Q}_1 + \mathcal{Q}_2$ *is approach measurable function.*

Proof: Since $\mathcal{Q}_1 : (\aleph_1, F_1, \ \mu_{\delta 1}) \to (\aleph_2, F_2, \ \mu_{\delta 2}), \mathcal{Q}_2 : (\aleph_1, F_1, \ \mu_{\delta 1}) \to (\aleph_2, F_2, \mu_{\delta 2})$ is two approach measurable functions then

$$\mathcal{Q}_1(\mathcal{A}) \in F_2 \text{ for each } \mathcal{A} \in F_1 \tag{4.9}$$

$$\mathcal{Q}_2(\mathcal{A}) \in F_2 \text{ for each } \mathcal{A} \in F_1 \tag{4.10}$$

Since $(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{A}) = \mathcal{Q}_1(\mathcal{A}) + \mathcal{Q}_2(\mathcal{A})$
Since $\mathcal{Q}_1(\mathcal{A}), \mathcal{Q}_2(\mathcal{A})$ sets then we can express by:

$$\mathcal{Q}_1(\mathcal{A}) + \mathcal{Q}_2(\mathcal{A}) = \mathcal{Q}_1(\mathcal{A}) \cup \mathcal{Q}_2(\mathcal{A})$$

By (4.9), (4.10) then $\mathcal{Q}_1(\mathcal{A}) \cup \mathcal{Q}_2(\mathcal{A}) \in F_2$ since F_2 is $\sigma-$ field
Therefor $(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{A}) \in F_2$
Since $(\aleph_1, F_1, \ \mu_{\delta 1}), (\aleph_2, F_2, \ \mu_{\delta 2})$ two standard AMS.
And By Theorem 4.2 we get $\mathcal{Q}_1 + \mathcal{Q}_2$ is approach measurable functions. □

5. Conclusion

In this study, we presented a new definition of measurable functions by using the approach structure, demonstrated some properties of this concept, clarified its relationship between this concept and open function and we also gave a condition to prove that the sum of approach measurable functions is also approach measurable function.

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Ali A. Hammood,
Department of Mathematics,
College of Science, Mustansiriyah University,
Iraq.
E-mail address: al83_math@uomustansiriyah.edu.iq

and

Arkan J Mohammed,
Department of Mathematics,
College of Science, Mustansiriyah University,
Iraq.
E-mail address: drarkanjasim@uomustansiriyah.edu.iq

and

Boushra Y. Hussein,
Department of Mathematics,
College of Education, University of Al-Qadisiyah,
Iraq.
E-mail address: boushra.alshebani@qu.edu.iq