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Existence of Solutions for Fuzzy Fractional Differential Equations under Caputo Fractional Derivatives Approach

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ABSTRACT: This paper investigates the existence of solutions for a class of fuzzy differential equations with nonlocal derivatives. By utilizing an extended version of Krasnosel'skii's fixed point theorem within the context of fuzzy metric spaces, we demonstrate that the problem has a fuzzy solution defined over a specified interval. The approach includes analyzing the corresponding integral problem to which the theorem is applied. The paper concludes with a discussion on potential physical applications.

Key Words: $\psi-$fractional integral, $\psi-$Caputo fractional derivative, fixed point, Carathéodory function.

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1. Introduction

Uncertainty quantification seeks to reduce ambiguities in mathematical, computational, and real-world applications through rigorous quantitative analysis. Uncertainty emerges in models and measurements as a result of multiple contributing factors, such as incomplete data, measurement errors, or inherent randomness, and has significant implications in fields like quantum theory, engineering, and biology. For instance, the Dirac equation has been analyzed under the generalized uncertainty principle [14], and a generalized momentum operator incorporating the extended uncertainty principle has been developed [11]. At the same time, nonlocal derivatives have gained prominence for their ability to model complex phenomena in science and engineering, such as electromagnetic processes, diffusion in heterogeneous media, biological systems, and electrical circuits [4,17]. In 2010, Agarwal et al. [1] introduced the concept of solutions to uncertain fractional differential equations (UFDEs) with nonlocal derivatives, combining uncertainty and memory effects. This innovative approach has inspired the development of hybrid operators that integrate the features of uncertain and arbitrary-order equations [4]. Recent research has focused on UFDEs and and connected integro-differential systems, often employing Riemann-Liouville or Caputo derivatives [2,19]. For example, Arshad and Lupulescu [6,5] established existence and uniqueness results for UFDEs using the Riemann-Liouville derivative, providing a foundation for further exploration in this area. A common strategy for studying differential problems involves transforming them into integral problems and applying fixed point theory. Results such as Banach and Krasnosel'skii theorems have been widely used to analyze arbitrary-order differential equations [10,15]. For UFDEs, extending fixed point theorems is crucial due to the unique properties of fuzzy sets and functions. Agarwal et al. [1] generalized Schauder's fixed point theorem to semilinear spaces, enabling the study of UFDEs. Similarly, Krasnosel'skii-type results have been extended to semilinear Banach spaces and applied to UFDEs [18,16], demonstrating the versatility of these tools in addressing uncertainty and memory effects. Throughout

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this work, we examine a class of nonlinear uncertain differential equations with nonlocal derivatives, which are broader in scope than those studied in [16]. Using a Krasnosel'skii-type fixed point theorem in fuzzy metric spaces, we prove the existence of at least one solution under specific conditions. Our approach leverages the cone structure of the base space and H-differences, extending previous results [26] to uncertain differential equations. This provides a more general framework than [16], which was limited to compact mappings. By combining uncertainty quantification with nonlocal derivatives, our results offer new insights into the analysis of complex systems and pave the way for further advancements in this interdisciplinary field.
The motivation for this work stems from the need to address the limitations of existing methods in handling uncertainty and nonlocal effects simultaneously. Traditional approaches often treat these aspects separately, leading to incomplete or overly restrictive models. By integrating uncertainty quantification with nonlocal derivatives, we provide a unified framework that captures the interplay between these two critical features. This is particularly relevant in applications such as modeling heterogeneous materials, where both uncertainty and memory effects play a significant role. For example, in the study of multiexponential signal decay processes, the ability to account for uncertainty and nonlocal interactions can lead to more accurate and robust predictions. Furthermore, our work contributes to the growing body of literature on fuzzy differential equations by introducing a more flexible and general approach. Unlike previous studies that rely on compact mappings, our framework accommodates a broader class of operators, making it applicable to a wider range of problems. This flexibility is achieved by leveraging the cone structure of the base space and H-differences, which allow for a more nuanced analysis of the underlying dynamics. As a result, our findings not only extend existing theoretical results but also open up new avenues for practical applications in fields such as physics, engineering, and biology.
This work closes the gap between uncertainty quantification and nonlocal derivatives, delivering a comprehensive framework for the analysis of complex systems. By proving the existence of solutions to a class of nonlinear uncertain differential equations, we set the stage for future research. Our results stress the importance of incorporating uncertainty and memory effects into mathematical models and demonstrate the value of fixed point theory in overcoming these challenges. With the increasing demand for accurate models, our approach paves the way for advancements in the study of uncertain differential equations and their practical uses.
The remainder of this article is organized as follows: Section 2 introduces the $\psi-$fractional integral and the $\psi-$Caputo derivative for fuzzy functions, along with some properties of the fractional integral and derivative for fuzzy functions. Additionally, Section 3 establishes necessary comparison theorems in real-valued differential equations using the $\psi-$Caputo fractional derivative to prove the existence and uniqueness of the solution for the proposed problem . In Section 4, provides some several applications of our results.

2. Preliminaries

To define the central issue and key findings, it is necessary to introduce several critical concepts and notations.For further elaboration, we direct readers to [7,9].

In this study, $\mathcal{Y}$ is used to denoteThe domain of fuzzy intervals. For an element $z \in \mathcal{Y}$, the ζ-cuts are given by:

$$[z]^\zeta := \{t \in \mathbb{R} \mid z(t) \geq \zeta\}, \quad \text{for } \zeta \in (0,1],$$

and
the support of z is expressed as:

$$[z]^0 := \overline{\{t \in \mathbb{R} \mid z(t) > 0\}}.$$

We denote the ζ-cuts of z,

$$[z]^\zeta = \left[\underline{z}^\zeta, \bar{z}^\zeta\right],$$

will also be used.

For any $z_1, z_2 \in \mathcal{Y}$ and a scalar $\lambda \in \mathbb{R}$,the addition operation $z_1 + z_2$ and the scalar multiplication operation λz_1 are defined as:

$$[z_1 + z_2]^\zeta = [z_1]^\zeta + [z_2]^\zeta, \quad \text{and} \quad [\lambda z_1]^\zeta = \lambda [z_1]^\zeta,$$

for all $\zeta \in [0, 1]$. These definitions are based on the conventional operations of interval addition and scalar multiplication for real intervals, respectively.

A metric $\mathfrak{D} : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+ \cup \{0\}$ can be introduced on the space $\mathcal{Y}$, based on the Pompeiu-Hausdorff distance between the ζ-cuts, defined as follows:

$$\begin{aligned}\mathfrak{D}(z_1, z_2) &:= \sup_{\zeta \in [0,1]} d_H \left([z_1]^\zeta, [z_2]^\zeta\right) \\ &= \sup_{\zeta \in [0,1]} \max \left\{ \left|\underline{z_1}^\zeta - \underline{z_2}^\zeta\right|, \left|\bar{z_1}^\zeta - \bar{z_2}^\zeta\right| \right\}, \quad z_1, z_2 \in \mathcal{Y}.\end{aligned}$$

Several interesting properties are satisfied by this distance, including translation invariance and positive homogeneity:

- $\mathfrak{D}(z_1 + r, z_2 + r) = \mathfrak{D}(z_1, z_2)$, for any $z_1, z_2, r \in \mathcal{Y}$.
- $\mathfrak{D}(kz_1, kz_2) = |k|\mathfrak{D}(z_1, z_2)$, for any $z_1, z_2 \in \mathcal{Y}$ and $k \in \mathbb{R}$.
- $\mathfrak{D}(z_1 + z_2, z + s) \leq \mathfrak{D}(z_1, z) + \mathfrak{D}(z_2, s)$, for any $z_1, z_2, z, s \in \mathcal{Y}$.
- $\mathfrak{D}(\alpha z, \gamma z) = |\alpha - \gamma|\mathfrak{D}(z, \tilde{0})$, for any $z \in \mathcal{Y}$ and $\alpha, \gamma \geq 0$.

Moreover, $(\mathcal{Y}, \mathfrak{D})$ forms a complete metric space.

Since $\mathcal{Y}$ does not possess the structure of a vector space, classical fixed-point theorems in Banach spaces are not applicable in this setting. To address this limitation, we introduce the concept of a semilinear Banach space, which provides a formal framework capable of encompassing the collection of all fuzzy intervals. This approach is consistent with earlier work on semilinear spaces, as explored in [13]. We define $\mathcal{E}$ as a semilinear metric space if $\mathcal{E}$ is a semilinear space endowed with a metric $d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_+$ that satisfies, for all $\mathfrak{x}, r, \gamma \in \mathcal{E}$:

- $d(\zeta, r) = d(\mathfrak{x} + \gamma, r + \gamma)$ (translation invariance),
- $\lambda d(\mathfrak{x}, r) = d(\lambda \mathfrak{x}, \lambda r)$, for $\lambda \geq 0$ (positive homogeneity).

Let $\tilde{0}$ represent the neutral element of $\mathcal{E}$. The norm on $\mathcal{E}$ is defined as $\|z\| := \lceil(z, \tilde{0})$. In semilinear metric spaces, the operations of addition and scalar multiplication are continuous. When $\mathcal{E}$ is a complete metric space, it is called a semilinear Banach space. For instance, $\mathcal{Y}$ cannot be a Banach space because it is not a vector space. If $\mathfrak{x} + r = \gamma + r$ implies $\mathfrak{x} = \gamma$ for all $\mathfrak{x}, r, \gamma \in \mathcal{E}$, then $\mathcal{E}$ is said to have the cancellation property.

For a fixed $a > 0$, the space $C(\mathfrak{I} = [0, a], \mathcal{Y})$ consists of all continuous fuzzy interval-valued functions defined on the compact interval $\mathfrak{I}$. This space is a semilinear space and possesses the cancellation property. In addition, the metric

$$\mathfrak{h}_0(z_1, z_2) := \max_{t \in \mathfrak{I}} \mathfrak{D}(z_1(t), z_2(t)), \quad z_1, z_2 \in C(\mathfrak{I}, \mathcal{Y}),$$

endows this space with a complete structure [9], making it an example of a semilinear Banach space.

The elements of $\mathcal{Y}$ are not required to be continuous in the membership degree ζ. By focusing on the subset of fuzzy intervals $z \in \mathcal{Y}$ for which the mapping $\zeta \mapsto [z]^\zeta$ is continuous on $[0, 1]$ (with respect to the Pompeiu-Hausdorff metric on compact intervals), we introduce the space $\mathcal{Y}^c$. Equipped with the metric $\mathfrak{D}$, the space $(\mathcal{Y}^c, \mathfrak{D})$ forms a complete metric space [25].

Definition 1 *[25] A subset $\mathcal{X} \subseteq \mathcal{Y}^c$ is said to be compact-supported if there exists a compact set $N \subseteq \mathbb{R}$ such that $[z]^0 \subseteq N$ for every $z \in \mathcal{X}$.*

Definition 2 *[25] For $\mathcal{X} \subseteq \mathcal{Y}^c$, if for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying*

$$|\zeta - \zeta_0| < \delta \Rightarrow \mathfrak{D}\left([z]^{\zeta}, [z]^{\zeta_0}\right) < \varepsilon, \quad \forall z \in \mathcal{X},$$

then $\mathcal{X}$ is termed level-equicontinuous at $\zeta_0 \in [0,1]$.

We say that $\mathcal{X}$ is level-equicontinuous on $[0,1]$ if the property outlined in Definition 2 is satisfied for all $\zeta \in [0,1]$.

Theorem 1 *[25] A subset $\mathcal{X} \subseteq \mathcal{Y}^c$ with compact support is relatively compact in $(\mathcal{Y}^c, \mathfrak{D})$ if and only if it satisfies level-equicontinuity on $[0,1]$.*

Next, we revisit the extension of the well-known Schauder fixed point theorem to semilinear spaces, as discussed in [3].

Theorem 2 *Consider a semilinear Banach space $\mathcal{E}$ with the cancellation property, and let $\mathcal{N}$ be a nonempty, bounded, closed, and convex subset of $\mathcal{E}$. If $P : \mathcal{N} \to \mathcal{N}$ is a compact mapping, then P guarantees the existence of at least one fixed point in $\mathcal{N}$.*

In [18], Long et al. proposed a generalization of the Krasnosel'skii fixed point theorem for semilinear Banach spaces. Here, we establish an analogous result for fuzzy spaces, using a proof that incorporates H-differences and differs somewhat from the original approach.

Theorem 3 *[8] (Fixed point result of Krasnosel'skii type for fuzzy metric spaces)*
Assume $\mathcal{G}$ is a nonempty, closed, and convex subset of $C(I, \mathcal{Y}^c)$. Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ map $\mathcal{G}$ into $\mathcal{N}$ and satisfy:

(1) $\mathcal{Q}_1$ is continuous and compact,

(2) $\mathcal{Q}_2$ is a contraction mapping,

(3) $\mathcal{Q}_1 z + \mathcal{Q}_2 y \in \mathcal{G}$ for all $z, y \in \mathcal{G}$.

Then, there exists $y \in \mathcal{G}$ such that $\mathcal{Q}_1 y + \mathcal{Q}_2 y = y$, proving that $\mathcal{Q}_1 + \mathcal{Q}_2$ has a fixed point in $\mathcal{G}$.

Proof: Let $y \in \mathcal{G}$ be an arbitrarily fixed element. We define a self-mapping $\mathcal{Q} : \mathcal{G} \to \mathcal{G}$ as follows:

$$x \to \mathcal{Q}(x) := \mathcal{Q}_1 y + \mathcal{Q}_2 x,$$

We verify that $\mathcal{Q}$ satisfies the conditions of a contraction mapping

$$\begin{aligned}\mathfrak{h}_0(\mathcal{Q}(x), \mathcal{Q}(z)) &= \mathfrak{h}_0(\mathcal{Q}_1 y + \mathcal{Q}_2 x, \mathcal{Q}_1 y + \mathcal{Q}_2 z)\\ &= \mathfrak{h}_0(\mathcal{Q}_2 x, \mathcal{Q}_2 z) \leq \kappa \mathfrak{h}_0(x, z),\end{aligned}$$

with $0 \leq \kappa < 1$ and $\mathcal{G}$ being a complete metric space, we examine the equation $z = \mathcal{Q}_1 y + \mathcal{Q}_2 z$. This leads to the conclusion that a unique solution exists in $\mathcal{G}$. For a fixed y, let $\Phi(y) \in \mathcal{G}$ be the unique solution of this equation, i.e., $\Phi(y) = \mathcal{Q}_1 y + \mathcal{Q}_2 \Phi(y)$. This implies $(I \ominus \mathcal{Q}_2)\Phi(y) = \mathcal{Q}_1 y$. Hence, $I \ominus \mathcal{Q}_2 : \Phi(\mathcal{G}) \to C(I, \mathcal{Y}^c)$ exists, and $\mathcal{Q}_1(\mathcal{G}) \subseteq (I \ominus \mathcal{Q}_2)(\Phi(\mathcal{G}))$. We demonstrate its injectivity so that

$$(I \ominus \mathcal{Q}_2)^{-1} : (I \ominus \mathcal{Q}_2)(\Phi(\mathcal{G})) \to \Phi(\mathcal{G}),$$

exists, which implies $\Phi(y) = (I \ominus \mathcal{Q}_2)^{-1} \mathcal{Q}_1 y$.
We now demonstrate that $I \ominus \mathcal{Q}_2 : \Phi(\mathcal{G}) \to C(I, \mathcal{Y}^c)$ is continuous.
Let $\hat{z}, \bar{z} \in \mathcal{G}$ be fixed, and consider $\Phi(\hat{z}), \Phi(\bar{z}) \in \Phi(\mathcal{G})$. Then, we

$$\begin{aligned}\mathfrak{h}_0(\Phi(\hat{z}), \Phi(\bar{z})) &= \mathfrak{h}_0((I \ominus \mathcal{Q}_2)\Phi(\hat{z}) + \mathcal{Q}_2\Phi(\hat{z}), (I \ominus \mathcal{Q}_2)\Phi(\bar{z}) + \mathcal{Q}_2\Phi(\bar{z})),\\ &\leq \mathfrak{h}_0((I \ominus \mathcal{Q}_2)\Phi(\hat{z}), (I \ominus \mathcal{Q}_2)\Phi(\bar{z})) + \mathfrak{h}_0(\mathcal{Q}_2\Phi(\hat{z}), \mathcal{Q}_2\Phi(\bar{z})),\\ &\leq \mathfrak{h}_0((I \ominus \mathcal{Q}_2)\Phi(\hat{z}), (I \ominus \mathcal{Q}_2)\Phi(\bar{z})) + \kappa \mathfrak{h}_0(\Phi(\hat{z}), \Phi(\bar{z})).\end{aligned}$$

since $\mathcal{Q}_2$ is a contraction.
Hence,

$$\mathfrak{h}_0((I \ominus \mathcal{Q}_2)\Phi(\hat{z}), (I \ominus \mathcal{Q}_2)\Phi(\bar{z})) \geq (1-\kappa)\mathfrak{h}_0(\Phi(\hat{z}), \Phi(\bar{z})). \tag{2.1}$$

From the inequality (2.1), it follows that $I \ominus \mathcal{Q}_2$ is injective on $\Phi(\mathcal{G})$.
Therefore, $I \ominus \mathcal{Q}_2$ is bijective and continuous from $\Phi(\mathcal{G})$ to $(I \ominus \mathcal{Q}_2)(\Phi(\mathcal{G}))$.
This confirms that $(I \ominus \mathcal{Q}_2)^{-1}$ exists as a mapping from $(I \ominus \mathcal{Q}_2)(\Phi(\mathcal{G}))$ to $\Phi(\mathcal{G})$. Moreover, from the inequality (2.1), $(I \ominus \mathcal{Q}_2)^{-1}$ is continuous on $(I \ominus \mathcal{Q}_2)(\Phi(\mathcal{G}))$.
As a result, we have

$$\begin{aligned}&\mathfrak{h}_0\left((I \ominus \mathcal{Q}_2)^{-1}(I \ominus \mathcal{Q}_2)\Phi(\hat{z}), (I \ominus \mathcal{Q}_2)^{-1}(I \ominus \mathcal{Q}_2)\Phi(\bar{z})\right) = \mathfrak{h}_0(\Phi(\hat{z}), \Phi(\bar{z})),\\ &\leq \frac{1}{1-\kappa}\mathfrak{h}_0((I \ominus \mathcal{Q}_2)\Phi(\hat{z}), (I \ominus \mathcal{Q}_2)\Phi(\bar{z})).\end{aligned}$$

Given that $\Phi(y) = (I \ominus \mathcal{Q}_2)^{-1}\mathcal{Q}_1 y$ and, by hypothesis, $(I \ominus \mathcal{Q}_2)^{-1}\mathcal{Q}_1$ is continuous and compact, we invoke Theorem 2 to deduce that Φ has a fixed point r_0 in $\mathcal{G}$, which satisfies $\Phi(r_0) = r_0$. Since, $\Phi(r_0)$ is the unique solution to $z = \mathcal{Q}_1 r_0 + \mathcal{Q}_2 z$, then, we have $r_0 = \mathcal{Q}_1 r_0 + \mathcal{Q}_2 r_0$. □

Definition 3 *[3] For $t \in (0,a)$ and $z \in \mathbb{E}$, the fuzzy ψ-Riemann-Liouville fractional integral of order $\beta > 0$ of z, is defined by*

$$\mathcal{I}^{\beta,\psi}z(t) = \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}z(s)ds,$$

under the assumption that the integral on the right is well-defined for the interval $(0,a)$. Where $\mathbb{E} := C((0,a],\mathcal{Y}) \cap L^1((0,a],\mathcal{Y})$ and $\mathcal{H}_{t,s} = \psi(t) - \psi(s)$.

Definition 4 *[3] For $z \in \mathbb{E}$. If the function $t \mapsto \int_0^t \psi'(s)\mathcal{H}_{t,s}^{-\beta}z(s)\,ds$ is Hukuhara differentiable on $(0,a]$, then the fuzzy ψ-Riemann-Liouville fractional (arbitrary order) derivative of order $\beta \in (0,1)$ of z at t is defined by*

$$D^{\beta,\psi}z(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{-\beta}z(s)ds,$$

which, by assumption, is a fuzzy interval. $D^{\beta,\psi}z(t) \in \mathcal{Y}$.

3. Main results

In the following section, we examine a class of uncertain differential equations characterized by non-local dynamics, represented through fractional (arbitrary-order) operators,

$$D^{\beta,\psi}z = \mathcal{W}(t,z) + \mathcal{Z}(t,z), \tag{3.1}$$

With the derivation order $0 < \beta < 1$, the fuzzy function $\mathcal{W} : \mathfrak{I} \times \mathcal{Y}^c \to \mathcal{Y}^c$ is both continuous and compact, and $\mathcal{Z} : \mathfrak{I} \times \mathcal{Y}^c \to \mathcal{Y}^c$ satisfies

$$\mathfrak{D}(\mathcal{Z}(t,z_1), \mathcal{Z}(t,z_2)) \leq L\mathfrak{D}(z_1,z_2), \quad z_1, z_2 \in \mathcal{Y}^c, \text{ for all } L \geq 0. \tag{3.2}$$

In the following, we revisit the concept of a solution to equation (3.1).

Definition 5 *[16] We say that the fuzzy function $z \in \mathbb{E}$ is a solution to (3.1) if:*

(i) The function z admits a fractional derivative $D^{\beta,\psi}z$ on $(0,a]$, which is continuous.

(ii) $D^{\beta,\psi}z(t) = \mathcal{W}(t,z(t)) + \mathcal{Z}(t,z(t)), \quad \forall t \in (0,a]$.

The remark that follows introduces a method for determining solutions to equation (3.1) via the solution of a related integral equation of arbitrary order.

Remark 1 *[16]Consider $z \in C(\mathfrak{I}, \mathcal{Y}^c)$ as a solution to the following fuzzy arbitrary-order integral equation:*

$$z(t) = \mathcal{I}^{\beta,\psi}(\mathcal{W}(t, z(t)) + \mathcal{Z}(t, z(t))),$$

and that $\mathcal{W}(\cdot, z(\cdot)), \mathcal{Z}(\cdot, z(\cdot)) \in \mathbb{E}$.
Therefore, z is a solution to the nonlocal uncertain differential equation (3.1).

Here, we introduce the critical tools that will allow us to prove the main existence results. Let $I_{\mathrm{h}} := [0, \mathrm{h}]$, where $\mathrm{h} \leq a$, and $\tau_0 > 0$ be such that

$$\mathfrak{M} := \sup\left\{\mathfrak{D}(\tilde{0}, \mathcal{W}(t, z)) \mid t \in I_{\mathrm{h}}, \mathfrak{D}(z, \tilde{0}) \leq \tau_0\right\} < +\infty.$$

Observe that h can be chosen small enough to ensure that

$$\frac{\mathfrak{M}\mathrm{h}^{\beta}}{\Gamma(\beta+1)} \leq \tau_0.$$

We define the set as follows

$$\mathbb{Y} := \left\{z \in C(I_{\mathrm{h}}, \mathcal{Y}^c) \mid \mathfrak{D}(\tilde{0}, z(t)) \leq \tau_0, \text{ for all } t \in I_{\mathrm{h}}, \text{ and } z(0) = \tilde{0}\right\}.$$

It is easy to verify that $\mathbb{Y} \subseteq C(I_{\mathrm{h}}, \mathcal{Y}^c)$ is bounded, closed, and convex. Recall also that $C(I_{\mathrm{h}}, \mathcal{Y}^c)$ is a semilinear Banach space. On $\mathbb{Y}$, we define two mappings related to $\mathcal{W}$ and $\mathcal{Z}$.

The first mapping, $\mathcal{O} : \mathbb{Y} \to C(I_{\mathrm{h}}, \mathcal{Y}^c)$, is defined by:

$$(\mathcal{O}z)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s, z(s))ds, \text{ for } t \in I_{\mathrm{h}} \tag{3.3}$$

To facilitate the procedure, we establish notation that addresses the constraints on the function f, which is involved in the nonlinearity of the equation.

The condition (**H**) is introduced for the function $\mathcal{W} \in C(\mathfrak{I} \times \mathcal{Y}^c, \mathcal{Y}^c)$, which requires the following constraint:
for all pair of points $(t, z_1), (t, z_2) \in \mathfrak{I} \times C(\mathfrak{I}, \mathcal{Y}^c)$, the following inequality is satisfied:

$$\mathfrak{D}(\mathcal{W}(t, z_1(t)), \mathcal{W}(t, z_2(t))) \leq \mathfrak{P}(t)\mathfrak{e}\left(\mathfrak{h}_0(z_1, z_2)\right), \tag{3.4}$$

where $\mathfrak{e}$ is a continuous function on $[0, \infty)$ such that $\mathfrak{e}(0) = 0$, and $\mathfrak{P} : \mathfrak{I} \to \mathbb{R}^+$ fulfills $\mathcal{I}^{\beta,\psi}\mathfrak{P}(t) < \mathcal{B}$ for any $t \in \mathfrak{I}$.

Lemma 1 *Assume that $\mathcal{W}$ satisfies condition* (**H**) *on $\mathfrak{I} \times \mathcal{E}^c$. Then, the following statements hold:*

(i) $\mathcal{O}$ is well-defined on $C(I_{\mathrm{h}}, \mathcal{Y}^c)$.

(ii) $\mathcal{O}$ is continuous on $C(I_{\mathrm{h}}, \mathcal{Y}^c)$.

Proof: First, we establish that the mapping $\mathcal{O}$ is well-defined. By definition, it follows directly that $(\mathcal{O}z)(0) = \tilde{0}$ for every $z \in \mathbb{Y}$. Next, for a fixed $z \in \mathbb{Y}$, we prove that $\mathcal{O}z \in C(I_{\mathrm{h}}, \mathcal{Y}^c)$.

We also establish that $\mathcal{O}z$ is uniformly continuous on the interval I_{h}. Let $t, t' \in I_{\mathrm{h}}$ be fixed, satisfying

$t < t'$. For these points, we infer that

$$\begin{aligned}
&\mathfrak{D}\left((\mathcal{O}z)(t), (\mathcal{O}z)(t')\right) \\
&= \frac{1}{\Gamma(\beta)}\mathfrak{D}\left(\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s))ds, \int_0^{t'} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds\right), \\
&= \frac{1}{\Gamma(\beta)}\mathfrak{D}\left(\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s))ds, \int_0^{t} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds\right. \\
&\left.+\int_t^{t'} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds\right), \\
&\leq \frac{1}{\Gamma(\beta)}\left[\mathfrak{D}\left(\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s))ds, \int_0^{t} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds\right)\right. \\
&\left.+\mathfrak{D}\left(\int_t^{t'} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds, \tilde{0}\right)\right], \\
&\leq \frac{1}{\Gamma(\beta)}\left[\int_0^t \mathfrak{D}\left(\psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s)), \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))\right)ds\right. \\
&\left.+\int_t^{t'} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathfrak{D}(\mathcal{W}(s,z(s)),\tilde{0})ds\right], \\
&\leq \frac{\mathfrak{M}}{\Gamma(\beta)}\left[\int_0^t \left|\psi'(s)\mathcal{H}_{t,s}^{\beta-1} - \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\right|ds + \int_t^{t'} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}ds\right], \\
&\leq \frac{\mathfrak{M}}{\Gamma(\beta+1)}\left(2\mathcal{H}_{t',t}^{\beta} + \mathcal{H}_{t,0}^{\beta} - \mathcal{H}_{t',0}^{\beta}\right), \\
&\leq \frac{2\mathfrak{M}}{\Gamma(\beta+1)}\mathcal{H}_{t',t}^{\beta}.
\end{aligned}$$

Thus, $\mathfrak{D}\left((\mathcal{O}z)(t), (\mathcal{O}z)(t')\right) \to 0$ as $|t-t'| \to 0$, proving that $\mathcal{O}z$ is continuous on I_{h} for $z \in \mathbb{Y}$. Additionally, we have

$$\begin{aligned}
\mathfrak{D}((\mathcal{O}z)(t), \tilde{0}) &\leq \frac{1}{\Gamma(\beta)}\int_0^t \mathfrak{D}\left(\psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s)), \tilde{0}\right)ds, \\
&\leq \frac{\mathfrak{M}\mathrm{h}^{\beta}}{\Gamma(\beta+1)}, \\
&\leq \tau_0,
\end{aligned}$$

for all $t \in I_{\mathrm{h}}$.
As a result, $\mathcal{O}$ is a self-mapping $\mathcal{O} : \mathbb{Y} \to \mathbb{Y}$. We now prove that $\mathcal{O}$ is continuous. Let $z_n, z \in \mathbb{Y}$, $n = 1, 2, \ldots$, such that $z_n \xrightarrow{n\to+\infty} z$ in $C(\mathfrak{I}, \mathcal{Y}^c)$, i.e., $\mathfrak{h}_0(z_n, z) \underset{n\to\infty}{\longrightarrow} 0$. It follows that, for any $t \in \mathfrak{I}$, we deduce that

$$
\begin{aligned}
&\mathfrak{h}_0\left(\mathcal{O}z_n, \mathcal{O}z\right) \\
&= \sup_{t\in J} \mathfrak{D}\left(\left(\mathcal{O}z_n\right)(t), (\mathcal{O}z)(t)\right), \\
&= \sup_{t\in I_{\mathrm{h}}} \mathfrak{D}\left(\frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}\left(s, z_n(s)\right)ds, \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s))ds\right), \\
&\leq \frac{1}{\Gamma(\beta)}\sup_{t\in I_{\mathrm{h}}}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathfrak{D}\left(\mathcal{W}\left(s, z_n(s)\right), \mathcal{W}(s,z(s))\right)ds, \\
&\leq \frac{1}{\Gamma(\beta)}\sup_{t\in I_{\mathrm{h}}}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathfrak{P}(s)\mathfrak{e}\left(\mathfrak{h}_0\left(z_n, z\right)\right)ds, \\
&\leq \sup_{t\in I_{\mathrm{h}}} \mathcal{I}^{\beta,\psi}\mathfrak{P}(t)\mathfrak{e}\left(\mathfrak{h}_0\left(z_n, z\right)\right).
\end{aligned}
$$

Under the conditions of (**H**), $\mathfrak{e}(0) = 0$ and $\mathfrak{e}$ is continuous on $[0,\infty)$, ensuring $\mathfrak{e}(r) \underset{r\to 0^+}{\longrightarrow} 0$. Since $\mathfrak{h}_0\left(z_n, z\right) \underset{n\to\infty}{\longrightarrow} 0$ and $\mathcal{I}^{\beta,\psi}\mathfrak{P}(t) < \mathcal{B}$, we conclude that $\mathcal{O}z_n \to \mathcal{O}z$ as $n\to\infty$, which shows that $\mathcal{O}$ is continuous. □

Theorem 4 *Let $\beta \in (0,1)$ be fixed, and assume that the fuzzy function $\mathcal{W} : \mathfrak{I}\times\mathcal{Y}^c \to \mathcal{Y}^c$ is continuous and compact, satisfying* (**H**). *Suppose further that $\mathcal{Z} : \mathfrak{I}\times\mathcal{Y}^c \to \mathcal{Y}^c$ satisfies (3.2). Under these conditions, the uncertain differential equation of nonlocal type (3.1) admits at least one solution on the interval I_h, which is continuous, where h is a positive number such that $h < a$.*

Proof: Using Remark 1, we are led to the study of the uncertain integral equation

$$
z(t) = \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s))ds + \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{Z}(s,z(s))ds,
$$

□

Through Lemma 1, the mapping $\mathcal{O}$, defined by Eq. (3.3), is properly defined, and its continuity on $C\left(I_{\mathrm{h}}, \mathcal{Y}^c\right)$ is verified. Next, we demonstrate that $\mathcal{O}$ is a compact mapping. Fix an arbitrary $z\in\mathbb{Y}$, and let $t, t' \in I_{\mathrm{h}}$ with $t \leq t'$. Then

$$
\begin{aligned}
&\mathfrak{D}\left((\mathcal{O}z)(t), (\mathcal{O}z)\left(t'\right)\right) \\
&= \frac{1}{\Gamma(\beta)}\mathfrak{D}\left(\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s))ds, \int_0^{t'} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds\right), \\
&\leq \frac{1}{\Gamma(\beta)}\mathfrak{D}\left(\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,z(s))ds, \int_0^{t} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds\right), \\
&+ \frac{1}{\Gamma(\beta)}\mathfrak{D}\left(\int_0^t \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds, \int_0^{t'} \psi'(s)\mathcal{H}_{t',s}^{\beta-1}\mathcal{W}(s,z(s))ds\right), \\
&\leq \frac{\mathfrak{M}}{\Gamma(\beta+1)}\left(\mathcal{H}_{t,0}^{\beta} - \mathcal{H}_{t',0}^{\beta} + 2\mathcal{H}_{t',t}^{\beta}\right).
\end{aligned}
$$

This shows that $\mathcal{O}(\mathbb{Y})$ is equicontinuous in $C\left(I_{\mathrm{h}}, \mathcal{Y}^c\right)$. Next, to show the relative compactness of $\mathcal{O}(\mathbb{Y})(t)$ in $\mathcal{Y}^c$, we use Theorem 1, hence, equivalently, we show that $\mathcal{O}(\mathbb{Y})(t)$ is compact-supported and level-equicontinuous in $\mathcal{Y}^c$.

We now establish the relative compactness of $\mathcal{O}(\mathbb{Y})(t)$ in $\mathcal{Y}^c$ by invoking Theorem 1. This is equivalent to proving that $\mathcal{O}(\mathbb{Y})(t)$ is compact-supported and level-equicontinuous in $\mathcal{Y}^c$.

$$
v = \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,x(s))ds, \text{ for some } x\in\mathbb{Y}.
$$

By assumption, $\mathcal{W}$ is a compact mapping, which implies that $\mathcal{W}(I_{\mathrm{h}} \times \mathbb{Y})$ is relatively compact in $\mathcal{Y}^c$. According to Theorem 1, $\mathcal{W}(I_{\mathrm{h}} \times \mathbb{Y})$ is level-equicontinuous. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathfrak{D}\left([\mathcal{W}(s,x(s))]^{\zeta_1}, [\mathcal{W}(s,x(s))]^{\zeta_2}\right) < \frac{\Gamma(\beta+1)\varepsilon}{2\mathrm{h}^{\beta}}, \forall (s,x) \in I_{\mathrm{h}} \times \mathbb{Y},$$

provided $|\zeta_1 - \zeta_2| < \delta$. Following the approach in [3], for $|\zeta_1 - \zeta_2| < \delta$, we conclude that

$$\begin{aligned}
\mathfrak{D}\left([\mu]^{\zeta_1}, [\mu]^{\zeta_2}\right) &= \mathfrak{D}\left([\mathcal{O}(x)(t)]^{\zeta_1}, [\mathcal{O}(x)(t)]^{\zeta_2}\right), \\
&\leq \frac{1}{\Gamma(\beta)} \mathfrak{D}\left(\left[\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,x(s))ds\right]^{\zeta_1}, \left[\int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,x(s))ds\right]^{\zeta_2}\right), \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1} \mathfrak{D}\left([\mathcal{W}(s,x(s))]^{\zeta_1}, [\mathcal{W}(s,x(s))]^{\zeta_2}\right) ds, \\
&\leq \varepsilon.
\end{aligned}$$

Therefore, $\mathcal{O}(\mathbb{Y})(t)$ is level-equicontinuous in $\mathcal{Y}^c$. Due to the relative compactness of $\mathcal{W}(I_{\mathrm{h}} \times \mathbb{Y})$, we can assert the existence of a compact set $\mathcal{X} \subset \mathbb{R}^n$ with $[\mathcal{W}(s,x(s))]^0 \subseteq \mathcal{X}$ for all $(s,x) \in I_{\mathrm{h}} \times \mathbb{Y}$. Thus, we conclude that

$$\begin{aligned}
\left[\frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{W}(s,x(s))ds\right]^0 &= \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}[\mathcal{W}(s,x(s))]^0 ds \\
\subseteq \frac{M}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1} ds &= \frac{M\mathcal{H}_{t,0}^{\beta}}{\Gamma(\beta+1)}.
\end{aligned}$$

As a result, $\mathcal{O}(\mathbb{Y})(t)$ is compact-supported as a subset of $\mathcal{Y}^c$. Applying the Arzelà-Ascoli theorem, we deduce the relative compactness of $\mathcal{O}(\mathbb{Y})$ in $C(I_h, \mathcal{Y}^c)$. Additionally, we define the mapping $\widetilde{\mathcal{O}} : \mathbb{Y} \to C(I_{\mathrm{h}}, \mathcal{Y}^c)$ as follows:

$$(\widetilde{\mathcal{O}}z)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathcal{Z}(s,z(s))ds, \quad t \in I_{\mathrm{h}}. \tag{3.5}$$

Therefore,

$$\begin{aligned}
\mathfrak{D}((\widetilde{\mathcal{O}}z_1)(t), (\widetilde{\mathcal{O}}z_2)(t)) &\leq \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathfrak{D}(\mathcal{Z}(s,z_1(s)), \mathcal{Z}(s,z_2(s)))ds, \\
&\leq \frac{L}{\Gamma(\beta)} \int_0^t \psi'(s)\mathcal{H}_{t,s}^{\beta-1}\mathfrak{D}(z_1(s), z_2(s))ds, \\
&\leq \frac{L\mathcal{H}_{t,0}^{\beta}}{\Gamma(\beta+1)}\mathfrak{h}_0(z_1,z_2), \\
&\leq \frac{L\mathrm{h}^{\beta}}{\Gamma(\beta+1)}\mathfrak{h}_0(z_1,z_2).
\end{aligned}$$

Thus, if h is small enough such that $Lh^{\beta} < \Gamma(\beta+1)$, the mapping $\widetilde{\mathcal{O}}$ is contractive.
From Theorem 3, the mapping $\mathcal{O} + \widetilde{\mathcal{O}}$ has at least one fixed point in $\mathbb{Y}$. It follows that there exists at least one continuous fuzzy solution z to (3.1) on I_h, for some $0 < h \leq a$.

Remark 2 *The compactness requirement In $\mathcal{W}$ in Theorem 4 can be relaxed. Assume $\mathcal{W}(I_{\mathrm{h}} \times F)$ is level-equicontinuous, $\mathfrak{P}$ is bounded on I_{h}, and there exists $r_0 \in \mathcal{Y}^c$ such that $\cup_{t\in I_{\mathrm{h}}} [\mathcal{W}(t,r_0)]^0$ is bounded.*

Let B be a bounded set in $I_{\mathrm{h}} \times F$, and assume $\mathcal{W}$ is bounded in $\mathcal{Y}^c$. We now examine the relative compactness of $\mathcal{W}(B) \subseteq \mathcal{W}(I_{\mathrm{h}} \times F)$.

When $t \in I_{\mathrm{h}}$ and $r \in F$, it is deduced that

$$\begin{aligned}\mathfrak{D}\left([\mathcal{W}(t,r)]^0,[\mathcal{W}(t,r_0)]^0\right) &\leq \sup_{\zeta\in[0,1]} \mathfrak{D}\left([\mathcal{W}(t,r)]^\zeta,[\mathcal{W}(t,r_0)]^\zeta\right),\\ &= \mathfrak{D}\left(\mathcal{W}(t,r),\mathcal{W}(t,r_0)\right),\\ &\leq \mathfrak{P}(t)\mathfrak{e}\left(\mathfrak{h}_0\left(\mathfrak{C}_r,\mathfrak{C}_{r_0}\right)\right),\end{aligned}$$

where we define the auxiliary constant function for each $z \in \mathcal{Y}^c$, as follows:

$$\begin{aligned}\mathfrak{C}_z : I_{\mathrm{h}} \to &\quad \mathcal{Y}^c,\\ t \to &\,\mathfrak{C}_z(t) := z.\end{aligned}$$

Since $\mathcal{W}$ is bounded and $r \in F$, then $\mathfrak{D}(r, r_0)$ is bounded and, thus, $\mathfrak{h}_0(\mathfrak{C}_r, \mathfrak{C}_{r_0})$ is bounded. On the other hand, since $\mathfrak{e}$ is continuous, then $w(\mathfrak{h}_0(\mathfrak{C}_r, \mathfrak{C}_{r_0}))$ is bounded.By assumption, $\mathfrak{P}$ is bounded on I_{h}, which implies that$\mathfrak{D}\left([\mathcal{W}(t,r)]^0,[\mathcal{W}(t,r_0)]^0\right)$ is bounded for all $r \in F$.
Therefore, there exists $\eta > 0$ satisfying

$$\mathfrak{D}\left([\mathcal{W}(t,r)]^0,[\mathcal{W}(t,r_0)]^0\right) \leq \eta, \forall t \in I_{\mathrm{h}}, \forall r \in F. \tag{3.6}$$

Denote $\mathcal{W}(t,r) =: \mathfrak{p}$, and $\mathcal{W}(t,r_0) =: \mu$.
With these assumptions, we can confirm the existence of
$\mathcal{B} > 0$ with

$$[\mathcal{W}(t,r_0)]^0 = \left[\underline{\mu}^0, \bar{\mu}^0\right] \subseteq [-\mathcal{B}, \mathcal{B}].$$

So, by (3.6), we have

$$\mathfrak{D}\left([\mathfrak{p}]^0,[\mu]^0\right) = \max\left\{\left|\underline{\mathfrak{p}}^0-\underline{\mu}^0\right|,\left|\bar{\mathfrak{p}}^0-\bar{\mu}^0\right|\right\} \leq \eta.$$

Then, $\left|\underline{\mathfrak{p}}^0-\underline{\mu}^0\right| \leq \eta$ and $\left|\bar{\mathfrak{p}}^0-\bar{\mu}^0\right| \leq \eta$.
Thus, we conclude that

$$\begin{aligned}\underline{\mathfrak{p}}^0 &\in \left[\underline{\mu}^0-\eta, \underline{\mu}^0+\eta\right] \subseteq [-\mathcal{B}-\eta, \mathcal{B}+\eta],\\ \bar{\mathfrak{p}}^0 &\in \left[\bar{\mu}^0-\eta, \bar{\mu}^0+\eta\right] \subseteq [-\mathcal{B}-\eta, \mathcal{B}+\eta].\end{aligned}$$

Hence,

$$[\mathcal{W}(t,r)]^0 = [\mathfrak{p}]^0 \subseteq [-\mathcal{B}-\eta, \mathcal{B}+\eta], \forall t \in I_{\mathrm{h}}, \forall r \in F.$$

Consequently, there exists a compact set $\eta_0 := [-\mathcal{B}-\eta, \mathcal{B}+\eta]$ such that $[\mathcal{W}(I_{\mathrm{h}} \times F)]^0 \subseteq \eta_0$.

Remark 3 *Under the assumption that $\mathfrak{e}$ is nondecreasing and the inequality*

$$\mathfrak{D}(\mathcal{W}(t,z_1),\mathcal{W}(t,z_2)) \leq \mathfrak{P}(t)\mathfrak{e}(\mathfrak{D}(z_1,z_2)), \text{ for every } t \in \mathfrak{I}, z_1, z_2 \in \mathcal{Y}^c, \tag{3.7}$$

The previous results remain valid. As observed in [26] for the classical case, if $\mathfrak{P}(t) = \theta$, with $\theta > 0$ a constant, then inequality (3.7) reduces to the Osgood condition. Specifically, if $\mathfrak{P}(t) = \theta$ with $\theta > 0$ and $\mathfrak{e}(\mathfrak{D}(z_1,z_2)) = \mathfrak{D}(z_1,z_2)$, meaning $\mathfrak{e}$ is the identity, then inequality (3.7) represents the Lipschitzian nature of $\mathcal{W}$ in the second variable.

4. Applications in physics

The use of arbitrary-order calculus in physical problems has been widely emphasized in the literature. As a specific illustration of the article's results, we focus on the model introduced in [20], which connects nuclear magnetic resonance to the Bloch equation, a first-order system employed for material modeling. By extending this model to the fractional domain, it becomes possible to analyze heterogeneous materials, providing significant advantages for studying multiexponential signal decay processes. In particular, [20] employs a ψ-Caputo fractional derivative to describe a single-spin system in a static magnetic field at resonance.

We now analyze a ψ-Riemann-Liouville-type equation incorporating uncertainty, given by:

$$\begin{cases} D^{\beta,\psi}\mathfrak{E}_z(t) = \gamma_t^{1-\beta}(A_0 \ominus A_1\mathfrak{E}_z(t)), \\ D^{\beta,\psi}\mathfrak{E}_x(t) = \gamma_t^{1-\beta}(B_0\mathfrak{E}_y(t) \ominus B_1\mathfrak{E}_x(t)), \\ D^{\beta,\psi}\mathfrak{E}_y(t) = \gamma_t^{1-\beta}((-B_0)\,\mathfrak{E}_x(t) \ominus B_1\mathfrak{E}_y(t)), \end{cases} \tag{4.1}$$

In this setting, $\mathfrak{E}_x(t)$, $\mathfrak{E}_y(t)$, and $\mathfrak{E}_z(t)$ characterize the system's magnetization coordinates, and A_0, A_1, B_0, B_1 are suitable (uncertain) constants linked to the equilibrium magnetization, spin-lattice relaxation time, spin-spin relaxation time, and resonant frequency. The parameter $\gamma_t > 0$ is introduced to ensure consistency with dimensionality, as described in [12].

The first equation in this system fits within the framework being studied. Similarly, comparable results can be obtained for the space $C(\mathfrak{I}, \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})$ with necessary adjustments. This allows problem (4.1) to be expressed as a higher-dimensional extension of problem (3.1). by choosing $\mathcal{W} \equiv 0$ and

$$\mathcal{Z}\left(t, z_1, z_2, z_3\right) = \gamma_t^{1-\beta}\left(B_0 z_2 \ominus B_1 z_1, \left(-B_0\right) z_1 \ominus B_1 z_2, A_0 \ominus A_1 z_3\right),$$

provided $\mathcal{Z}$ is well-defined and adheres to the specified conditions.

5. Conclusion

In conclusion, this paper explored the existence of solutions for a class of fuzzy differential equations with nonlocal derivatives. By applying an extended version of Krasnosel'skii's fixed point theorem within the framework of fuzzy metric spaces, we demonstrated that the problem has a fuzzy solution defined over a specific interval. The approach allowed us to analyze the corresponding integral problem to which the theorem was applied. Finally, this study opens the door to potential applications in physical fields, where fuzzy differential equations could provide more suitable models for uncertain or imprecise systems.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. Agarwal, R. P., Lakshmikantham, V., & Nieto, J. J. (2010). On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Analysis, 72*, 2859-2862.
2. Agarwal, R. P., Arshad, S., O'Regan, D., & Lupulescu, V. (2012). Fuzzy fractional integral equations under compactness type condition. *Fractional Calculus & Applied Analysis, 15*, 572-590.
3. Agarwal, R. P., Arshad, S., O'Regan, D., & Lupulescu, V. (2013). A Schauder fixed point theorem in semilinear spaces and applications. *Fixed Point Theory and Applications, 2013*, 306.
4. Agarwal, R. P., Baleanu, D., Nieto, J. J., Torres, D. F., & Zhou, Y. (2017). A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *Journal of Computational and Applied Mathematics, 339*, 3-29.
5. Arshad, S., & Lupulescu, V. (2011). On the fractional differential equations with uncertainty. *Nonlinear Analysis, 74*, 3685-3693.
6. Arshad, S. (2013). On existence and uniqueness of solution of fuzzy fractional differential equations. *Iranian Journal of Fuzzy Systems, 10*, 137-151.
7. Bede, B. (2013). *Mathematics of fuzzy sets and fuzzy logic.* Springer.
8. Burton, T. A. (1998). A fixed point theorem of Krasnoselskii. *Applied Mathematics Letters, 11*(1), 85-88.
9. Diamond, P., & Kloeden, P. (1994). *Metric spaces of fuzzy sets.* World Scientific.
10. Eiman, K., Shah, M., & Sarwar, D. (2020). Study on Krasnoselskii's fixed point theorem for Caputo-Fabrizio fractional differential equations. *Advances in Differential Equations.*
11. El-Nabulsi, R. A. (2020). Inverse-power potentials with positive-bound energy spectrum from fractal, extended uncertainty principle and position-dependent mass arguments. *The European Physical Journal Plus, 135*(9), 1-15.
12. Gómez-Aguilar, J. F., & Baleanu, D. (2014). Fractional transmission line with losses. *Zeitschrift für Naturforschung A, 69*(10-11), 539-546.
13. Godini, G. (1985). A framework for best simultaneous approximation, Normed almost linear spaces. *Journal of Approximation Theory, 43*, 338-358.

14. Haseli, S. (2020). Quantum-memory-assisted entropic uncertainty relation with moving quantum memory inside a leaky cavity. *The European Physical Journal Plus, 135*(9), 1-11.
15. Hammad, H. A., & De la Sen, M. (2022). Involvement of the fixed point technique for solving a fractional differential system. *AIMS Mathematics, 7*(4), 7093-7105.
16. Khastan, A., Nieto, J. J., & Rodríguez-López, R. (2014). Schauder fixed-point theorem in semilinear spaces and its application to fractional differential equations with uncertainty. *Fixed Point Theory and Applications, 2014*(1), 1-14.
17. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations.* Elsevier.
18. Long, H. V., & Dong, N. P. (2018). An extension of Krasnoselskii's fixed point theorem and its application to nonlocal problems for implicit fractional differential systems with uncertainty. *Journal of Fixed Point Theory and Applications.*
19. Long, H. V. (2018). On random fuzzy fractional partial integro-differential equations under Caputo generalized Hukuhara differentiability. *Computational and Applied Mathematics, 37*(3), 2738-2765.
20. Magin, R., Feng, X., & Baleanu, D. (2009). Solving the fractional order Bloch equation. *Concepts in Magnetic Resonance Part A, 34A*(1), 16-23.
21. Melliani, S., Mfadel, A. E., Chadli, L. S., & Elomari, M. H. (2021). Stability of intuitionistic fuzzy nonlinear fractional differential equations. *Notes Instuitionistic Fuzzy Sets, 27*(1), 83-100.
22. El Mfadel, A., Melliani, S., & Elomari, M. H. (2021). Notes on local and nonlocal intuitionistic fuzzy fractional boundary value problems with Caputo fractional derivatives. *Journal of Mathematics, 2021*(1), 4322841.
23. El Mfadel, A., Melliani, S., & Elomari, M. H. (2021). On the existence and uniqueness results for fuzzy linear and semilinear fractional evolution equations involving Caputo fractional derivative. *Journal of Function Spaces, 2021*(1), 4099173.
24. Elmfadel, A., Melliani, S., & Elomari, M. (2021). On the existence result of fuzzy fractional boundary value problems. *International Journal On Optimization and Applications, 1.*
25. Román-Flores, H., & Rojas-Medar, M. (2002). Embedding of level-continuous fuzzy sets on Banach spaces. *Information Sciences, 144*, 227-247.
26. Yu, C., & Gao, G. (2005). Existence of fractional differential equations. *Journal of Mathematical Analysis and Applications, 310*, 26-29.

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