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On the Study of Algebraic Structures: Multiset Dimensions in Zero-Divisor Graphs Associated with Rings

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ABSTRACT: This article explores the multiset dimension (Mdim) in zero divisor graphs (ZD-graphs) of commutative rings with unity. Given a finite ring R, its zero divisors form the set L(R), which defines the ZD-graph. We establish general bounds for Mdim in ZD-graphs and extend these results to various rings, including Gaussian integers, Ring of Z_n modulo n, and quotient polynomial rings. Notably, we provide a complete characterization of Mdim for Z_n for all n. Moreover, this study helps characterizing rings based on the Mdim of associated ZD-graphs. Our findings reveal structural patterns among rings with identical Mdim, emphasizing its role in isomorphism and enhancing the algebraic understanding of these graphs.

Key Words: Rings, zero-divisor graphs, multiset dimensions, bounds, graph theory, algebraic structures, ring theory, mathematical analysis, graph bounds, structural properties

Contents

1. Introduction

The interplay between graph theory and algebra has led to significant insights, particularly through the study of ZD-graphs associated with commutative rings. Beck [1] pioneered this connection by introducing the concept of ZD-graphs denoted by Z^o (R), focusing on the correspondence between ring elements and graph nodes, where a zero vertex is connected to all other remaining vertices. Subsequently, Anderson and Livingston [2] extended this notion to consider ZD-graphs where each vertex stands for a nonzero zero divisor. In this formulation, an undirected graph is formed by considering ring elements x and y as nodes connected by an edge if their product equals zero. Notably, Anderson and Livingston's investigation concentrated on finite rings, establishing connections between ring properties and graph theoretic properties such as completeness or star structure. This approach, denoted as Z(R), slightly deviates from Beck's original definition of ZD-graphs, wherein zero is not considered a vertex. Anderson and Livingston's findings shed light on the relationship between ring properties and the structural characteristics of Z(R), offering valuable insights into both algebraic and graph theoretic domains.

Expanding upon this foundation, Redmond [3] extended the study of ZD-graphs to include noncommutative rings, introducing various methods to characterize ZD-graphs for both undirected and directed graphs. Redmond [4] further enriched this exploration by introducing ideal-based ZD-graphs for commutative rings. This novel approach involves substituting elements with zero products with elements

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belonging to a specific ideal I of the ring R, yielding a graph denoted as $\Gamma_I(R)$. Throughout the paper L(R) denotes the set of zero divisors of ring R and Z(R) will be type of ZD-graph under consideration.

Throughout the literature, numerous variations of ZD-graphs have been proposed, including total graphs, unit graphs, Jacobson graphs, and ZD-graphs based on equivalence classes. Each variant offers a distinct perspective on the algebraic and structural properties inherent in commutative rings, enriching the study of ZD-graphs and their applications in both algebra and graph theory. These works can be found in sources such as [5-9]. Readers may refer to [10, 11] for a deeper understanding of graph theory, and for some fundamental definitions of ring theory, [12, 13] can be consulted.

Simanjuntak et al. [15] introduced a new variant of the metric dimension (md), known as the *multiset dimension* (Mdim), where a multiset of distances between a vertex v and all vertices in a resolving set (RS) W is calculated, including their multiplicities. The multiset dimension is defined as the minimum cardinality of such a resolving set.

Assume v is a vertex of a graph G and $B \subseteq V(G)$. Then, the representation of the multiset of v with respect to B is defined as the multiset of distances between v and the vertices in B. This representation is denoted by $r_m(v \mid B)$. For every pair of distinct vertices u and v, B is called a multiset resolving set (M-RS) of G if $r_m(u \mid B) \neq r_m(v \mid B)$.

The cardinality of an M-RS is called the multiset basis of G, and the minimum cardinality among all multiset bases is called the multiset dimension of G, denoted by Mdim(G). If G does not contain any M-RS, we write Mdim(G) $= \infty$.

The key point of this article is that the apparent simplifications are an oversimplification of the task of identifying graph vertices using the multiset representation.

The practical applications of our findings extend to various fields. For instance, insights from [24] and [25] illustrate the utility of advanced graph analysis techniques. Our study enhances this by providing new tools for understanding the structural properties of algebraic graphs. In recent years, there has been growing interest in the study of multiset dimensions of graphs, particularly in the context of zero-divisor graphs associated with rings. The concept of multiset dimension provides a nuanced understanding of the structural properties of graphs. Several studies have focused on the local multiset dimension of various graph families, highlighting its significance in graph theory. For instance, Alfarisi et al. explored the local multiset dimension of amalgamation graphs, providing valuable insights into their structural characteristics [26]. Further, Alfarisi, Susilowati, and Dafik investigated the local multiset dimension of some specific families of graphs, contributing to the broader understanding of this concept [27]. Additionally, Alfarisi et al. discussed both the multiset dimension and local multiset dimension of graphs, offering foundational perspectives on these measures [28]. These studies underscore the importance of examining multiset dimensions to uncover deeper properties of graphs, motivating our investigation into certain bounds for multiset dimensions in zero-divisor graphs associated with rings. Recent studies, such as [29] and [30], highlight advanced techniques in graph analysis. These works emphasize the importance of multi-dimensional and self-supervised approaches to understanding complex networks. Our investigation of multiset dimensions in zero-divisor graphs contributes to this broader effort by revealing key structural properties of algebraic graphs.

The article presents significant contributions to the field. Primarily, it expands the notion of Mdim to ZD-graphs, with a special emphasis on the ZD-graph representing the ring of integers Z_n modulo n. This expansion facilitates varied descriptions of rings by examining the vertices within their respective ZD-graphs, integrating metric and multiset representations effectively. Through rigorous proofs, it establishes that rings can be effectively characterized by their multiset dimensions. Additionally, it offers insights into various rings, demonstrating that their ZD-graphs can have bounded multiset dimensions determined by the graph's diameter. Moreover, the article presents a straightforward method for computing the Mdim of ZD-graphs for rings of integers modulo n, enhancing accessibility to this important metric. The novelty of this research lies in its exploration of multiset dimensions in the context of graph theory, a relatively underexplored area. By unveiling the multiset dimensions of ZD-graphs, the research significantly enhances our grasp of the inherent structural and algebraic characteristics found within these graphs. This progress bodes well for various practical applications such as network design, social networking, and communication systems. A deeper comprehension of graph structures is imperative for maximizing performance and efficiency in these domains.

2. Preliminaries

In the preliminary section of this research article, we establish foundational concepts and terminology pertinent to graph theory and its applications within algebraic structures. A graph, formally defined as an ordered pair G=(V,E), comprises a set of vertices or nodes (V) and a set of edges (E). The order of a graph refers to the cardinality of its vertex set, while its size pertains to the cardinality of its edge set. The distance between two nodes u' and v' is defined as the length of the shortest path connecting them, while the distance from a node w to an edge e'=u' v' is determined as the minimum of the distances from w to u' and v'. Graphs exhibit diverse structural properties, including regularity and completeness. A graph is considered regular if every vertex has the same degree, specified as deg(r)=c for a fixed c in the positive integers. Complete graphs, denoted as k_m, establish connections between all pairs of vertices, where m denotes the number of vertices. Complete bipartite graphs, typically represented as $k_{m,n}$, are partitioned into two distinct sets of vertices, X and Y, with each vertex in X connected to every vertex in Y. A significant graph theoretic concept is that of a cut vertex, which arises when the removal of a vertex from a connected graph results in two or more disconnected components.

Recent research has extended graph theory into algebraic realms, particularly concerning ZD-graphs. Redmond's investigations in [17] and [18] explored ZD-graphs of noncommutative and commutative rings, respectively. Subsequent studies, including those by Siddiqui et al. in [20] and Pirzada and Aijaz in [21], have further delved into metric parameters for ZD-graphs, examining dimensions and bounds associated with these algebraic structures. Such endeavors not only enrich the theoretical understanding of graph theory but also offer practical insights into the structural properties of algebraic systems. Simanjuntak et al. [15] found some sharp bounds for Mdim of arbitrary graphs in term of its md, order, or diameter. Siamanjuntak also provided some necessary conditions for a graph to have finite Mdim, with an example of an infinite family of graphs where those necessary conditions are also sufficient. It was also shown that the Mdim of any graph other than a path is at least 3 and two families of graphs having the Mdim 3 were proved. Here, we consider some results from [15] as follows:

Theorem 2.1 [15]: For any integer $m \geq 3, n \geq 6$, $\mathrm{Mdim}(P_m) = 1$, $\mathrm{Mdim}(k_m) = \infty$, and $\mathrm{Mdim}(C_n) = 3$. Moreover, $\mathrm{Mdim}(P_m) = 1$ if and only if $G \cong P_m$.

Moreover, for a complete bipartite graph $K_{m,n}$, we get different Mdim for different choices of values of m and n.

Theorem 2.2 [15]: For any complete bipartite graph $K_{m,n}$, the $\mathrm{Mdim}(K_{m,n})$ is given below.

$\mathrm{Mdim}(K_{m,n}) = 1$	for $m = 1$ and $n = 1, 2$ for $m = 2$ and $n = 1$
$\mathrm{Mdim}(K_{m,n}) = \infty$	for $m = 1$ and $n \geq 3$ for $m = 2$ and $n \geq 2$

Moreover, Mdim for a single vertex graph G is supposed to be zero and for an empty graph it is undefined. Our discussion commences with the subsequent observation.

For the readers some notations and terms should be kept in mind. Within the next section 3, We explore the polynomial representations of rings, specifically focusing on the algebraic significance of variables. In our study, r represents elements within the ring. For example, in the ring $\mathbb{Z}_2 \times \mathbb{Z}_4$, an element r can be expressed as (a, b), where $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_4$. The variable X denotes the indeterminate in the polynomial ring. For instance, in $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(X^2)$, X is the variable, and the ideal (X^2) implies that any polynomial expression is considered modulo X^2, making $X^2 = 0$. By clearly defining these variables and their roles, we provide a solid foundation for understanding the algebraic structures we are analyzing and how they influence the bounds of multiset dimensions in zero-divisor graphs.

3. Results and Discussion

Theorem 3.1: Let R be a finite commutative ring with unity. If R is an integral domain (ID), then $\mathrm{Mdim}(\mathbb{Z}(R))$ is undefined.

Proof:

It is well-known that if R is an integral domain (ID), then the set of zero divisors $\mathbb{Z}(R)$ is not defined, implying that the multiset dimension (Mdim) of the ZD-graph $\mathbb{Z}(R)$ is undefined, and conversely. Now, the following result provides the Mdim of the ZD-graph of ring R when the set $\mathbb{Z}(R)$ is isomorphic to the path graph P_m.

Proposition 3.1: Consider R as a finite commutative ring having unity. Then $\text{Mdim}(\mathbb{Z}(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3(r)/(r^2), \mathbb{Z}_2(r)/(r^3)$, or $\mathbb{Z}_4(r)/(2r, r^2 - 2)$.

Proof: Suppose that $\text{Mdim}(\mathbb{Z}(R)) = 1$. The ZD-graphs of rings $\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3(r)/(r^2), \mathbb{Z}_2(r)/(r^3)$, or $\mathbb{Z}_4(r)/(2r, r^2 - 2)$ are path graphs, and we know that path graphs are the only graphs whose Mdim $= 1$ by Theorem 2.1. Since $|L(R)|$ is at most three whenever $\mathbb{Z}(R) \cong P_n$, using [**?**], Lemma 2.6, $\mathbb{Z}(R)$ is either P_2 or P_3.

Case I: Let's consider the case where $\mathbb{Z}(R) \cong P_2$, moreover, let $|L(R)| = a, b$ satisfying $a \cdot b = 0$. Rings that fulfill this property encompass $\mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3(r)/(r^2)$.

Case II: In the case where $\mathbb{Z}(R)$ is isomorphic to P_3, considering $|L(R)| = a, b, c$, satisfying $a \cdot b = 0$ and $b \cdot c = 0$. Rings satisfying these conditions include $\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_2(r)/(r^3)$, and $\mathbb{Z}_4(r)/(2r, r^2 - 2)$ [14]. Conversely, the ZD-graphs of the mentioned rings are either P_2 or P_3 [14]. Thus, by Theorem 2.1, $\text{Mdim}(\mathbb{Z}(R)) = 1$.

Proposition 3.2: Let R be a finite commutative ring with unity and R is isomorphic to one of the following rings, $\mathbb{Z}_3 \times \mathbb{Z}_3, K_4(r)/(r^2), \mathbb{Z}_4(r)/(r^2 + r + 1), \mathbb{Z}_4(r)/(2, r)^2, \mathbb{Z}_2[r, s]/(r, s)^2$. Then $\mathbb{Z}(R) \cong C_m$, and $\text{Mdim}(\mathbb{Z}(R))$ is infinite.

Proof: Assuming R is a commutative ring with unity and $\mathbb{Z}(R)$ is a cyclic graph, according to [**?**], Theorem 2.4, it follows that the maximum length of the cyclic graph is 4. Since $\mathbb{Z}(R) \cong C_m$ and m does not exceed 4, so by Theorem 2.1, $\text{Mdim}(\mathbb{Z}(R)) = \infty$.

The corresponding ZD-graphs for the rings mentioned above can be observed in Figure 1.

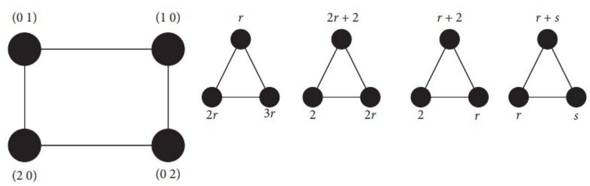

Figure 1: ZD-graphs of $\mathbb{Z}_3 \times \mathbb{Z}_3, K_4(r)/(r^2), \mathbb{Z}_4(r)/(r^2 + r + 1), \mathbb{Z}_4(r)/(2, r)^2, \mathbb{Z}_2[r, s]/(r, s)^2$.

Theorem 3.2: Let R be a finite commutative ring with unity. If R is isomorphic to one of the following rings $\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times (\mathbb{Z}_4[X])/(x^2), \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\text{Mdim}(\mathbb{Z}(R)) = 3$.

Proof: Consider the ring R is isomorphic to one of the following rings $\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times (\mathbb{Z}_4[X])/(x^2), \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The ZD-graphs of the above rings are shown in Figure 2 and Figure 3. Then there exists a minimal resolving set for $\mathbb{Z}(R)$, say $\{v_1, v_2, v_3\}$, by [3], Theorem 2.3. Now, by [23], Theorem 2.3, $\text{Mdim}(\mathbb{Z}(R)) = 3$.

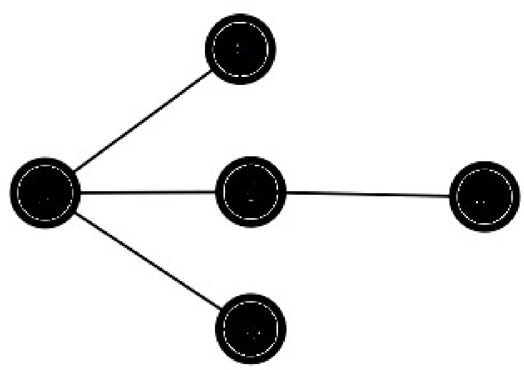

Figure 2: Graph of $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(x^2)$

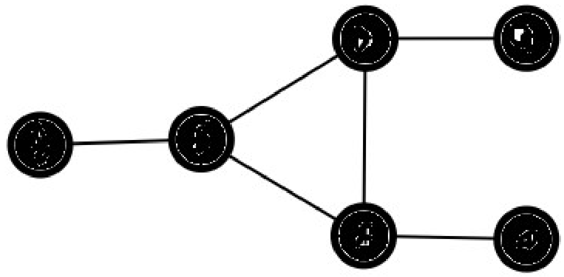

Figure 3: Graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Theorem 3.3: Let R be a finite commutative ring with unity, such that each element r in the zero-divisor set $L(R)$ is nilpotent. Then:

(a) If $|L(R)| \geq 3$ and $L(R)^2 = \{0\}$, then $\mathrm{Mdim}(\mathbb{Z}(R)) = \infty$.

(b) If $|L(R)| \geq 3$ and $L(R)^2 \neq 0$, then $\mathrm{Mdim}(\mathbb{Z}(R))$ is finite.

Proof: (a) Assuming that $|L(R)| \geq 3$ and $L(R)^2 = \{0\}$, it follows that the product of any pair of elements r and s in $L(R)$ is zero, i.e., $r \cdot s = 0$ for all $r, s \in L(R)$. By [**?**], Theorem 2.8, this condition implies that $\mathbb{Z}(R)$ forms a complete graph. Therefore, according to Theorem 2.1, the Mdim of $\mathbb{Z}(R)$ is infinite.

(b) Considering $|L(R)| \geq 3$ and $L(R)^2 \neq 0$, there exists an element $r \in L(R)$ such that $r^2 = 0$, indicating the occurrence of some element $s \in L(R)$, thereby the distance between r and s is at least 2. Consequently, $L(R)/(r, s)$ serves as a multiset generator for any vertex s adjacent to r. Thus, the Mdim of $\mathbb{Z}(R)$ is finite in this case.

Corollary 3.1: Let R be a finite commutative ring with unity such that $|L(R)| \geq 3$. If $\mathbb{Z}(R)$ has a cut vertex but no vertex of degree 1, then $\mathrm{Mdim}(\mathbb{Z}(R)) = \infty$. **Corollary 3.1:** Let R be a finite commutative ring with unity such that $|L(R)| \geq 3$. If $\mathbb{Z}(R)$ has a cut vertex but no vertex of degree 1, then $\mathrm{Mdim}(\mathbb{Z}(R)) = \infty$.

Proof: For the given ring R, suppose the ZD-graph Z(R) has a cut vertex but no vertex with degree 1. According to ([25], Theorem 3), this implies that R exhibits an isomorphism with one of the following rings:

$$\frac{\mathbb{Z}_2[r,s]}{(r^2, s^2 - rs)}, \quad \frac{\mathbb{Z}_4[r]}{(r^2 + 2r)}, \quad \frac{\mathbb{Z}_4[r,s]}{(r^2, s^2 - rs, rs - 2, 2r, 2s)},$$

$$\frac{\mathbb{Z}_8[r,s]}{(2r, r^2 + 4)}, \quad \frac{\mathbb{Z}_2[r,s]}{(r^2, s^2)}, \quad \frac{\mathbb{Z}_4[r]}{(r^2)}, \quad \frac{\mathbb{Z}_4[r,s]}{(r^2, s^2, rs - 2, 2r, 2s)}.$$

One should note that Theorem 3 in [25] establishes that if the ZD-graph has a cut vertex but no vertex with degree one, the authors provide a list of rings, having associated ZD-graphs following this property. Figure 4 represents $\mathbb{Z}(R)$ for the first four rings, and Figure 5 represents $\mathbb{Z}(R)$ for the remaining three rings. Therefore, it is concluded that $\mathrm{Mdim}(\mathbb{Z}(R)) = \infty$.

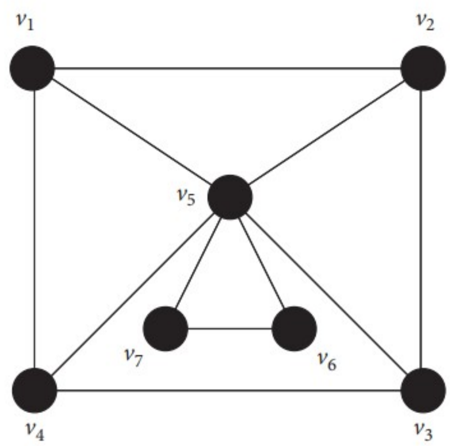

Figure 4: $\frac{\mathbb{Z}_2[r,s]}{(r^2,s^2-rs)}$, $\frac{\mathbb{Z}_4[r]}{(r^2+2r)}$, $\frac{\mathbb{Z}_4[r,s]}{(r^2,s^2-rs,rs-2,2r,2s)}$, $\frac{\mathbb{Z}_8[r,s]}{(2r,r^2+4)}$

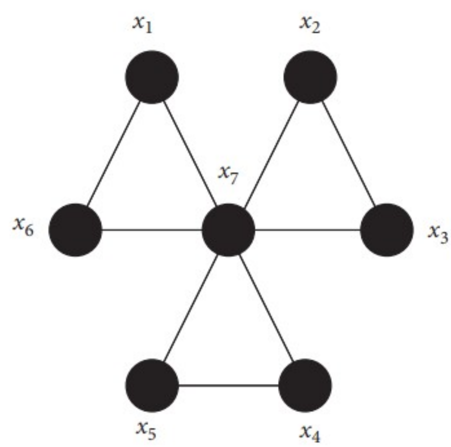

Figure 5: $\frac{\mathbb{Z}_2[r,s]}{(r^2,s^2)}$, $\frac{\mathbb{Z}_4[r]}{(r^2)}$, $\frac{\mathbb{Z}_4[r,s]}{(r^2,s^2,rs-2,2r,2s)}$

Corollary 3.2: Let R be a finite commutative ring with unity and R be a local ring having no cycles in associated $Z(R)$, then $\text{Mdim}(Z(R)) = 1$.
Proof. Let R be a local ring having no cycle in the associated $Z(R)$, then by [**?**, Theorem 2.1], $Z(R) \cong P_2$ or P_3. Hence, $\text{Mdim}(Z(R)) = 1$.
Theorem 3.5: Let $\mathbb{Z}_n$ be a ring of integers modulo n. Assuming p and q as distinct primes, we have $\text{Mdim}(\mathbb{Z}(\mathbb{Z}_n))$ as follows:

n	**Mdim**$(\mathbb{Z}(\mathbb{Z}_n))$
p	undefined
p^2	∞
pq	∞
2^2	0
23	1
3^2	1

Table 1: Multiset dimension of $\mathbb{Z}(\mathbb{Z}_n)$ for various n

Proof: To prove the theorem, it is imperative to address multiple cases individually.

(i) When $n = 2^2$, then $Z(\mathbb{Z}_n)$ consists of a single vertex. So $\text{Mdim}(Z(\mathbb{Z}_n)) = 0$. Now consider $n = 3^2, 2^3$, then $Z(\mathbb{Z}_n)$ consists of two and three vertices respectively, and $Z(R) \cong P_n$ (where $n = 2$ or 3). By Theorem 2.1, we know that $\text{Mdim}(P_m) = 1$, which implies that $\text{Mdim}(Z(\mathbb{Z}_n)) = 1$.

Suppose $n = pq$, where p and q are distinct primes. We can partition the vertices into two sets:

$$U = \{\lambda p \in Z(\mathbb{Z}_n) \mid (\lambda, q) = 1\}, \quad V = \{\lambda q \in Z(\mathbb{Z}_n) \mid (\lambda, p) = 1\}.$$

This partition clearly demonstrates that $Z(\mathbb{Z}_n)$ is bipartite. Furthermore, it is evident that $x \cdot y = 0$ for every $x \in U$ and $y \in V$. Therefore, $Z(\mathbb{Z}_n)$ forms a complete bipartite graph. Consequently, when $n = pq$, $Z(\mathbb{Z}_n) \cong K_{q-1,p-1}$. Thus, by Theorem 2.2, $\text{Mdim}(Z(\mathbb{Z}_n)) = \infty$.

(ii) Consider $p > 3$, if we take n different from case (i), i.e., if $n = p$, then $Z(\mathbb{Z}_n) = \varnothing$ so $\text{Mdim}(Z(\mathbb{Z}_n)) = \infty$. If $n = p^2, p^n$ or 2^2p or any other n, then $Z(\mathbb{Z}_n) \cong K_{p-1}, K_{n,m}$ (not complete), and $K_{q-1,p-1}$. By Theorem 2.2, $\text{Mdim}(Z(\mathbb{Z}_n)) = \infty$.

Table 2: Multiset dimension of $Z(\mathbb{Z}_n)$

n	$\lvert V\rvert$	$\lvert E\rvert$	Diameter	Girth	$Z(\mathbb{Z}_n)$	$\text{Mdim}(Z(\mathbb{Z}_n))$
p	0	0	0	Undefined	$Z(\mathbb{Z}_n) = \emptyset$	∞
2^2	1	0	0	Undefined	•	0
3^2	2	1	1	Undefined	•—•	1
p^2, $p \geq 5$	$p-1$	$\binom{p-1}{2}$	1	3	Complete graph K_{p-1}	∞
2^3	3	2	2	Undefined	•—•—•	1
p^n, $n \geq 3$	$p^{n-1}-1$	$\sum_{i=1}^{p-1}(p-1)^{\lfloor i/2 \rfloor}$	2	3	Quasi-local: $\mathbb{P}^{n-1}$ is attached to everything	∞
2^2p, $p \geq 3$	$2p+1$	$4p-4$	3	4	Bipartite graph (but not complete)	∞
pq	$q-1+p-1$	$(q-1)(p-1)$	2	4	Complete bipartite graph $K_{q-1,p-1}$	1 for (p=3, q=2) and ∞ for all other values
All others			2	3		

4. Bound between multiset dimensions and diameter of Zero-divisor graphs:

Within this section, we explore the relationship between Mdim and the diameter of ZD-graphs. Through our analysis, we uncover insightful bounds that enhance our understanding of these graph properties and their structural nuances.

Lemma 4.1: Suppose G is a ZD-graph with a diameter of at most 2, and $Z(R) \ncong P_n$, then the multiset dimension of $Z(R)$ is infinite.

Proof: Consider the ZD-graph G with a maximum diameter of 2, and assume $Z(R)$ is not a path graph. Consequently, $Z(R)$ can take the form of a cycle with a maximum of 5 vertices, a complete graph, the Petersen graph, or a star graph. Then by using Theorem 2.1 and 2.2 and Theorem 3.1 [15], the result follows.

Corollary 4.1: Let $Z(R)$ be a ZD-graph of a ring R such that $|L(R)| \geq 3$ and diameter d. Then,

$$\text{Mdim}(Z(R)) > f(n, d).$$

(For positive integers n, d, $f(n,d)$ is defined to be the least positive integer k for which

$$\frac{(k+d-1)!}{k!(d-1)!} + k \geq n)$$

Proof: Consider a commutative ring R with identity and let $Z(R)$ denote its zero-divisor graph. For positive integers n and d, the function $f(n,d)$ is defined to be the least positive integer k for which

$$\frac{(k+d-1)!}{k!(d-1)!} + k \geq n.$$

We aim to prove that the multiset dimension $\mathrm{Mdim}(Z(R))$ is at least some function $f(n,d)$.

For $n = 3$, the diameter d of the zero-divisor graph can be at most 2, as for any three vertices, there exists a pair with a distance of at most 2 between them. Thus, $\mathrm{Mdim}(Z(R)) = 2$.

For $n > 3$, let's examine the definition of $f(n,d)$. We want to find the least positive integer k satisfying the inequality mentioned above. This inequality essentially gives us a lower bound on the cardinality of S (resolving set), which is k. Thus, $\mathrm{Mdim}(Z(R)) \geq k$.
Therefore, $\mathrm{Mdim}(Z(R))$ is at least $f(n,d)$. Alternatively, the proof is followed by using Theorem 2.3 and Theorem 2.4 [**?**].

Theorem 4.2: Consider a ZD-graph $Z(R)$ with n vertices, and having diameter k. If $\mathrm{Mdim}(Z(R)) = 3$, then

$$n \leq \frac{k^2(k+3) + 2(k+6)}{6}$$

Proof: We can deduce from the observation that no vertex in a ZD-graph of multiset dimension 3 can have the representation as $(0,1,1)$. Likewise, no vertex may have the representation $(1,1,1)$.

The interpretation of a vertex $v \in V$ with respect to the resolving set W (outside of W) will take the form:

$$\{1^{a_1}, 2^{a_2}, \ldots, k^{a_k}\}, \quad \text{such that } a_1 + a_2 + \cdots + a_k = 3.$$

The number of such solutions is given by:

$$\binom{k+2}{k-1} = \frac{k^2(k+3) + 2k}{6}$$

Since the representation $(1,1,1)$ should not be included, the possible number of multiset representations for vertices outside the resolving set is:

$$\frac{k^2(k+3) + 2k}{6} - 1$$

As $(n-3)$ vertices lie outside the resolving set, we have:

$$\frac{k^2(k+3) + 2k}{6} - 1 \geq n - 3$$

Hence,

$$n \leq \frac{k^2(k+3) + 2k}{6} - 1 + 3 = \frac{k^2(k+3) + 2(k+6)}{6}$$

Open Problem: Find a family of ZD-graphs of rings which satisfy the equality in Theorem 4.2?

5. Methodology

To investigate the multiset dimensions of ZD-graphs associated with rings, we employed a comprehensive methodology involving data visualization, categorization, and dimension analysis. Initially, we utilized both MATLAB and Python algorithms to visualize the ZD-graphs corresponding to various commutative rings, including the ring Z_n of integers modulo n, the ring of Gaussian integers modulo m, and quotient polynomial rings. This visualization process allowed us to gain insights into the structural properties of the graphs and identify key characteristics. Subsequently, we undertook a rigorous categorization process to classify the ZD-graphs based on their structural features. This categorization facilitated the systematic analysis of multiset dimensions across different types of ZD-graphs, enabling us to identify patterns and variations among them.
Following the categorization phase, we conducted in-depth dimension analysis to determine the multiset dimensions of the categorized ZD-graphs. This involved implementing algorithms to calculate the multiset dimensions and establish relationships between the dimensions and other graph parameters, such as diameter and maximum degree. Throughout the methodology, meticulous attention was paid to ensure the accuracy and reliability of the analysis results. The use of both MATLAB and Python algorithms provided flexibility and robustness in data processing and visualization. Additionally, the categorization

and dimension analysis procedures were carried out systematically to ensure comprehensive coverage and thorough exploration of the research objectives.
Overall, the methodology employed in this study facilitated a detailed investigation into the multiset dimensions of ZD-graphs associated with rings, enabling us to uncover valuable insights into the algebraic and graph-theoretic properties of these structures.

6. Discussion

The exploration of Mdim in ZD-graphs presents a compelling avenue for understanding the structural properties of rings and their graphical representations. In this section, we delve into the implications of our findings and discuss the potential applications and future directions of this study. Our analysis reveals that the multiset dimension serves as a distinctive feature that can aid in characterizing rings based on their associated zero divisor graphs. By establishing general bounds and exploring the behavior of Mdim across different types of rings, we can discern characteristic patterns and attributes. This classification approach offers a systematic method for identifying and distinguishing rings based on their algebraic structures. The ability to characterize rings through multiset dimensions has practical implications in various domains. In algebraic cryptography, for instance, understanding the Mdim of rings can inform the selection of cryptographic protocols, ensuring the security and efficiency of cryptographic systems. Additionally, in computational algebra, knowledge of the structural properties of rings via ZD-graphs can streamline algorithms for solving algebraic equations and optimizing computational processes. By examining the relationship between multiset dimensions and zero divisor graphs, we deepen our understanding of algebraic structures and their graphical representations. This study sheds light on the intricate interplay between algebraic concepts and graph theory, highlighting the rich connections between seemingly disparate mathematical disciplines. Moreover, the exploration of Mdim provides insights into the inherent properties and behaviors of rings, paving the way for further research and inquiry. Looking ahead, there are several avenues for future research in this area. Expanding our analysis to encompass a broader range of rings and exploring additional properties of zero divisor graphs could yield further insights into the relationship between algebraic structures and graphical representations. Moreover, investigating the applicability of Mdim in other contexts, such as network theory or combinatorics, could uncover new connections and applications beyond the realm of algebra. In summary, the study of multiset dimensions in zero divisor graphs offers a fruitful avenue for exploring the structural properties of rings and their graphical representations. By leveraging Mdim as a characterization tool, we can identify rings based on their distinctive features and discern characteristic patterns across different types of rings. This discussion not only enhances our understanding of algebraic structures but also opens up new avenues for practical applications and future research endeavors.

7. Conclusion

In this study, we have investigated the characterization of rings through the analysis of multiset dimensions associated with their ZD-graphs. Specifically, we have explored the multiset dimensions of ZD-graphs corresponding to various commutative rings, such as the ring Z_n of integers modulo n, the ring of Gaussian integers modulo m, and polynomial rings. Additionally, we have generalized the multiset dimension for the ring Z_n of integers modulo n. Our investigation has culminated in the presentation of bounds relating the multiset dimension to the diameter of ZD-graphs. Moving forward, there are several avenues for future research in this domain. Firstly, extending the analysis to include noncommutative rings could offer valuable insights into the structural properties of ZD-graphs in a broader algebraic context. Moreover, exploring the multiset dimensions of ZD-graphs associated with other algebraic structures beyond rings could further enrich our understanding of their graph-theoretic properties. Additionally, investigating the relationships between multiset dimensions and other graph parameters, such as connectivity or chromatic numbers, could provide deeper insights into the interplay between algebra and graph theory in diverse settings. These future directions hold promise for advancing both theoretical understanding and practical applications in algebraic graph theory.

Declaration:

- **Availability of data and materials:** The data presented in this study are openly available in Arxive with link: https://arxiv.org/abs/2405.06180
- **Conflicts of interest:** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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