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Growth of solutions to the coupled nonlinear Klein-Gordon equations with distributed delay, strong damping and source terms*

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ABSTRACT: In this work, we are concerned for a coupled nonlinear Klein-Gordon equations with distributed delay and strong damping and source terms, under suitable assumptions we will show the exponential growth of solutions.

Key Words: Viscoelastic equation, exponential growth, strong damping, nonlinear source, distributed delay.

Contents

1. Introduction

In this paper, we consider the following system

$$\begin{cases} |u_t|^{\eta} u_{tt} + m_1^2 u - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds \\ \quad +\mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x, t-\varrho) d\varrho = f_1(u,v), \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ |v_t|^{\eta} v_{tt} + m_2^2 v - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s)\Delta v(s)ds \\ \quad +\mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x, t-\varrho) d\varrho = f_2(u,v), \quad (x,t) \in \Omega \times \mathbb{R}_+, \\ u(x,t) = 0, \quad v(x,t) = 0, \quad x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), \quad v_t(x,-t) = k_0(x,t) \quad (x,t) \in \Omega \times (0,\tau_2), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \; x \in \Omega, \\ v(x,0) = v_0(x), \; v_t(x,0) = v_1(x), \; x \in \Omega, \end{cases} \tag{1.1}$$

where

$$\begin{cases} f_1(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|u|^p.u.|v|^{p+2}, \\ f_2(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|v|^p.v.|u|^{p+2}, \end{cases} \tag{1.2}$$

and $m_1, m_2, \omega_1, \omega_2, \mu_1, \mu_3, a_1, b_1 > 0$, $\eta \geq 0$ for $N = 1, 2$ and $\frac{2}{N-2} \geq \eta > 0$ for $N \geq 3$, and τ_1, τ_2 are the time delay with $0 \leq \tau_1 < \tau_2$, and μ_2, μ_4 are a L^∞ functions, and g, h are a differentiable functions.
It is well known that viscous materials are the opposite of elastic materials which have the capacity to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of other applied sciences.

Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory called viscoelastic damping term, where the kernel of the term memory is the function h see [10]. Numerous results appeared on the existence and long time behavior of solutions. In this direction, see [10,11,14,13,15,12] and the references therein.

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If $\eta \geq 0$, this type of problem has been studied by many authors. For more depth, here are some papers that focused on the study of this damping. See for example [6]
In many works on this field, under assumptions of the kernel g. For the problem (1.1), $\eta = 0$ and with $\mu_1 \neq 0$, for example in [5] Kafini et al. proved a blow up result for the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^{\infty} g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}.u, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \end{cases} \tag{1.3}$$

where g satisfies $\int_0^{\infty} g(s)ds < (2p-4)/(2p-3)$, initial data was supported with negative energy like that $\int u_0 u_1 dx > 0$.
If $(w > 0)$. In [14], Song et al. considered with the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^{\infty} g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}.u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x). \end{cases} \tag{1.4}$$

Under suitable assumptions on g, that there were solutions of (1.4) with initial energy, they showed the blow up in a finite time. For the same problem (1.4), in [15], Song et al proved that there were solutions of (1.4) with positive initial energy that blow up in finite time. In [6] the authors considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(s)\Delta u(t-s)ds - \varepsilon_1 \Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.5}$$

they showed a blow up result if $p > m$, and established the global existence. For more details see [2,7].
In the case of coupled of equations, in [1], the authors are studied the following system of equations:

$$\begin{cases} u_{tt} - \Delta u + u_t|u_t|^{m-2} = f_1(u,v), \\ v_{tt} - \Delta v + v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{1.6}$$

with nonlinear functions f_1 and f_2 satisfying appropriate conditions. Under certain restrictions imposed on the parameters and the initial data, they obtained numerous results on the existence of weak solutions. They also showed that any weak solution with negative initial energy blows up for a finite period of time by using the same techniques as in [4]. And in [3], the authors considered the system:

$$\begin{cases} u_{tt} - \Delta u + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u,v), \\ v_{tt} - \Delta v + (a|u|^\theta + b|v|^\vartheta)v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{1.7}$$

they stated and proved the blows up in finite time of solution, under some restrictions on the initial data and (with positive initial energy) for some conditions on the functions f_1 and f_2.
A complement the work of [9], we are working to prove under appropriate assumptions the solution of problem (1.1) grows exponentially,

$$\lim_{t\to\infty} \|u_t\|_{2(p+2)}^{2(p+2)} + \|\nabla u\|_{2(p+2)}^{2(p+2)} \to \infty \tag{1.8}$$

In the following, let $c, c_i > 0, i = 1,2,3$, and we prove the exponential growth result under the following suitable assumptions.
(A1) $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ are a differentiable and decreasing functions such that

$$\begin{aligned} g(t) \geq 0 \quad , \quad 1 - \int_0^{\infty} g(s)\,ds = l_1 > 0, \\ h(t) \geq 0 \quad , \quad 1 - \int_0^{\infty} h(s)\,ds = l_2 > 0. \end{aligned} \tag{1.9}$$

(**A2**) There exists a constants $\xi_1, \xi_2 > 0$ such that

$$\begin{aligned} g'(t) &\leq -\xi_1 g(t) \quad , \quad t \geq 0, \\ h'(t) &\leq -\xi_2 h(t) \quad , \quad t \geq 0. \end{aligned} \tag{1.10}$$

(**A3**) $\mu_2, \mu_4 : [\tau_1, \tau_2] \to \mathbb{R}$ are a L^∞ functions so that

$$\begin{aligned} (\frac{2\delta-1}{2}) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1 \quad , \quad \delta > \frac{1}{2}, \\ (\frac{2\delta-1}{2}) \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho < \mu_3 \quad , \quad \delta > \frac{1}{2}. \end{aligned} \tag{1.11}$$

2. Main results

In this section, we prove the blow up result of solution of problem (1.1). First, as in [8], we introduce the new variables

$$\begin{aligned} y(x, \rho, \varrho, t) &= u_t(x, t - \varrho\rho), \\ z(x, \rho, \varrho, t) &= v_t(x, t - \varrho\rho), \end{aligned}$$

then, we obtain

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ y(x, 0, \varrho, t) = u_t(x, t), \end{cases} \tag{2.1}$$

and

$$\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \\ z(x, 0, \varrho, t) = v_t(x, t). \end{cases} \tag{2.2}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s) |u(t) - u(s)|^2 ds dx. \tag{2.3}$$

Therefore, problem (1.1) takes the form:

$$\begin{cases} |u_t|^\eta u_{tt} + m_1^2 u - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t-s) \Delta u(s) ds \\ \quad + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho = f_1(u, v), \qquad x \in \Omega, t \geq 0, \\ |v_t|^\eta v_{tt} + m_2^2 v - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t-s) \Delta v(s) ds \\ \quad + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z(x, 1, \varrho, t) d\varrho = f_2(u, v), \qquad x \in \Omega, t \geq 0, \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \end{cases} \tag{2.4}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \;\; v(x,t) = 0 \quad x \in \partial\Omega, \\ y(x, \rho, \varrho, 0) = f_0(x, \varrho\rho), \;\; z(x, \rho, \varrho, 0) = k_0(x, \varrho\rho), \\ u(x, 0) = u_0(x), \;\; u_t(x, 0) = u_1(x), \\ v(x, 0) = v_0(x), \;\; v_t(x, 0) = v_1(x), \end{cases} \tag{2.5}$$

where

$$(x, \rho, \varrho, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Theorem 2.1 *Assume (1.9),(1.10), and (1.11) holds. Let*

$$\begin{cases} -1 < p < \dfrac{4-n}{n-2}, & n \geq 3; \\ p \geq -1, & n = 1, 2. \end{cases} \tag{2.6}$$

Then for any initial data

$$(u_0, u_1, v_0, v_1, f_0, k_0) \in \mathcal{H},$$

where

$$\begin{aligned} \mathcal{H} &= H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)) \\ &\quad \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)), \end{aligned}$$

the problem (2.4) has a unique solution

$$u \in C([0,T]; \mathcal{H}),$$

for some $T > 0$.

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time, we will make use of arguments in [13].

Theorem 2.2 *Suppose that (1.9),(1.10),(1.11), and (2.6) holds. If $u_0, v_0 \in W$, $u_1, v_1 \in H_0^1(\Omega)$ and $y, z \in L^2(\Omega \times (0,1) \times (\tau_1, \tau_2))$*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{2.7}$$

where C_* is the best Poincare's constant. Then the local solution (u, v, y, z) is global in time.

To achieve our goal, we need the following lemmas.

Lemma 2.1 *There exists a function $F(u, v)$ such that*

$$\begin{aligned} F(u,v) &= \frac{1}{2(p+2)}\left[uf_1(u,v) + vf_2(u,v)\right] \\ &= \frac{1}{2(p+2)}\left[a_1|u+v|^{2(p+2)} + 2b_1|uv|^{p+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \quad \frac{\partial F}{\partial v} = f_2(u,v),$$

we take $a_1 = b_1 = 1$ for convenience.

Lemma 2.2 *[9] There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(p+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(p+2)}\left(|u|^{2(\rho+2)} + |v|^{2(p+2)}\right). \tag{2.8}$$

We define the energy functional

Lemma 2.3 *Assume (1.9),(1.10),(1.11), and (2.6) hold, let (u, v, y, z) be a solution of (2.4), then $E(t)$ is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) + \frac{m_1^2}{2}\|u\|_2^2 + \frac{m_2^2}{2}\|v\|_2^2 \\ &\quad + \frac{1}{2}l_1\|\nabla u\|_2^2 + \frac{1}{2}l_2\|\nabla v\|_2^2 + \frac{1}{2}(go\nabla u) + \frac{1}{2}(ho\nabla v) \\ &\quad + \frac{1}{2}K(y,z) - \int_\Omega F(u,v)dx, \end{aligned} \tag{2.9}$$

satisfies

$$\begin{aligned} E'(t) \quad &\leq \quad -c_3\{\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_2^2 + \|v\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ &\quad + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z^2(x,1,\varrho,t) d\varrho dx\} \quad \leq 0, \end{aligned} \tag{2.10}$$

where

$$K(y,z) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho\{|\mu_2(\varrho)| y^2(x,\rho,\varrho,t) + |\mu_4(\varrho)| z^2(x,\rho,\varrho,t)\} d\varrho d\rho dx. \tag{2.11}$$

Proof: By multiplying $(2.4)_1, (2.4)_2$ by u_t, v_t and integrating over Ω, we get

$$\begin{aligned} \frac{d}{dt} \quad & \{\frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) + \frac{m_1^2}{2}\|u\|_2^2 + \frac{m_2^2}{2}\|v\|_2^2 + \frac{1}{2} l_1 \|\nabla u\|_2^2 \\ & + \frac{1}{2} l_2 \|\nabla v\|_2^2 + \frac{1}{2}(go\nabla u) + \frac{1}{2}(ho\nabla v) - \int_\Omega F(u,v) dx\}, \\ = \quad & -\mu_1 \|u_t\|_2^2 - m_1^2 \|u\|_2^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho dx \\ & -\mu_3 \|v_t\|_2^2 - m_2^2 \|v\|_2^2 - \int_\Omega v_t \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z(x,1,\varrho,t) d\varrho dx \\ & + \frac{1}{2}(g' o \nabla u) - \frac{1}{2} g(t) \|\nabla u\|_2^2 - \omega_1 \|\nabla u_t\|_2^2 \\ & + \frac{1}{2}(h' o \nabla v) - \frac{1}{2} h(t) \|\nabla v\|_2^2 - \omega_2 \|\nabla v_t\|_2^2. \end{aligned} \tag{2.12}$$

We obtain (2.10). □

Now we define the functional

$$\begin{aligned} \mathbb{H}(t) = -E(t) \quad &= \quad -\frac{1}{\eta+2}\|u_t\|_{\eta+2}^{\eta+2} - \frac{1}{\eta+2}\|v_t\|_{\eta+2}^{\eta+2} - \frac{m_1^2}{2}\|u\|_2^2 - \frac{m_2^2}{2}\|v\|_2^2 - \frac{1}{2} l_1 \|\nabla u\|_2^2 \\ & \quad -\frac{1}{2} l_2 \|\nabla v\|_2^2 - \frac{1}{2}(go\nabla u) - \frac{1}{2}(ho\nabla v) - \frac{1}{2} K(y,z) \\ & \quad + \frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{2.13}$$

Theorem 2.3 *Assume (1.9)-(1.11), and (2.6) hold. Assume further that $E(0) < 0$, and*

$$2\,(p+2) \succ \frac{\eta+2}{\eta+1}, \tag{2.14}$$

then the solution of problem (2.4) growth exponentially.

Proof: From (2.9), we have

$$E(t) \leq E(0) \leq 0. \tag{2.15}$$

Therefore

$$\begin{aligned} \mathbb{H}'(t) = -E'(t) \quad &\geq \quad c_3(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ & \quad + \|v\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z^2(x,1,\varrho,t) d\varrho dx). \end{aligned} \tag{2.16}$$

Hence

$$\begin{aligned} \mathbb{H}'(t) &\geq c_3 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \geq 0, \\ \mathbb{H}'(t) &\geq c_3 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| z^2(x,1,\varrho,t) d\varrho dx \geq 0, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} 0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) &\leq \frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}], \\ &\leq \frac{c_1}{2(p+2)}[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \end{aligned} \tag{2.18}$$

We set

$$\begin{aligned} \mathcal{K}(t) &= \mathbb{H}(t) + \frac{\varepsilon}{\eta+1} \int_\Omega (u\,|u_t|^\eta\, u_t + v\,|v_t|^\eta\, v_t) dx + \frac{\varepsilon}{2}\int_\Omega (\mu_1 u^2 + \mu_3 v^2) dx \\ &+ \frac{\varepsilon}{2}\int_\Omega (\omega_1 (\nabla u)^2 + \omega_2 (\nabla v)^2) dx. \end{aligned} \tag{2.19}$$

Where $\varepsilon > 0$ to be assigned later.
By multiplying $(2.4)_1, (2.4)_2$ by u, v and with a derivative of (2.19), we get

$$\begin{aligned} \mathcal{K}'(t) &= \mathbb{H}'(t) + \frac{\varepsilon}{\eta+1}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_2^2 + \|v\|_2^2\right) - \varepsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \\ &+ \varepsilon \int_\Omega \nabla u \int_0^t g(t-s)\nabla u(s) ds dx + \varepsilon \int_\Omega \nabla v \int_0^t h(t-s) \nabla v(s) ds dx \\ &- \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v z(x,1,\varrho,t) d\varrho dx \\ &+ \varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{2.20}$$

Using Young's inequality, we get

$$\begin{aligned} \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx &\leq \varepsilon\{\delta_1 (\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \|u\|_2^2 \\ &+ \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx\}. \end{aligned} \tag{2.21}$$

$$\begin{aligned} \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v z(x,1,\varrho,t) d\varrho dx &\leq \varepsilon\{\delta_2 (\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho) \|v\|_2^2 \\ &+ \frac{1}{4\delta_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z^2(x,1,\varrho,t) d\varrho dx\}, \end{aligned} \tag{2.22}$$

and, we have

$$\begin{aligned} \varepsilon \int_0^t g(t-s) ds \int_\Omega \nabla u . \nabla u(s) dx ds &= \varepsilon \int_0^t g(t-s) ds \int_\Omega \nabla u . (\nabla u(s) - \nabla u(t)) dx ds \\ &+ \varepsilon \int_0^t g(s) ds \|\nabla u\|_2^2, \\ &\geq \frac{\varepsilon}{2}\int_0^t g(s) ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2}(go\nabla u). \end{aligned} \tag{2.23}$$

$$\begin{aligned}\varepsilon\int_0^t h(t-s)ds\int_\Omega \nabla v.\nabla v(s)dxds &= \varepsilon\int_0^t h(t-s)ds\int_\Omega \nabla v.(\nabla v(s)-\nabla v(t))dxds\\ &+\varepsilon\int_0^t h(s)ds\|\nabla v\|_2^2,\\ &\geq \frac{\varepsilon}{2}\int_0^t h(s)ds\|\nabla v\|_2^2-\frac{\varepsilon}{2}(ho\nabla v).\end{aligned}\tag{2.24}$$

We obtain, from (2.20),

$$\begin{aligned}\mathcal{K}'(t) &\geq \mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}+\|u\|_2^2+\|v\|_2^2)\\ &-\varepsilon((1-\frac{1}{2}\int_0^t g(s)ds)\|\nabla u\|_2^2+(1-\frac{1}{2}\int_0^t h(s)ds)\|\nabla v_2^2\|)\\ &-\varepsilon\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2-\varepsilon\delta_2(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2\\ &-\frac{\varepsilon}{2}(go\nabla u)-\frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ &-\frac{\varepsilon}{2}(ho\nabla v)-\frac{\varepsilon}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\\ &+\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}].\end{aligned}\tag{2.25}$$

Therefore, using (2.17) and by setting δ_1,δ_1 so that, $\frac{1}{4\delta_1 c_3}=\frac{\kappa}{2}$, and $\frac{1}{4\delta_2 c_3}=\frac{\kappa}{2}$, substituting in (2.25), we get

$$\begin{aligned}\mathcal{K}'(t) &\geq [1-\varepsilon\kappa]\mathbb{H}'(t)+\frac{\varepsilon}{\eta+1}(\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}+\|u\|_2^2+\|v\|_2^2)\\ &-\varepsilon[(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2-\varepsilon[(1-\frac{1}{2}\int_0^t h(s)ds)]\|\nabla v\|_2^2\\ &-\varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2-\frac{\varepsilon}{2}(go\nabla u)\\ &-\varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2-\frac{\varepsilon}{2}(ho\nabla v)\\ &+\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}].\end{aligned}\tag{2.26}$$

For $0<a<1$, from (2.13)

$$\begin{aligned}\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}] &= \varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}]\\ &+\varepsilon 2(p+2)(1-a)\mathbb{H}(t)\\ &+\varepsilon(p+2)(1-a)(\|u_t\|_2^2+\|v_t\|_2^2)\\ &+\varepsilon(p+2)(1-a)(1-\int_0^t g(s)ds)\|\nabla u\|_2^2\\ &+\varepsilon(p+2)(1-a)(1-\int_0^t h(s)ds)\|\nabla v\|_2^2\\ &-\varepsilon(p+2)(1-a)(go\nabla u)\\ &-\varepsilon(p+2)(1-a)(ho\nabla v)\\ &+\varepsilon(p+2)(1-a)K(y,z).\end{aligned}\tag{2.27}$$

Substituting in (2.26), we get

$$\begin{aligned}\mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]\mathbb{H}'(t)+\varepsilon[(p+2)(1-a)+1](\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}+\|u\|_2^2+\|v\|_2^2)\\ &+\varepsilon[(p+2)(1-a)(1-\int_0^t g(s)ds)-(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2\\ &+\varepsilon[(p+2)(1-a)(1-\int_0^t h(s)ds)-(1-\frac{1}{2}\int_0^t h(s)ds)]\|\nabla v\|_2^2\\ &-\varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2-\varepsilon\frac{1}{2c_3\kappa}(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2\\ &+\varepsilon(p+2)(1-a)K(y,z)+\varepsilon[(p+2)(1-a)-\frac{1}{2}](go\nabla u+ho\nabla v)\\ &+\varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}]+\varepsilon 2(p+2)(1-a)\mathbb{H}(t).\end{aligned} \tag{2.28}$$

Using Poincaré's inequality, we obtain

$$\begin{aligned}\mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]\mathbb{H}'(t)+\varepsilon[(p+2)(1-a)+1](\|u_t\|_{\eta+2}^{\eta+2}+\|v_t\|_{\eta+2}^{\eta+2}+\|u\|_2^2+\|v\|_2^2)\\ &+\varepsilon\{[(p+2)(1-a)-1]-(\int_0^t g(s)ds)[(p+2)(1-a)-\frac{1}{2}]\\ &-\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\}\|\nabla u\|_2^2\\ &+\varepsilon\{[(p+2)(1-a)-1]-(\int_0^t h(s)ds)[(p+2)(1-a)-\frac{1}{2}]\\ &-\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2}|\mu_4(s)|ds)\}\|\nabla v\|_2^2\\ &+\varepsilon(p+2)(1-a)K(y,z)+\varepsilon[(p+2)(1-a)-\frac{1}{2}](go\nabla u+ho\nabla v)\\ &+\varepsilon c_0 a[\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}]\\ &+\varepsilon 2(p+2)(1-a)\mathbb{H}(t).\end{aligned} \tag{2.29}$$

In this stage, we take $a>0$ small enough so that

$$\alpha_1=(p+2)(1-a)-1>0,$$

and we assume

$$\max\{\int_0^\infty g(s)ds,\int_0^\infty h(s)ds\}<\frac{(p+2)(1-a)-1}{((p+2)(1-a)-\frac{1}{2})}=\frac{2\alpha_1}{2\alpha_1+1}, \tag{2.30}$$

then, we choose κ so large that

$$\begin{aligned}\alpha_2 &= \{(p+2)(1-a)-1)-\int_0^t g(s)ds((p+2)(1-a)-\frac{1}{2})\\ &\quad -\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\}>0,\\ \alpha_3 &= \{(p+2)(1-a)-1)-\int_0^t h(s)ds((p+2)(1-a)-\frac{1}{2})\\ &\quad -\frac{c}{2\kappa}(\int_{\tau_1}^{\tau_2}|\mu_4(s)|ds)\}>0.\end{aligned}$$

We fixed κ and a, we appoint ε small enough so that

$$\alpha_4 = 1 - \varepsilon\kappa > 0,$$

and, from (2.19)

$$\begin{aligned} \mathcal{K}(t) &\leq \frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}], \\ &\leq \frac{c_1}{2(p+2)}[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \end{aligned} \tag{2.31}$$

Thus, for some $\beta > 0$, estimate (2.29) becomes

$$\begin{aligned} \mathcal{K}'(t) \geq\ & \beta\{\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ & + (go\nabla u) + (ho\nabla v) + K(y,z) \\ & + [\|u\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}]\}. \end{aligned} \tag{2.32}$$

By (2.8), for some $\beta_1 > 0$, we obtain

$$\begin{aligned} \mathcal{K}'(t) \geq\ & \beta_1\{\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ & + (go\nabla u) + (ho\nabla v) + K(y,z) \\ & + [\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]\}, \end{aligned} \tag{2.33}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{2.34}$$

Next, using Holder's and Young's inequalities, we have

$$\begin{aligned} \int_\Omega (u\,|u_t|^\eta\, u_t + v\,|v_t|^\eta\, v_t)dx \leq\ & C\left[\|u\|_{2(p+2)}^\theta + \|u_t\|_{\eta+2}^\mu \right. \\ & \left. + \|v\|_{2(p+2)}^\theta + \|v_t\|_{\eta+2}^\mu\right], \end{aligned}$$

where $\dfrac{1}{\mu} + \dfrac{1}{\theta} = 1$.

We take $\mu = (\eta+2)$, to get

$$\theta = \frac{(\eta+2)}{(\eta+1)} \leq 2\,(p+2)\,. \tag{2.35}$$

Subsequently, by using (2.14) and the algebraic inequality

$$B^\zeta \leq (B+1) \leq \left(1 + \frac{1}{b}\right)(B+b)\,,\ \forall B \succ 0, 0 \prec \zeta \prec 1, b \succ 0, \tag{2.36}$$

we obtain

$$\begin{aligned} \|u\|_{2(p+2)}^{\frac{\eta+2}{(\eta+1)}} &\leq K\left(\|u\|_{2(p+2)}^{2(p+2)} + H\,(t)\right). \\ \|v\|_{2(p+2)}^{\frac{\eta+2}{(\eta+1)}} &\leq K\left(\|v\|_{2(p+2)}^{2(p+2)} + H\,(t)\right),\ \forall t \geq 0. \end{aligned} \tag{2.37}$$

Therefore

$$\left|\int_\Omega (u\,|u_t|^\eta\, u_t + v\,|v_t|^\eta\, v_t)dx\right| \leq c_{13}\left\{\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right\}. \tag{2.38}$$

Hence, using Young's and Poincaré's inequalities, from (2.19) we have

$$\begin{aligned}
\mathcal{K}(t) &= (\mathbb{H}^{1-\alpha} + \frac{\varepsilon}{\eta+1}\int_\Omega (u\left|u_t\right|^\eta u_t + v\left|v_t\right|^\eta v_t)dx + \frac{\varepsilon}{2}\int_\Omega(\mu_1 u^2 + \mu_3 v^2)dx \\
&\quad + \frac{\varepsilon}{2}\int_\Omega(\omega_1 \nabla u^2 + \omega_2 \nabla v^2)dx), \\
&\leq c\{\mathbb{H}(t) + |\int_\Omega (u\left|u_t\right|^\eta u_t + v\left|v_t\right|^\eta v_t)dx| + \|u\|_2 + \|\nabla u\|_2 \\
&\quad + \|v\|_2 + \|\nabla v\|_2\}, \\
&\leq c[\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2, \\
&\leq c[\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|u\|_2^2 + \|v\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (go\nabla u) \\
&\quad + (ho\nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}].
\end{aligned} \tag{2.39}$$

For some $c > 0$. From inequalities (2.32) and (2.39) we obtain the differential inequality

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}(t), \tag{2.40}$$

where $\lambda > 0$, depending only on β and c.
A simple integration of (2.40), we obtain

$$\mathcal{K}(t) \geq \mathcal{K}(0)e^{(\lambda t)}, \forall t > 0. \tag{2.41}$$

From (2.19) and (2.31), we have

$$\mathcal{K}(t) \leq \frac{c_1}{2(p+2)}[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \tag{2.42}$$

By (2.41) and (2.42), we have

$$\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \geq Ce^{(\lambda t)}, \forall t > 0.$$

Therefore, we conclude that the solution is growths exponentially. This completes the proof. □

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