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α_i-properties, selection principles and the ideals

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ABSTRACT: Kočinac [8] introduced several α_i-properties as a selection principles and there were motivated by Arhangel'skii [1] α_i-local properties. In this paper, we identify some classes $\mathcal{A}$ and $\mathcal{B}$ of open covers in topological spaces, topological groups, hyperspaces and abstract boundedness for which the Kočinac $\alpha_i(\mathcal{A}, \mathcal{B})$-properties are closely related and often equivalent to $S_1(\mathcal{A}, \mathcal{B})$, using the notion of an ideal. Further we introduce the ideal form of Hurewicz-bounded topological group and characterize it using these α_i-properties.

Key Words: Selection principles, α_i-properties, γ-cover, ω-cover, topological group, hyperspace, boundedness, ideal, topological space.

Contents

1. Introduction

In 1972, Arhangel'skii [1] introduced the α_i-properties in the following way. A countable collection $\mathbb{A}$ of convergent sequences of X is said to be a sheaf in X if all members of $\mathbb{A}$ converge to the same point of X, which is said to be the vertex of the sheaf $\mathbb{A}$. Let $\mathbb{A}$ be a sheaf in X with vertex $x \in X$. Then there exists a sequence B converging to x such that:
(α_1): if $A \in \mathbb{A}$, then $|A \setminus B| < \aleph_0$, where, for a set C, $|C|$ denotes the cardinality of C,
(α_2): if $A \in \mathbb{A}$, then $A \cap B$ is an infinite subsequence of A and B,
(α_3): $|\{A \in \mathbb{A}, A \cap B$ is an infinite subsequence of A and $B\}| = \aleph_0$,
(α_4): $|\{A \in \mathbb{A}, A \cap B \neq \emptyset\}| = \aleph_0$.

Motivated by these α_i-properties, Kočinac [8] introduced following α_i-properties in the form of selection principles.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets of families of subsets of an infinite set X. The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 1, 2, 3, 4$, denotes the following selection principles:

For each sequence $(A_n : n \in \mathbb{N})$ of infinite elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

1. $\alpha_1(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;
2. $\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
3. $\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
4. $\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

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The $S_1(\mathcal{A},\mathcal{B})$ [18] denotes the following selection principle:

$S_1(\mathcal{A},\mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

The following implications directly follows from the definitions:

$\alpha_1(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_2(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_3(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_4(\mathcal{A},\mathcal{B})$ and $S_1(\mathcal{A},\mathcal{B}) \Rightarrow \alpha_4(\mathcal{A},\mathcal{B})$.

Tsaban [29] investigated the Kočinac α_i-properties and made some observations on these properties.

Recently, the investigation of relative selection principles was commenced by Kočinac, Konca and Singh ([9,21,22,23,24,25,26,27,28]). Kočinac identified some classes $\mathcal{A}$ and $\mathcal{B}$ for which $\alpha_i(\mathcal{A},\mathcal{B})$-properties are equivalent to $S_1(\mathcal{A},\mathcal{B})$ in topological spaces, topological groups and hyperspaces. The purpose of this paper is to identify some more classes $\mathcal{A}$ and $\mathcal{B}$ for which these $\alpha_i(\mathcal{A},\mathcal{B})$-properties are closely related and often equivalent to $S_1(\mathcal{A},\mathcal{B})$, using the notion of an ideal.

The paper is organized so that introduction is followed by six sections. In Section-2, we familiarize the reader with the basic notions of different types of open covers and their analogues open covers using ideals. In Section-3, we give several characterization of α_i-properties in terms $S_1(\mathcal{A},\mathcal{B})$ selection principle and the game $G_1(\mathcal{A},\mathcal{B})$. Briefly we find some classes $\mathcal{A}$ and $\mathcal{B}$ of open covers in topological spaces such that the α_i-properties, $S_1(\mathcal{A},\mathcal{B})$ and the game $G_1(\mathcal{A},\mathcal{B})$ becomes equivalent. In Section-4, we define the ideal version of Hurewicz bounded topological group and characterize it these α_i-properties. Further several classes of $\mathcal{A}$ and $\mathcal{B}$ in topological group also find such that α_i-properties and $S_1(\mathcal{A},\mathcal{B})$ becomes equivalent. In Section-5, it is shown that $(\mathbb{K}(X),\mathsf{V}^+)$ satisfies $\alpha_i(\Omega,\mathcal{I}-\Gamma)$, $i = 2,3,4$ if and only if $(\mathbb{K}(X),\mathsf{V}^+)$ satisfies $S_1(\Omega,\mathcal{I}-\Gamma)$ if and only if X satisfies $S_1(\mathcal{K},\mathcal{I}-\Gamma_k)$ also the same analogue result is shown for $(\mathbb{F}(X),\mathsf{V}^+)$. In Section-6 we find some classes $\mathcal{A}$ and $\mathcal{B}$ of open covers in abstract boundedness spaces such that the α_i-properties, $S_1(\mathcal{A},\mathcal{B})$ and the game $G_1(\mathcal{A},\mathcal{B})$ becomes equivalent.

2. Preliminaries

Throughout the paper X is a Hausdorff topological space, $\mathbb{N}$ denotes the set of all positive integers and by an open cover we mean the countable open cover unless otherwise stated.

There is an infinite long game corresponding to $S_1(\mathcal{A},\mathcal{B})$ denoted by $G_1(\mathcal{A},\mathcal{B})$ [18]. Two Players, ONE and TWO, who play a round for each natural number n. In the n-th round ONE chooses a set A_n from $\mathcal{A}$ and TWO responds by choosing an element $b_n \in A_n$. TWO wins the play $(A_1, b_1, ..., A_n, b_n, ...)$ if $(b_n : n \in \mathbb{N}) \in \mathcal{B}$; otherwise ONE wins.

It can easily be seen that if ONE does not have a winning strategy in the game $G_1(\mathcal{A},\mathcal{B})$, then the corresponding selection hypothesis $S_1(\mathcal{A},\mathcal{B})$ is true. However the converse implication is not always true (see [8]).

In topological spaces, various special families of open covers have been studied in the literature in relations of selection principles, We refer the interested reader to the survey papers [19,10,30]. We now recall some classes of open covers which we will use through out the paper.

An open cover $\mathcal{U}$ of a space X is called an ω-cover [6] (k-cover) if every finite (compact) subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$. An open cover $\mathcal{U}$ of a space X is called an γ-cover [6] if $\mathcal{U}$ is infinite and each $x \in X$ belongs to all but finitely many $U \in \mathcal{U}$. An open cover $\mathcal{U}$ of a space X is called an γ_k-cover if $\mathcal{U}$ is infinite and each compact subset of X is contained in all but finitely many $U \in \mathcal{U}$ and X is not a member of $\mathcal{U}$.

A family $\mathcal{I}$ of subsets of a non-empty set Y is said to be an ideal in Y if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$. An ideal is said to be admissible ideal or free ideal of Y if $\{y\} \in \mathcal{I}$ for each $y \in Y$. If $\mathcal{I}$ is a proper ideal in Y, then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. Throughout the paper $\mathcal{I}$ will stand for proper admissible ideal of $\mathbb{N}$. We denote the ideal of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_{fin}$.

Now we consider the following covers defined by Das et. al [4,5] using the ideals of $\mathbb{N}$. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space X is said to be an $\mathcal{I}-\gamma$-cover [4] if for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin U_n\} \in \mathcal{I}$. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space X is said to be an $\mathcal{I}-\gamma_k$-cover [5] if $X \notin \mathcal{U}$ and for each compact subset C of X, the set $\{n \in \mathbb{N} : C \nsubseteq U_n\} \in \mathcal{I}$. An open cover $\mathcal{U}$ of a X is said to be $\mathcal{I}$-groupable [4] ($\mathcal{I}$-ω-groupable, $\mathcal{I}$-k-groupable) if it can be expressed in the form $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, where $\mathcal{U}_n$'s are finite, pairwise disjoint and for each $x \in X$ (any finite $F \subset X$, any compact $K \subset X$), $\{n \in \mathbb{N} : x \notin \cup\mathcal{U}_n\} \in \mathcal{I}$ ($\{n \in \mathbb{N} : F \nsubseteq \cup\mathcal{U}_n\} \in \mathcal{I}$, $\{n \in \mathbb{N} : K \nsubseteq \cup\mathcal{U}_n\} \in \mathcal{I}$). When $\mathcal{I}$

is an admissible ideal, every γ-cover of X is $\mathcal{I}-\gamma$-cover and every γ_k-cover of X is $\mathcal{I}-\gamma_k$-cover but the converse is not true (see [Example 2.1, [4]]).

A subset $\mathcal{V}$ of a cover $\mathcal{U}=\{U_n : n\in\mathbb{N}\}$ of a space X is $\mathcal{I}$-dense [4] in $\mathcal{U}$ if the set $M=\{m_1<m_2<m_3<m...\}$ of indices of elements from $\mathcal{V}$ belongs to $F(\mathcal{I})$ and further if $f:\mathbb{N}\to M$ be the bijection given by $f(i)=m_i$, then $f(A)\in\mathcal{I}$ if and only if $A\in\mathcal{I}$.

Throughout the paper each infinite subset of $\mathcal{I}-\gamma$-cover and $\mathcal{I}-\gamma_k$-cover are taken to be $\mathcal{I}$-dense. Since every $\mathcal{I}$-dense subset of $\mathcal{I}-\gamma$-cover of X is $\mathcal{I}-\gamma$-cover of X (see [Lemma 2.2, [4]]) also every $\mathcal{I}$-dense subset of $\mathcal{I}-\gamma_k$-cover of X is $\mathcal{I}-\gamma_k$-cover of X (see Lemma 4.1, [5]). Thus throughout the paper it is clear that each infinite subset of $\mathcal{I}-\gamma$-cover and $\mathcal{I}-\gamma_k$-cover are $\mathcal{I}-\gamma$-cover and $\mathcal{I}-\gamma_k$-cover respectively.

We use the following notations:

1. $\mathcal{O}$ denote the collection of open covers of X;
2. Ω denote the collection of ω-covers of X;
3. $\mathcal{K}$ denote the collection of k-covers of X;
4. Γ denote the collection of γ-covers of X;
5. Γ_k denote the collection of γ_k-covers of X;
6. $\mathcal{I}-\Gamma$ denote the collection of $\mathcal{I}-\gamma$-covers of X;
7. $\mathcal{I}-\Gamma_k$ denote the collection of $\mathcal{I}-\gamma_k$-covers of X;
8. $\mathcal{O}^{\mathcal{I}-gp}$ denote the collection of $\mathcal{I}$groupable-covers of X.

3. Open covers and α_i-properties using ideals

In this section we identify some classes $\mathcal{A}$ and $\mathcal{B}$ for which these $\alpha_i(\mathcal{A},\mathcal{B})$-properties, i=2, 3, 4, are equivalent to $S_1(\mathcal{A},\mathcal{B})$ and $G_1(\mathcal{A},\mathcal{B})$ using the ideal notions in topological spaces.

Theorem 3.1 *For a space X the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\Omega,\mathcal{I}-\Gamma)$*;*
2. *X satisfies* $\alpha_3(\Omega,\mathcal{I}-\Gamma)$*;*
3. *X satisfies* $\alpha_4(\Omega,\mathcal{I}-\Gamma)$*;*
4. *X satisfies* $S_1(\Omega,\mathcal{I}-\Gamma)$*;*
5. *ONE has no winning strategy in the game* $G_1(\Omega,\mathcal{I}-\Gamma)$ *on X.*

Proof: (3) $\Rightarrow$ (4). Let $(\mathcal{U}_n : n\in\mathbb{N})$ be a sequence of ω-covers of X. Assume that for each $n\in\mathbb{N}$, $\mathcal{U}_n=\{U_{n,m} : m\in\mathbb{N}\}$. Define $\mathcal{V}_n=\{U_{1,m_1}\cap U_{2,m_2}...\cap U_{n,m_n} : n<m_1<m_2<...<m_n, U_{i,m_i}\in\mathcal{U}_i, i\leq n\}\setminus\{\phi\}$. Then for each $n\in\mathbb{N}$, $\mathcal{V}_n$ is a ω-cover of X. Since $\mathcal{I}$-dense subset of $\mathcal{I}-\gamma$-cover is $\mathcal{I}-\gamma$-cover (see [4]) and by (3), there is an increasing sequence $\{n_1<n_2<...\}$ in $\mathbb{N}$ such that $\mathcal{V}=\{V_{n_i} : i\in\mathbb{N}\}$ is a $\mathcal{I}-\gamma$-cover of X and for each $i\in\mathbb{N}$, $V_{n_i}\in\mathcal{V}_{n_i}$. Let for each $i\in\mathbb{N}$, $V_{n_{i+1}}=U_{1,m_1}\cap U_{2,m_2}...\cap U_{n_{i+1},m_{n_{i+1}}}$, $U_{j,m_j}\in\mathcal{U}_j$, $j\leq n_{i+1}$. Let $n_0=0$. For each $i\geq 0$, each n with $n_i<n\leq n_{i+1}$. Let $O_n=U_{n,m_{n_{i+1}}}$. Then for each $n\in\mathbb{N}$, $O_n\in\mathcal{U}_n$ and the set $\{O_n : n\in\mathbb{N}\}$ is a $\mathcal{I}-\gamma$-cover of X. So X satisfies $S_1(\Omega,\mathcal{I}-\Gamma)$.

(4) $\Rightarrow$ (5). Consider the following strategy σ for ONE in $G_1(\Omega,\mathcal{I}-\Gamma)$. Let in the first round ONE chooses $\sigma(\phi)=\{U_{(1)},U_{(2)},...,U_{(n)},...\}$, a ω-cover of X. Suppose for each finite sequence t of natural numbers of length $\leq m$, U_t has been defined. Define now $\{U_{(n_1,n_2,...,n_m,k)} : k\in\mathbb{N}\}$ as the set $\sigma(U_{(n_1)},U_{(n_1,n_2)},...,U_{(n_1,n_2,...,n_m)})\setminus\{U_{(n_1)},U_{(n_1,n_2)},...,U_{(n_1,n_2,...,n_m)}\}$. Clearly for each finite sequence t of natural numbers, the set $\{U_{(t,n)} : n\in\mathbb{N}\}$ is a ω-cover of X. By (4) for each t choose $n_t\in\mathbb{N}$ such that $\{U_{(t,n_t)}$: t is a finite sequence of natural numbers $\}$ is an $\mathcal{I}-\gamma$-cover of X. Define inductively the sequence

$n_1 = n_\phi$, $n_{k+1} = n_{(n_1,n_2,..,n_k)}$ for $k \geq 1$. Since every infinite subset of $\mathcal{I}-\gamma$ cover is $\mathcal{I}$-dense and $\mathcal{I}$-dense subset of $\mathcal{I}-\gamma$-cover is $\mathcal{I}-\gamma$-cover (see [4]). Thus the sequence $U_{(n_1)}, U_{(n_1,n_2)}, ..., U_{(n_1,n_2,...,n_k)}, ...$ of moves of TWO is actually a $\mathcal{I}-\gamma$-cover of X which shows that σ is not a wining strategy for ONE in the game $G_1(\Omega, \mathcal{I}-\Gamma)$.

(5) $\Rightarrow$ (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-cover of X and let for each $n \in \mathbb{N}, \mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. From each $\mathcal{U}_n$ form a sequence of ω-covers of X in accordance with the strategy σ for ONE defined in the following way: ONE'S first move is $\sigma(\phi) = \mathcal{U}_1$. Let TWO now chooses a member $U_{1,m_{i_1}} \in \mathcal{U}_1$. Then ONE chooses the ω-cover $\mathcal{V}(1, m_{i_1}) = \{U_{1,m} \cap U_{2,m} : m \geq m_{i_1}\}$ and plays $\sigma(U_{1,m_{i_1}}) = \mathcal{V}(1, m_{i_1})$. If TWO now takes a set $U_{1,m_{i_2}} \cap U_{2,m_{i_2}} \in \mathcal{V}(1, m_{i_1})$ then ONE plays $\sigma(U_{1,m_{i_1}}, U_{2,m_{i_2}}) = \mathcal{V}(2, m_{i_2}) = \{U_{1,m} \cap U_{2,m} \cap U_{3,m} : m \geq m_{i_2}\}$, which is still ω-covers of X. TWO then chooses a set $U_{1,m_{i_3}} \cap U_{2,m_{i_3}} \cap U_{3,m_{i_3}} \in \sigma(U_{1,m_{i_1}}, U_{2,m_{i_2}})$ and so on. ONE and TWO proceed in this way for all $n \geq 2$. Since σ is not a winning strategy for ONE so there exists a σ-play :
$\sigma(\phi), U_{1,m_{i_1}}; \sigma(U_{1,m_{i_1}}), U_{1,m_{i_2}} \cap U_{2,m_{i_2}}; \sigma(U_{1,m_{i_1}}, U_{2,m_{i_2}}), U_{1,m_{i_3}} \cap U_{2,m_{i_3}} \cap U_{3,m_{i_3}}; ...$
lost by ONE. In other words, TWO's moves $U_{1,m_{i_1}}, U_{1,m_{i_2}} \cap U_{2,m_{i_2}}, U_{1,m_{i_3}} \cap U_{2,m_{i_3}} \cap U_{3,m_{i_3}}; ...$ form a sequence which forms a $\mathcal{I}-\gamma$-cover of X. Subsequently the sequence $U_{1,m_{i_1}}, U_{1,m_{i_2}}, U_{2,m_{i_2}}, U_{1,m_{i_3}}, U_{2,m_{i_3}}, U_{3,m_{i_3}}; ...$ forms a $\mathcal{I}-\gamma$-cover of X and obviously it contains infinitely many element from each $\mathcal{U}_n$, which shows that (1) holds. □

On the lines of preceding theorem, we can prove the following two theorems.

Theorem 3.2 *For a non-compact space X the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\mathcal{K}, \mathcal{I}-\Gamma)$;
2. *X satisfies* $\alpha_3(\mathcal{K}, \mathcal{I}-\Gamma)$;
3. *X satisfies* $\alpha_4(\mathcal{K}, \mathcal{I}-\Gamma)$;
4. *X satisfies* $S_1(\mathcal{K}, \mathcal{I}-\Gamma)$;
5. *ONE has no winning strategy in the game* $G_1(\mathcal{K}, \mathcal{I}-\Gamma)$ *on X.*

Theorem 3.3 *For a non-compact space X the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\mathcal{K}, \mathcal{I}-\Gamma_k)$;
2. *X satisfies* $\alpha_3(\mathcal{K}, \mathcal{I}-\Gamma_k)$;
3. *X satisfies* $\alpha_4(\mathcal{K}, \mathcal{I}-\Gamma_k)$;
4. *X satisfies* $S_1(\mathcal{K}, \mathcal{I}-\Gamma_k)$;
5. *ONE has no winning strategy in the game* $G_1(\mathcal{K}, \mathcal{I}-\Gamma_k)$ *on X.*

We also have following result.

Theorem 3.4 *For a space X and* $\mathcal{B} \in \{\mathcal{I}-\Gamma, \mathcal{I}-\Gamma_k\}$ *the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\Gamma_k, \mathcal{B})$;
2. *X satisfies* $\alpha_3(\Gamma_k, \mathcal{B})$;
3. *X satisfies* $\alpha_4(\Gamma_k, \mathcal{B})$;
4. *X satisfies* $S_1(\Gamma_k, \mathcal{B})$.

Proof: (3) $\Rightarrow$ (4). Consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of γ_k-covers of X. Let $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For all $n, m \in \mathbb{N}$ define $V_{n,m} = U_{1,m} \cap U_{2,m} \cap ... \cap U_{n,m}$. Then for each $n \in \mathbb{N}$, the set $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ is a γ_k-covers of X, since each $\mathcal{U}_n$ is a γ_k-cover of X. By (3) there is an increasing sequence $n_1 < n_2 < ...$ in $\mathbb{N}$ and a cover $\mathcal{V} = \{V_{n_i,m_i} : i \in \mathbb{N}\} \in \mathcal{B}$ such that for each $i \in \mathbb{N}$, $V_{n_i,m_i} \in \mathcal{V}_{n_i}$. Put $n_0 = 0$. For each $i \geq 0$, each j with $n_i < j \leq n_{i+1}$ and each $V_{n_{i+1},m_{i+1}} = U_{1,m_{i+1}} \cap ... \cap U_{n_{i+1},m_{i+1}}$, put $H_j = U_{j,m_{i+1}}$. For each $j \in \mathbb{N}$, $H_j \in \mathcal{U}_j$ and the set $\mathcal{H} = \{H_j : j \in \mathbb{N}\}$ is in $\mathcal{B}$, since $\mathcal{H}$ is refined by $\mathcal{V}$ which is in $\mathcal{B}$. So, X satisfies $S_1(\Gamma_k, \mathcal{B})$.

(4) $\Rightarrow$ (1). Consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of γ_k-covers of X. Let $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Choose an increasing sequence $k_1 < k_2 < ... < k_p$.. of positive integers and for each n and each k_i consider $\mathcal{V}(n, k_i) = \{U_{n,m} : m \geq k_i\}$. Then for each $n, i \in \mathbb{N}$, $\mathcal{V}(n, k_i)$ is a γ_k-covers of X. Apply (4) to the sequence $(\mathcal{V}(n, k_i) : n, i \in \mathbb{N})$, there is a sequence $(V_{n,k_i} : n, i \in \mathbb{N})$ such that for each $(n, i) \in \mathbb{N} \times \mathbb{N}$, $V_{n,k_i} \in \mathcal{V}(n, k_i)$ and the set $\mathcal{W} = \{V_{n,k_i} : n, i \in \mathbb{N}\} \in \mathcal{B}$. Hence the X satisfies $\alpha_2(\Gamma_k, \mathcal{B})$, since $\mathcal{W}$ is chosen in such a way that for each $n \in \mathbb{N}$, $\mathcal{U}_n \cap \mathcal{W}$ is infinite. □

In a similar way as Theorem 3.4, we can prove the following theorem.

Theorem 3.5 *For a space X the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\Gamma, \mathcal{I} - \Gamma)$;
2. *X satisfies* $\alpha_3(\Gamma, \mathcal{I} - \Gamma)$;
3. *X satisfies* $\alpha_4(\Gamma, \mathcal{I} - \Gamma)$;
4. *X satisfies* $S_1(\Gamma, \mathcal{I} - \Gamma)$.

4. $\mathcal{I}$-Hurewicz bounded topological groups

Now we shall present the ideal version of certain covers in topological groups using the α_i-properties. For instance we introduce the ideal version of Hurewicz-bounded topological group called $\mathcal{I}$-Hurewicz-bounded topological group and characterize it in the form of α_i-properties. Let $\mathbb{F}(X)$ and $\mathbb{K}(X)$ stand for the class of nonempty finite sets and compact sets of X, respectively.

Throughout this section $(G, *, \tau)$ *or* G denotes the topological group with the identity element e. Let $\mathcal{B}_e$ be a local base at e. For each $U \in \mathcal{B}_e$ with $U \neq G$ define

$o(U) = \{x * U : x \in G\}$

$\mathcal{O}(e) = \{o(U) : U \in \mathcal{B}_e\}$

$\omega(U) = \{F * U : F \in \mathbb{F}(G)\}$

$\Omega(e) = \{\omega(U) : U \in \mathcal{B}_e$ and there is no $F \in \mathbb{F}(G)$ with $F * U = G\}$

$k(U) = \{K * U : K \in \mathbb{K}(G)\}$

$\mathcal{K}(e) = \{k(U) : U \in \mathcal{B}_e$ and there is no $K \in \mathbb{K}(G)$ with $K * U = G\}$

It is clear from the definitions, $\mathcal{O}(e) \subset \mathcal{O}$; $\Omega(e) \subset \Omega$; $\mathcal{K}(e) \subset \mathcal{K}$.

In [2], Hurewicz-bounded topological groups have been studied. A topological group G is Hurewicz-bounded if for each sequence $(U_n : n \in \mathbb{N})$ of open neighborhoods of e, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of G such that for each $g \in G$, $g \in F_n * U_n$ for all but finitely many n. Equivalently, G is Hurewicz-bounded if it satisfies the selection principle $S_1(\Omega(e), \Gamma)$.

Now we define ideal version of a Hurewicz-bounded topological group (for the related study see [20,31]).

Definition 4.1 A topological group G is said to be $\mathcal{I}$-Hurewicz-bounded (in short, $\mathcal{I}$HB) if for each sequence $(U_n : n \in \mathbb{N})$ of open neighborhoods of e, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of G such that for each $g \in G$, $\{n \in \mathbb{N} : g \notin F_n * U_n\} \in \mathcal{I}$.

Equivalently, G is said to be $\mathcal{I}$HB if it satisfies the selection principle $S_1(\Omega(e), \mathcal{I} - \Gamma)$.

Theorem 4.1 *For a topological group G the following statements are equivalent:*

1. *G satisfies* $\alpha_2(\Omega(e), \mathcal{I} - \Gamma)$;

2. *G satisfies* $\alpha_3(\Omega(e), \mathcal{I}-\Gamma)$*;*
3. *G satisfies* $\alpha_4(\Omega(e), \mathcal{I}-\Gamma)$*;*
4. *G satisfies* $S_1(\Omega(e), \mathcal{I}-\Gamma)$*;*
5. *G satisfies* $S_1(\mathcal{K}(e), \mathcal{I}-\Gamma)$*.*

Proof: Since (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious and (3) $\Rightarrow$ (1) can similarly prove as the proof of (3) $\Rightarrow$ (4). Thus we only prove (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (4).

(3) $\Rightarrow$ (4). Consider the sequence $(U_n : n \in \mathbb{N})$ of elements of $\mathcal{B}_e$. For each $n \in \mathbb{N}$, let $V_n = U_1 \cap U_2 \cap ... \cap U_n$ be a member of $\mathcal{B}_e$. Apply (3) to the sequence $(V_n : n \in \mathbb{N})$, there is a sequence $n_1 < n_2 < ...$ in $\mathbb{N}$ and finite sets $F_{n_i} \subset G,\ i \in \mathbb{N}$, so that $\{F_{n_i} * V_{n_i} : n \in \mathbb{N}\}$ is a $\mathcal{I}-\gamma$ cover of G. If $n_0 = 0$, then for each positive integer n with $n_{i-1} < n \leq n_i,\ i \in \mathbb{N}$, put $F_n = F_{n_i}$ and U_n to be n-th component in the representation $U_1 \cap U_2 \cap ... \cap U_{n_i}$ of V_{n_i}. Evidently, $\{F_n * U_n : n \in \mathbb{N}\}$ is a $\mathcal{I}-\gamma$ cover of G. Thus the sequence $(F_n : n \in \mathbb{N})$ witnesses for $(U_n : n \in \mathbb{N})$ that G satisfies $S_1(\Omega(e), \mathcal{I}-\Gamma)$.

(5) $\Rightarrow$ (4). Consider the sequence $(U_n : n \in \mathbb{N})$ of elements of $\mathcal{B}_e$. For each $n \in \mathbb{N}$, pick $V_n \in \mathcal{B}_e$ so that $V_n^2 \subset U_n$. Apply (5) to the sequence $(V_n : n \in \mathbb{N})$, there is a sequence $(K_n : n \in \mathbb{N})$ of compact subsets of G such that $\{K_n * V_n : n \in \mathbb{N}\}$ is a $\mathcal{I}-\gamma$ cover of G. Now for each $n \in \mathbb{N}$, pick a finite set F_n in G such that $K_n \subset F_n * V_n$. Then for each n, $K_n * V_n \subset (F_n * V_n) * V_n \subset F_n * U_n$ and it concludes that $\{F_n * U_n : n \in \mathbb{N}\}$ is a $\mathcal{I}-\gamma$ cover of G. □

The proof of following theorem is similar to Theorem 4.1, thus omitted.

Theorem 4.2 *For a topological group G the following statements are equivalent:*

1. *G satisfies* $\alpha_2(\Omega(e), \mathcal{I}-\Gamma_k)$*;*
2. *G satisfies* $\alpha_3(\Omega(e), \mathcal{I}-\Gamma_k)$*;*
3. *G satisfies* $\alpha_4(\Omega(e), \mathcal{I}-\Gamma_k)$*;*
4. *G satisfies* $S_1(\Omega(e), \mathcal{I}-\Gamma_k)$*;*
5. *G satisfies* $S_1(\mathcal{K}(e), \mathcal{I}-\Gamma_k)$*.*

Now we prove some more characterization on $\mathcal{I}$HB using the $S_1(\mathcal{A}, \mathcal{B})$ selection principle. Let G and H be topological space with H a subspace of G. We shall use the following notion in our next results.

1. $\mathcal{O}_G$: The collection of open covers of G.
2. $\mathcal{O}_{GH}$: The collection covers of H by sets open in G.

Theorem 4.3 *Let H be a subgroup of topological group G. Then following are equivalent:*

1. $S_1(\Omega_G(e), \mathcal{O}_{GH}^{\mathcal{I}-gp})$*;*
2. $S_1(\Omega_H(e), \mathcal{O}_{H}^{\mathcal{I}-gp})$*.*

Proof: We only prove (1) $\Rightarrow$ (2). Let $(\omega(U_n) : n \in \mathbb{N})$ be a sequence in $\Omega_H(e)$. Since each $U_n \in \mathcal{B}_e$ (in H). Choose for each $n \in \mathbb{N}$, $T_n \in \mathcal{B}_e$ (in G) such that $U_n = T_n \cap H$. Now choose $S_n \in \mathcal{B}_e$ (in G) such that $S_n^{-1} * S_n \subset T_n$. Apply (1) to the sequence $(\omega(S_n) : n \in \mathbb{N})$ we get for each n a finite set $F_n \subset G$ such that for each element $x \in H$ there is an $\{n \in \mathbb{N} : x \notin F_n * S_n\} \in \mathcal{I}$. So $\{n \in \mathbb{N} : x \in F_n * S_n\} \in \mathcal{F}(\mathcal{I})$. For each n, and for each $f \in F_n$, choose a $h_f \in H$ as $h_f \in H \cap f * S_n$ if nonempty, $h_f = e$ otherwise. Then put $H_n = \{h_f : f \in F_n\}$, a finite subset of H. For each n, $H_n * U_n \in \omega(U_n) \in \Omega_H(e)$. Let $x \in H$. Then $x \in F_n * S_n$ for some n as every member of $\mathcal{F}(\mathcal{I})$ is nonempty. Choose $f_x \in F_n$ so that $x \in f_x * S_n$. Then evidently $H \cap f_x * S_n$ is nonempty, and so $h_{f_x} \in H$ is defined as an element of this intersection. Since $h_{f_x} \in f_x * S_n$, $f_x \in h_{f_x} * S_n^{-1}$, thus $x \in h_{f_x} * S_n^{-1} * S_n \subset h_{f_x} * T_n$. Now $h_{f_x}^{-1} * x \in H \cap T_n = U_n$, $x \in h_{f_x} * U_n \subset H_n * U_n$. Therefore $\{n \in \mathbb{N} : x \in F_n * S_n\} \subset \{n \in \mathbb{N} : x \in H_n * U_n\}$. This completes the proof. □

Corollary 4.1 *If G has property $S_1(\Omega_G(e), \mathcal{I} - \Gamma_G)$, then for each infinite subgroup H of G, has the property $S_1(\Omega_H(e), \mathcal{I} - \Gamma_H)$.*

5. Hyperspace and α_i-properties using ideals

In this section we shall present the ideal version of certain covers in hyperspaces using the α_i-properties.

For a space X, the collection of all closed subsets of X is denoted by 2^X. Let $\mathbb{F}(X)$ and $\mathbb{K}(X)$ stand for the class of nonempty finite sets and compact sets of X, respectively. If A is a subset of X and $\mathcal{A}$ is a family of subsets of X, then we write

$A^+ = \{F \in 2^X : F \subset A\}$, $\mathcal{A}^+ = \{A^+ : A \in \mathcal{A}\}$.

Notice that we use same symbol F to denote a closed subset of X and the point F in 2^X; from the context it will be clear what F is.

The *upper Fell topology* F^+ on 2^X is the topology whose base is the collection

$\{(K^C)^+ : K \in \mathbb{K}(X)\} \cup \{2^X\}$.

The *upper Vietoris topology* V^+ has basic open sets of the form U^+, U open in X. It is clear that $(\mathbb{K}(X), \mathsf{V}^+)$ and $(\mathbb{F}(X), \mathsf{V}^+)$ are considered as subspaces of $(2^X, \mathsf{V}^+)$.

Theorem 5.1 *For a space X and an open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X. The following statements hold:*

1. *$\mathcal{U}$ is a $\mathcal{I} - \gamma$-cover of X if and only if $\mathcal{U}^+$ is a $\mathcal{I} - \gamma$-cover of $(\mathbb{F}(X), \mathsf{V}^+)$;*

2. *$\mathcal{U}$ is a $\mathcal{I} - \gamma_k$-cover of X if and only if $\mathcal{U}^+$ is a $\mathcal{I} - \gamma_k$-cover of $(\mathbb{K}(X), \mathsf{V}^+)$;*

3. *$\mathcal{U}$ is a $\mathcal{I}$-ω-groupable cover of X if and only if $\mathcal{U}^+$ is a $\mathcal{I}$-ω-groupable cover of $(\mathbb{F}(X), \mathsf{V}^+)$;*

4. *$\mathcal{U}$ is a $\mathcal{I}$-k-groupable cover of X if and only if $\mathcal{U}^+$ is a $\mathcal{I}$-k-groupable cover of $(\mathbb{K}(X), \mathsf{V}^+)$.*

Proof: (1) Let $\mathcal{U}$ be a $\mathcal{I} - \gamma$-cover of X and let $F \in (\mathbb{F}(X), \mathsf{V}^+)$. The set $\{n \in \mathbb{N} : F \nsubseteq U_n\} \in \mathcal{I}$. This means $\{n \in \mathbb{N} : F \notin U_n^+\} \in \mathcal{I}$ for $U_n^+ \in \mathcal{U}^+$, that is $\mathcal{U}^+$ is a $\mathcal{I} - \gamma$-cover of $(\mathbb{F}(X), \mathsf{V}^+)$.

(2) Similar to (1).

(3) Let $\mathcal{U}$ is a $\mathcal{I}$-ω-groupable cover of X and let $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ be a partition of $\mathcal{U}$ witnessing that fact. Let $\{F_1, F_2, ..., F_p\}$ be a finite subset of $\mathbb{F}(X)$. Then $F = F_1 \cup F_2 \cup ... \cup F_p$ is a finite subset of X and thus $\{n \in \mathbb{N} : F \nsubseteq \cup\mathcal{U}_n\} \in \mathcal{I}$. Then $\{n \in \mathbb{N} : \{F_1, F_2, ..., F_p\} \notin \cup\mathcal{U}_n^+\} \in \mathcal{I}$. Thus $\mathcal{U}^+$ is a $\mathcal{I}$-ω-groupable cover of $(\mathbb{F}(X), \mathsf{V}^+)$.

(4) Similar to (3). □

To prove our next theorem we need following lemma from [8].

Lemma 5.1 *[8] For a space X and an open cover $\mathcal{W}$ of $(\mathbb{K}(X), \mathsf{V}^+)$ the following holds: $\mathcal{W}$ is an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) = \{U \subset X : U \text{ is open in } X \text{ and } U^+ \in W \text{ for some } W \in \mathcal{W}_n\}$ is a k-cover of X.*

Theorem 5.2 *For a space X the following statements are equivalent:*

1. *$(\mathbb{K}(X), \mathsf{V}^+)$ satisfies $\alpha_2(\Omega, \mathcal{I} - \Gamma)$;*

2. *$(\mathbb{K}(X), \mathsf{V}^+)$ satisfies $\alpha_3(\Omega, \mathcal{I} - \Gamma)$;*

3. *$(\mathbb{K}(X), \mathsf{V}^+)$ satisfies $\alpha_4(\Omega, \mathcal{I} - \Gamma)$;*

4. *$(\mathbb{K}(X), \mathsf{V}^+)$ satisfies $S_1(\Omega, \mathcal{I} - \Gamma)$;*

5. *X satisfies $S_1(\mathcal{K}, \mathcal{I} - \Gamma_k)$.*

Proof: The equivalence of (1), (2), (3) and (4) are follows from Theorem 3.1.

(4) $\Rightarrow$ (5). Consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of k-covers of X. Then $(\mathcal{U}_n^+ : n \in \mathbb{N})$ is a sequence of ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. For some $n \in \mathbb{N}$, let $\{K_1, K_2, ..., K_m\}$ be a finite subset of $\mathbb{K}(X)$. Then $K = K_1 \cup K_2 \cup ... \cup K_m$ is a compact subset of X and thus there is $U \in \mathcal{U}$ with $K \subset U$, that is for each $i \leq m$, $K_i \subset U$ and thus $K_i \in U^+$. Therefore $\{K_1, K_2, ..., K_m\} \subset U^+$ and $\mathcal{U}_n$ is an ω-cover of $\mathcal{K}(X)$. Apply (4) on each n, there is an element $U_n^+ \in \mathcal{U}_n^+$ such that the set $\mathcal{U}^+ = \{U_n^+ : n \in \mathbb{N}\}$ is a $\mathcal{I} - \gamma$-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. To prove (5), it is sufficient to prove that $(U_n : n \in \mathbb{N})$ is a $\mathcal{I} - \gamma_k$ cover of X. Then for each compact subset K of X, the set $\{n \in \mathbb{N} : K \notin U_n^+\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : K \in U_n^+\} \in \mathcal{F}(\mathcal{I})$. Let $K \in U_n^+$ for some $n \in \mathbb{N}$. Then $K \subset U_n$ for some $n \in \mathbb{N}$. Therefore $\{n \in \mathbb{N} : K \in U_n^+\} \subset \{n \in \mathbb{N} : K \subset U_n\} \in \mathcal{F}(\mathcal{I})$. Thus it shows that $(U_n : n \in \mathbb{N})$ is a $\mathcal{I} - \gamma_k$ cover of X.

(5) $\Rightarrow$ (4). Consider the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$. For each $n \in \mathbb{N}$, let $\mathcal{W}_n = \{W \subset X :$ W is open in X and $W^+ \in U$ for some $U \in \mathcal{U}_n\}$. By Lemma 5.1, each $\mathcal{W}_n$ is a k-cover of X. Apply (5) to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$, there is a sequence $(W_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $W_n \in \mathcal{W}_n$ and the set $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ is a a $\mathcal{I} - \gamma_k$ cover of X. For each $W_n \in \mathcal{W}$, pick $U_n \in \mathcal{U}_n$ so that $W_n^+ \in U_n$. To prove (4), it is sufficient to prove that $(U_n : n \in \mathbb{N})$ is a $\mathcal{I} - \gamma_k$ cover of $(\mathbb{K}(X), \mathsf{V}^+)$. Then for each $K \in \mathbb{K}(X)$, the set $\{n \in \mathbb{N} : K \subset W_n\} \in \mathcal{I}$. Consequently, $\{n \in \mathbb{N} : K \subseteq W_n\} \in \mathcal{F}(\mathcal{I})$. Let $K \subseteq W_n$ for some $n \in \mathbb{N}$. Then $K \in W_n^+ \in U_n$ for some $n \in \mathbb{N}$. Therefore $\{n \in \mathbb{N} : K \subseteq W_n\} \subset \{n \in \mathbb{N} : K \in U_n\} \in \mathcal{F}(\mathcal{I})$. Thus it shows that $(U_n : n \in \mathbb{N})$ is a $\mathcal{I} - \gamma_k$ cover of $(\mathbb{K}(X), \mathsf{V}^+)$. □

Lemma 5.2 *[8] For a space X and an open cover $\mathcal{W}$ of $(\mathbb{F}(X), \mathsf{V}^+)$ the following holds: $\mathcal{W}$ is an ω-cover of $(\mathbb{F}(X), \mathsf{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) = \{U \subset X : U$ is open in X and $U^+ \in W$ for some $W \in \mathcal{W}_n\}$ is an ω-cover of X.*

We can similarly prove the following theorem, by using Lemma 5.2 and Theorem 3.1.

Theorem 5.3 *For a space X the following statements are equivalent:*

1. $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_2(\Omega, \mathcal{I} - \Gamma)$;
2. $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_3(\Omega, \mathcal{I} - \Gamma)$;
3. $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_4(\Omega, \mathcal{I} - \Gamma)$;
4. $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $S_1(\Omega, \mathcal{I} - \Gamma)$;
5. *X satisfies* $S_1(\Omega, \mathcal{I} - \Gamma)$.

6. Abstract boundedness and α_i-properties using ideals

In 1949, S. T. Hu [7] were introduced and studied an abstract boundedness in any topological space.

A family $\mathbb{B}$ of nonempty closed subsets of a space X is said to be an abstract boundedness (or, boundedness) if it is closed for finite unions, closed hereditary and contains all singletons. If $\mathbb{B}$ is a boundedness in a space X and $\mathcal{U}$ is an open cover of X, then $\mathcal{U}$ is said to be a $\mathbb{B}$-cover if each $B \in \mathbb{B}$ is contained in an element of $\mathcal{U}$ and $X \notin \mathcal{U}$. $\mathcal{U}$ is called $\gamma_{\mathbb{B}}$-cover [16,13] if it is infinite and each $B \in \mathbb{B}$ is contained in all but finitely many elements of $\mathcal{U}$.

Now we introduce the ideal version of the $\gamma_{\mathbb{B}}$-cover called $\mathcal{I} - \gamma_{\mathbb{B}}$-cover. A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of a space X is said to be an $\mathcal{I}-\gamma_{\mathbb{B}}$-cover if for each $B \in \mathbb{B}$, the set $\{n \in \mathbb{N} : B \nsubseteq U_n\} \in \mathcal{I}$.

For a given boundedness $\mathbb{B}$ in a space, the collection of all $\mathbb{B}$-cover (resp. $\gamma_{\mathbb{B}}$, $\mathcal{I} - \gamma_{\mathbb{B}}$) denoted by $\mathcal{O}_{\mathbb{B}}$ (resp. $\Gamma_{\mathbb{B}}$, $\mathcal{I} - \Gamma_{\mathbb{B}}$) .

Lemma 6.1 *An $\mathcal{I}$-dense subset of an $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X is also an $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X.*

Proof: Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X and $\{U_{n_k} : k \in \mathbb{N}\}$ be an $\mathcal{I}$-dense subset of $\mathcal{U}$. Suppose that $\{U_{n_k} : k \in \mathbb{N}\}$ is not an $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X. Then there is a set $B \in \mathbb{B}$ for which the set $\{k \in \mathbb{N} : B \notin U_{n_k}\} \notin \mathcal{I}$. But $\{n_k \in \mathbb{N} : B \notin U_{n_k}\} \notin \mathcal{I}$ from the definition of $\mathcal{I}$-dense. Now observe that $\{n_k \in \mathbb{N} : B \notin U_{n_k}\} \notin \mathcal{I}$ from the definition of $\mathcal{I} \subset \{n \in \mathbb{N} : B \notin U_n\}$. Since $\mathcal{U}$ is an $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X, so $\{n \in \mathbb{N} : B \notin U_n\} \in \mathcal{I}$ and so $\{n_k \in \mathbb{N} : B \notin U_{n_k}\} \in \mathcal{I}$, a contradiction. □

If an infinite subset of $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X is $\mathcal{I}$-dense, then by above lemma, every infinite subset of $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X is $\mathcal{I} - \gamma_{\mathbb{B}}$-cover of X. We give several results related to boundedness without proof, since the proofs are quite similar to Theorem 3.1, Theorem 3.4 and Theorem 5.3.

Theorem 6.1 *For a space X the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
2. *X satisfies* $\alpha_3(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
3. *X satisfies* $\alpha_4(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
4. *X satisfies* $S_1(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
5. *ONE has no winning strategy in the game* $G_1(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma)$ *on X.*

Theorem 6.2 *If an infinite subset of* $\mathcal{I} - \gamma_{\mathbb{B}}$*-cover of X is* $\mathcal{I}$*-dense, then the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
2. *X satisfies* $\alpha_3(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
3. *X satisfies* $\alpha_4(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
4. *X satisfies* $S_1(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
5. *ONE has no winning strategy in the game* $G_1(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$ *on X.*

Theorem 6.3 *For a space X the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
2. *X satisfies* $\alpha_3(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
3. *X satisfies* $\alpha_4(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
4. *X satisfies* $S_1(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
5. *ONE has no winning strategy in the game* $G_1(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma)$ *on X.*

Theorem 6.4 *If an infinite subset of* $\mathcal{I} - \gamma_{\mathbb{B}}$*-cover of X is* $\mathcal{I}$*-dense, then the following statements are equivalent:*

1. *X satisfies* $\alpha_2(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
2. *X satisfies* $\alpha_3(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
3. *X satisfies* $\alpha_4(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
4. *X satisfies* $S_1(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$;
5. *ONE has no winning strategy in the game* $G_1(\Gamma_{\mathbb{B}}, \mathcal{I} - \Gamma_{\mathbb{B}})$ *on X.*

Consider $\mathbb{B}$ as a subspace of $(2^X, \mathsf{V}^+)$ and denote this space by $(\mathbb{B}(X), \mathsf{V}^+)$. Then we have the following.

Theorem 6.5 *For a space X the following statements are equivalent:*

1. $(\mathbb{B}(X), \mathsf{V}^+)$ *satisfies* $S_1(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma)$;
2. *X satisfies* $S_1(\mathcal{O}_{\mathbb{B}}, \mathcal{I} - \Gamma)$.

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