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Picture Fuzzy Normed Linear Operators

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ABSTRACT: Picture fuzzy set (PFS) is an instantaneous extension of Intuitionstic Fuzzy (IF) sets. In this article, we have introduced an idea of PF norm of a linear operator from a PFNL space to another PFNL space and also defined two varieties (strong and weak) of Picture Fuzzy-Normal Linear (PFNL) Bounded Operators. Also various types of continuity, i.e. strong continuity, week continuity, sequential continuity etc. of PFNL operators are discussed, and their relationships are studied subsequently. This study also examines the relation between continuity and boundedness of a PFNL operator. Finally, a result on fixed point theory in PFNL space is addressed.

Key Words: PFNL Space; PF Norm; PF Continuity; PFNL Bounded Operator, Fixed Point Theory.

Contents

1. Introduction

Setting a set value for the distance between two points is not a precise concept in real measurements. In many cases, the inexactness is modeled by taking the average of multiple measurements or an interval. Therefore, in these situations, the idea of a fuzzy standard seems more appropriate than a crisp norm, and as a result, the idea of fuzzy metric spaces [1] is developed. In 1992, Felbin [2] introduced fuzzy normed linear spaces (FNLSs) based on the work on fuzzy metric space. In 1993, he also addressed the completion of FNLSs and demonstrated that any fuzzy normed linear space with limited dimensions must be complete [3,4]. Later, several researchers studied on FNLSs and published some high-quality research articles [5,6,7,8]. Another fuzzy norm on a linear space was defined by Bag and Samanta [9]. Furthermore, they examined the characteristics of fuzzy normed linear spaces with finite dimensions and presented a decomposition theorem of fuzzy norms into a family of crisp norms. This work showed the path for several methods of FN space research. Intuitionistic FNLS, Neutrosophic NLS, n-FNLS and other advances [10,11,12,13,14,15,16,17,18,19,20] of FNLS have resulted from their publication. Also, the notions of norm and inner product on fuzzy linear spaces over fuzzy fields were first presented by Santhosh and Ramakrishnan [21].

In 2013, Cuong and Kreinovich [22] introduced the idea of neutral membership into the IFS theory and presented us with a lovely new notion of a set called a picture fuzzy set (PFS). It goes without saying that PFS is a generalization of IFS and other existence set theoretic structures [23,24,25,26,27]. As time passed, a number of researchers became interested in the theory of PFS and many kinds of research were conducted on it [28,29,30,31,32]. Undoubtedly PFNLS is a far more generalized idea than the other previously exist set structure. In 2024, we first studied the PFNL space (PFNLS) for the first time [33].

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In that article, we introduced the idea of a PFNL Operator with the help of PF norm and discussed some important properties of PFNL operators along with examples.

1.1. Scope and Objective of the Present Investigation

This article introduces the concept of boundedness of a linear operator from one PFNLS to another PFNLS and defines two varieties (strong and weak) of PFNL bounded linear operators. A study of some elegant outcomes has been attempted in this structure using the PF norm. Also, several types of continuity of PFNL operators are addressed. In addition, the relationship between the PF boundedness and the PF continuity is also examined. Our manuscript is organized as follows: In Section 3 some preliminaries regarding PFNLS are given, and a suitable example is provided for better understanding. In Section 4 continuity in PFNLS is addressed. various types of continuity, i.e. strong continuity, week continuity, sequential continuity etc. are discussed with a great extent. In Section 5, we have studied some ideas on PF boundedness. Here we have studied the relation between continuity and boundedness of a PFNL operator. In the next section, a result on fixed point theory in PFNL space is presented. Finally Section 7 concludes our article.

1.1.1. Preliminaries on PFNLS. We ask our readers to first read the concept of t-norm say, '$\odot$' and t-co-norm say '$\circ$', in order to realize PFNLS. Information about t norm and t co-norm can be found in any standard article, such as [16], because PFS is an ongoing evolution of FS and IFS. In 2013, Cuong et al. released the initial concept of PFS, a novel theory, in [22]. This section recapitulates the idea of PFNLS based on PFS theory, which was discussed in [33]. In addition, we have provided an example of PFNLS for smooth understanding of this article.

Definition 1.1. *[33] Suppose U is a linear space over $\mathbb{R}$. A picture fuzzy subset*

$$A = \{((\alpha, r); P(\alpha, r), Q(\alpha, r), R(\alpha, r)) : (\alpha, r) \in U \times \mathbb{R}^+\}$$

is called a PF norm on U w.r.t. continuous t-norm $\odot$ and t-co-norm $\circ$ respectively if the following holds:

(a) $P(\alpha, r) + Q(\alpha, r) + R(\alpha, r) \leq 1 \;\; \forall (\alpha, r) \in U \times \mathbb{R}^+$.

(b) $P(\alpha, r) > 0$.

(c) $P(\alpha, r) = 1$ *iff* $\alpha = 0$.

(d) $P(k\alpha, r) = P(\alpha, \frac{r}{|k|})\ , k \in \mathbb{R} \setminus \{0\}$.

(e) $P(\alpha, r) \odot P(\beta, s) \leq P(\alpha + \beta, r + s)$.

(f) $P(\alpha, .)$ *is non-decreasing mapping of* $\mathbb{R}^+$ *and* $\lim_{r \to \infty} P(\alpha, r) = 1$.

(g) $Q(\alpha, r) > 0$.

(h) $Q(\alpha, r) = 0$ *if and only if* $\alpha = 0$.

(i) $Q(k\alpha, r) = Q(\alpha, \frac{r}{|k|})\ , k \in \mathbb{R} \setminus \{0\}$.

(j) $Q(\alpha, r) \circ Q(\beta, s) \geq Q(\alpha + \beta, r + s)$.

(k) $Q(\alpha, .)$ *is non-increasing function of* $\mathbb{R}^+$ *and* $\lim_{r \to \infty} Q(\alpha, r) = 0$.

(l) $R(\alpha, r) > 0$.

(m) $R(\alpha, r) = 0$ *iff* $x = 0$.

(n) $R(k\alpha, r) = R(\alpha, \frac{r}{|k|})\ , k \in \mathbb{R} \setminus \{0\}$.

(o) $R(\alpha, r) \circ R(\beta, s) \geq R(\alpha + \beta, r + s)$.

(p) $R(\alpha,.)$ *is a non-increasing mapping of* $\mathbb{R}^+$ *and* $\lim_{r\to\infty} R(\alpha,r)=0$.

Here, $(U,A,\odot,\circ)$ is called a PFNLS. We will denote $(U,A,\odot,\circ)$ by (U,A) throughout this article.

Example 1.2. *Suppose* $U=(\mathbb{R},\|.\|)$ *is NLS with* $\|.\|=|x|\ \forall\ x\in\mathbb{R}$. *Considering* $a_1\odot a_2=\min\{a_1,a_2\}$ *and* $a_1\circ a_2=\max\{a_1,a_2\}\ \forall\ a_1,a_2\in[0,1]$ *we take* $P(\alpha,r)=\frac{r}{r+c|\alpha|}, Q(\alpha,r)=\frac{c|\alpha|}{r+c|\alpha|}, R(\alpha,r)=\frac{|\alpha|}{r},\ c>0$. *Let* $A=\{(\alpha,r);P(\alpha,r),Q(\alpha,r),R(\alpha,r)\}$. *Clearly* $(U,A,\odot,\circ)$ *is a PFNLS.*

2. Continuity in PFNLS

In this section we will study various types of continuity of an operator over PFNLS.

Definition 2.1. *Suppose* $(W,A_W,\odot,\circ)$ *and* $(Z,A_Z,\odot,\circ)$ *are two PFNLSs. A mapping* $\mathfrak{T}:(W,A_W,\odot,\circ)\to(Z,A_Z,\odot,\circ)$ *is said to be PF continuous at* $w_0\in W$ *if for all* $w\in W$ *for each* $\mu>0, m\in(0,1)$, $\exists\ \nu>0, n\in(0,1)$ *s.t.*

$$P_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),m)>(1-\mu),\ Q_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),m)<\mu,\ R_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),m)<\mu$$

whenever

$$P_W(w-w_0,n)>(1-\nu),\ Q_W(w-w_0,n)<\nu,\ R_W(w-w_0,n)<\nu$$

Definition 2.2. *Suppose* $(W,A_W,\odot,\circ)$ *and* $(Z,A_Z,\odot,\circ)$ *are two PFNLSs. A mapping* $\mathfrak{T}:(W,A_W,\odot,\circ)\to(Z,A_Z,\odot,\circ)$ *is said to be*

- *strongly PF continuous at* $w_0\in W$ *if for all* $w\in W$ *for each* $\mu>0\ \exists\ \nu>0$ *s.t.*

$$\begin{aligned}P_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),\mu)&\geq P_W(w-w_0,\nu),\\ Q_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),\mu)&\leq Q_W(w-w_0,\nu),\\ R_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),\mu)&\leq R_W(w-w_0,\nu).\end{aligned}$$

- *weakly PF continuous at* $w_0\in W$ *if for all* $w\in W$ *for each* $\mu>0,\ m\in(0,1)\ \exists\ \nu>0$ *s.t.*

$$\begin{aligned}P_W(w-w_0,\nu)\geq m&\to P_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),\mu)\geq m\\ Q_W(w-w_0,\nu)\leq m&\to Q_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),\mu)\leq m\\ R_W(w-w_0,\nu)\leq m&\to R_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),\mu)\leq m\end{aligned}$$

Definition 2.3. *Suppose* $(W,A_W,\odot,\circ)$ *and* $(Z,A_Z,\odot,\circ)$ *are two PFNLSs. A mapping* $\mathfrak{T}:(W,A_W,\odot,\circ)\to(Z,A_Z,\odot,\circ)$ *is said to be sequentially PF continuous at* $w_0\in W$ *if for any sequence* $\{w_n\}$ *in* W *satisfying* $w_n\to w_0$ *implies that* $\mathfrak{T}(w_n)\to\mathfrak{T}(w_0)$ *i.e.*

$$\begin{aligned}\lim_{n\to\infty}P_W(w_n-w_0,m)=1&\to\lim_{n\to\infty}P_Z(\mathfrak{T}(w_n)-\mathfrak{T}(w_0),m)=1\\ \lim_{n\to\infty}Q_W(w_n-w_0,m)=0&\to\lim_{n\to\infty}Q_Z(\mathfrak{T}(w_n)-\mathfrak{T}(w_0),m)=0\\ \lim_{n\to\infty}R_W(w_n-w_0,m)=0&\to\lim_{n\to\infty}R_Z(\mathfrak{T}(w_n)-\mathfrak{T}(w_0),m)=0,\end{aligned}$$

where $m>0$.

Theorem 2.4. *Suppose* $(W,A_W,\odot,\circ)$ *and* $(Z,A_Z,\odot,\circ)$ *are two PFNLSs and* $\mathfrak{T}:(W,A_W,\odot,\circ)\to(Z,A_Z,\odot,\circ)$ *is a linear operator. If* $\mathfrak{T}$ *is strongly PF continuous at a point* $w_0\in W$, *then* $\mathfrak{T}$ *is strongly PF continuous at* W.

Proof. Since $\mathfrak{T}$ is strongly PF continuous at $w_0\in W$, thus for every $\mu>0,\exists\nu>0$ s.t. $w\in W$ one can have $P_Z(\mathfrak{T}(w)-\mathfrak{T}(w_0),\mu)\geq P_W(w-w_0,\nu)$. Taking any $u\in W$ and replacing w by $w+w_0-u$ we have,

$$\begin{aligned}P_Z(\mathfrak{T}(w+w_0-u)-\mathfrak{T}(w_0),\mu)&\geq P_W(w+w_0-u-w_0,\nu)\\ \implies P_Z(\mathfrak{T}(w)-\mathfrak{T}(u),\mu)&\geq P_W(w-u,\nu)\end{aligned}$$

In a parallel way we can say

$$Q_Z(\mathfrak{T}(w) - \mathfrak{T}(u), \mu) \leq Q_W(w - u, \nu),$$
$$R_Z(\mathfrak{T}(w) - \mathfrak{T}(u), \mu) \leq R_W(w - u, \nu).$$

Since u is arbitrary, it implies that $\mathfrak{T}$ is strongly fuzzy continuous on W. ∎

The above result also holds for weak PF continuity, sequentially PF continuity of a linear operator $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$. Thus, we have the following results:

Theorem 2.5. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. If $\mathfrak{T}$ is weakly PF continuous at a point $w_0 \in W$, then $\mathfrak{T}$ is weakly PF continuous at W.*

Theorem 2.6. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. If $\mathfrak{T}$ is sequentially PF continuous at a point $w_0 \in W$, then $\mathfrak{T}$ is sequentially PF continuous at W.*

The proof for weakly PF continuous and sequential PF continuous cases are quite similar to the strongly PF continuous case and hence we are omitting it.

Theorem 2.7. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. Then strong continuity of $\mathfrak{T}$ implies sequential continuity of $\mathfrak{T}$ at a point $w_0 \in W$.*

Proof. Suppose $\{w_n\}$ is a convergent sequence that converges to $w_0 \in W$. Since $\mathfrak{T}$ is strongly continuous, then for all $w \in W$, for each $\mu > 0 \; \exists \; \nu > 0$, we have

$$P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \geq P_W(w - w_0, \nu),$$
$$Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \leq Q_W(w - w_0, \nu),$$
$$R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \leq R_W(w - w_0, \nu).$$

Now by putting $w = w_n$ we have

$$P_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), \mu) \geq P_W(w_n - w_0, \nu),$$
$$Q_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), \mu) \leq Q_W(w_n - w_0, \nu),$$
$$R_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), \mu) \leq R_W(w_n - w_0, \nu).$$

which clearly implies that $\mathfrak{T}(w_n) \to \mathfrak{T}(w_0)$ i.e. $\mathfrak{T}$ is sequentially convergent. ∎

However, the converse is not true, i.e. sequential continuity in a PFNLS does not imply strong continuity. Our following example demonstrates this.

Example 2.8. *Suppose $S = \langle \mathbb{R}, ||x||, \odot, \circ \rangle$ is a PFNLS, where $||x|| = |x|, a \odot b = \min\{a, b\}, a \circ b = \max\{a, b\} \; \forall \; a, b \in [0, 1]$. We define the functions $P_i, Q_i, R_i, i = 1, 2 : S \times \mathbb{R}^+ \to [0, 1]$ by*

$$P_1(x, r) = \frac{r}{r + |x|}, \; P_2(x, r) = \frac{r}{r + w|x|}, w > 0$$
$$Q_1(x, r) = \frac{|x|}{r + |x|}, \; Q_2(x, r) = \frac{|x|}{r + w|x|}, w > 0$$
$$R_1(x, r) = \frac{|x|}{r}, \; R_2(x, r) = \frac{|x|}{wr}, w > 0$$

It is clear that $\langle S, A_{S_i}, \odot, \circ\rangle$, $i = 1, 2$ are PFNLSs. Suppose $g : \langle S, A_{S_1}, \odot, \circ\rangle \to \langle S, A_{S_2}, \odot, \circ\rangle$ is a mapping defined as $g(s) = \frac{s^5}{1+s^2}$ $\forall\, s \in S$. Suppose $s_0 \in S$ and $\{s_n\}$ be a sequence in S such that $s_n \to s_0$ in $\langle S, A_{S_1}, \odot, \circ\rangle$. Then one can easily verify that $\forall\, r > 0$,

$$\lim_{n\to\infty} P_1(s_n - s_0, r) = 1, \lim_{n\to\infty} Q_1(s_n - s_0, r) = 0, \lim_{n\to\infty} R_1(s_n - s_0, r) = 0$$

Now we calculate $P_2(g(s_n)-g(s_0), r), Q_2(g(s_n)-g(s_0), r), R_2(g(s_n)-g(s_0), r)$. In every term of $Q_2(g(s_n)-g(s_0), r), R_2(g(s_n)-g(s_0), r)$ there is a term (s_n-s_0) in the numerator that becomes zero as n tends to ∞. Thus, $\lim_{n\to\infty} Q_2(g(s_n) - g(s_0), r) = 0$, $\lim_{n\to\infty} R_2(g(s_n) - g(s_0), r) = 0$. In a parallel manner, it is seen that $\exists$ also a term (s_n-s_0) in denominator in $P_2(g(s_n)-g(s_0), r)$. As a result, we get $\lim_{n\to\infty} P_2(g(s_n)-g(s_0), r) = 1$ and hence g is sequentially convergent on S. On the other hand, assume g is strongly continuous in S i.e. $\forall\, s_0 \in S$ and for each $r > 0$ such that,

$$\begin{aligned} P_1(s - s_0, t) &\le P_2(g(s_n) - g(s_0), r),\\ Q_1(s - s_0, t) &\ge Q_2(g(s_n) - g(s_0), r),\\ R_1(s - s_0, t) &\ge R_2(g(s_n) - g(s_0), r). \end{aligned}$$

Now

$$\begin{aligned} P_2((g(s_n) - g(s_0), r) &\ge P_1(s - s_0, t)\\ \implies \frac{r}{r + w\left|\frac{s^5}{1+s^2} - \frac{s^5}{1+s^2}\right|} &\ge \frac{t}{t + |s - s_0|}\\ \implies t &\le \frac{r}{w} h\left(\frac{1}{s}, s_0\right) \end{aligned}$$

Now

$$\implies \inf_{s\in S} h\left(\frac{1}{s}, s_0\right) \ge \frac{wt}{r},$$

where h is a polynomial in s, $s \neq s_0$ with degree ≤ 1. Thus, $\frac{wt}{r} = 0$. Since $w, r > 0$, it implies that $t = 0$ which is a contradiction to the fact $t > 0$. So, g is not strongly continuous.

Theorem 2.9. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *are two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is a linear operator. Then continuity of* $\mathfrak{T}$ *implies sequential continuity of* $\mathfrak{T}$ *and vice-versa at a point* $w_0 \in W$.

Proof. Suppose that $\mathfrak{T}$ is continuous at $w_0 \in W$ and $\{w_n\}$ is a convergent sequence that converges to w_0 in W. Then for all $w \in W$, for each $\mu \in (0, 1)$ and $r > 0$, $\exists\, \nu \in (0, 1)$ and $m > 0$ such that,

$$\begin{aligned} P_W(w - w_0, m) > (1 - \nu) &\Rightarrow P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) > (1 - \mu),\\ Q_W(w - w_0, m) < \nu &\Rightarrow Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) < \mu,\\ R_W(w - w_0, m) < \nu &\Rightarrow R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) < \mu \end{aligned}$$

Since $\{w_n\} \to w_0$, thus $\exists\, n_0 \in \mathbb{N}$ such that,

$$P_W(w_n - w_0, m) > (1 - \nu), Q_W(w_n - w_0, m) < \nu, R_W(w_n - w_0, m) < \nu$$

Hence,

$$\begin{aligned} P_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), r) &> (1 - \mu),\\ Q_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), r) &< \mu,\\ R_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), r) &< \mu \end{aligned}$$

which clearly shows that $\mathfrak{T}$ is sequentially continuous at $w_0 \in W$. Conversely suppose that $\mathfrak{T}$ is sequentially continuous at $w_0 \in W$ and if possible $\mathfrak{T}$ is not continuous at w_0 i.e. $\exists\ \mu \in (0,1)$ and $r > 0$ such that for any $\nu \in (0,1)$ and $m > 0$ such that

$$\begin{aligned} P_W(w - w_0, m) > (1-\nu) &\Rightarrow P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) \leq (1-\mu), \\ Q_W(w - w_0, m) < \nu &\Rightarrow Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) \geq \mu, \\ R_W(w - w_0, m) < \nu &\Rightarrow R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) \geq \mu \end{aligned}$$

So, for $\nu = 1 - \frac{1}{1+p}, m = \frac{1}{p+1},\ p \in \mathbb{N}, \exists\ w_p$ such that

$$\begin{aligned} P_W(w_p - w_0, \frac{1}{p+1}) > \frac{1}{p+1} &\text{ but } P_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) \leq (1-\mu), \\ Q_W(w_p - w_0, \frac{1}{p+1}) < 1 - \frac{1}{p+1} &\text{ but } Q_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) \geq \mu, \\ R_W(w_p - w_0, \frac{1}{p+1}) < 1 - \frac{1}{p+1} &\text{ but } R_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) \geq \mu. \end{aligned}$$

Considering $m > 0,\ \exists\ p_0$ s.t. $\frac{1}{p+1} < m\ \forall\ p \geq p_0$, we have

$$\begin{aligned} \lim_{n\to\infty} P_W(w_p - w_0, m) = 1; \\ \lim_{n\to\infty} Q_W(w_p - w_0, m) = 0, \\ \lim_{n\to\infty} R_W(w_p - w_0, m) = 0 \end{aligned}$$

this lead to $w_p \to w_0$, although

$$\begin{aligned} P_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) &\leq (1-\mu), \\ Q_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) &\geq \mu, \\ R_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) &\geq \mu. \end{aligned}$$

i.e. $\mathfrak{T}(w_p)$ does not converge $\mathfrak{T}(w_0)$ which gives a contradiction. Therefore, the result follows. ∎

3. PFN bounded operator

Throughout this section, we will discuss the idea of boundedness and isometry property of PFNL operators between PFNLSs. Also we will study some relationships between different types of PFNL bounded operators.

Definition 3.1. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. Then $\mathfrak{T}$ is called strongly PFN bounded if there exists a non-zero constant real number r such that for each $w \in W$ and $\forall\ s > 0$,*

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s), \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s), \\ R_Z(\mathfrak{T}(w), s) &\leq R_W(rw, s). \end{aligned}$$

It is quite clear that the zero operators and the identity operators are, by definition, a strongly PFN bounded operator.

Example 3.2. *Suppose $U = (\mathbb{R}, \|.\|)$ is a normed linear space with $\|.\| = |x|\ \forall\ x \in \mathbb{R}$. Considering $a_1 \odot a_2 = \min\{a_1, a_2\}$ and $a_1 \circ a_2 = \max\{a_1, a_2\}\ \forall\ a_1, a_2 \in [0,1]$ we take $P(\alpha, t) = \frac{t}{t + cm_1|\alpha|}, Q(\alpha, t) = \frac{c|\alpha|}{t + c|\alpha|}, R(\alpha, t) = \frac{|\alpha|}{t},\ c > 0,\ m_1$ is fixed real number. We take $A = \{(\alpha, t); P(\alpha, t), Q(\alpha, t), R(\alpha, t)\}$. Clearly $(U, A, \odot, \circ)$ is a PFNLS. Now, we choose $P_1(\alpha, t) = \frac{t}{t + cm_2|\alpha|}, Q_1(\alpha, t) = \frac{c|\alpha|}{t + c|\alpha|}, R_1(\alpha, t) = \frac{|\alpha|}{t},\ c > 0$ where m_2 is a fixed real number and $m_1 > m_2$. We take $B = \{(\alpha, t); P_1(\alpha, t), Q_1(\alpha, t), R_1(\alpha, t)\}$. Clearly $(U, B, \odot, \circ)$ is also a PFNLS. Now we consider a linear operator $\mathfrak{T} : (U, A, \odot, \circ) \to (U, B, \odot, \circ)$ given by $\mathfrak{T}(x) = rx,\ r \in \mathbb{R} \setminus \{0\}$. Clearly $\mathfrak{T}$ is a strongly PFN bounded operator.*

Definition 3.3. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. Then $\mathfrak{T}$ is called weakly PFN bounded if for $\mu \in (0,1)$ there exists a non zero constant real number r such that for each $w \in W$ and $s > 0$,*

$$\begin{aligned} P_Z(rw, s) \geq 1-\mu \quad &\Rightarrow \quad P_W(\mathfrak{T}(w), s) \geq 1-\mu, \\ Q_Z(rw, s) \leq \mu \quad &\Rightarrow \quad Q_W(\mathfrak{T}(w), s) \leq \mu, \\ R_Z(rw, s) \leq \mu \quad &\Rightarrow \quad R_W(\mathfrak{T}(w), s) \leq \mu, \end{aligned}$$

Definition 3.4. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. Then $\mathfrak{T} : W \to Z$ is called PFNL isometry if for each $w \in W, s > 0$ such that*

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &= P_W(w, s), \\ Q_Z(\mathfrak{T}(w), s) &= Q_W(w, s), \\ R_Z(\mathfrak{T}(w), s) &= R_W(w, s). \end{aligned}$$

Theorem 3.5. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. If $\mathfrak{T}$ is strongly PF continuous at a point $w_0 \in W$, then $\mathfrak{T}$ is strongly PF bounded.*

Proof. Firstly, suppose that $\mathfrak{T}$ is strongly PF-bounded. Then there exists a constant $r \neq 0,\ r \in \mathbb{R}$ such that for each $w \in W$ and $\forall\ s > 0$,

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s), \\ \Longrightarrow P_Z(\mathfrak{T}(w) - \mathfrak{T}(\overline{0}), s) &\geq P_W(w - \overline{0}, \frac{s}{r}) \\ \Longrightarrow P_Z(\mathfrak{T}(w) - \mathfrak{T}(\overline{0}), \mu) &\geq P_W(w - \overline{0}, \nu) \end{aligned}$$

where we assume $s = \mu$, $\frac{s}{r} = \nu$. In a parallel way we can see

$$\begin{aligned} Q_Z(\mathfrak{T}(w) - \mathfrak{T}(\overline{0}), \mu) \leq Q_W(w - \overline{0}), \nu), \\ R_Z(\mathfrak{T}(w) - \mathfrak{T}(\overline{0}), \mu) \leq R_W(w - \overline{0}), \nu). \end{aligned}$$

It is clear that $\mathfrak{T}$ is strongly PF continuous at $\overline{0}$, hence $\mathfrak{T}$ is strongly PF continuous in W. On the other hand, consider $\mathfrak{T}$ to be strongly PF continuous on W. Using the continuity property of $\mathfrak{T}$ at $\overline{0}$ we have for $\mu = 1, \exists\ \nu > 0, \forall\ w \in W$ such that

$$P_Z(\mathfrak{T}(w) - \mathfrak{T}(\overline{0}), 1) \geq P_W(w - \overline{0}, \nu).$$

Now for $w = 0$, the case is trivial, and hence we have omitted it. Suppose that $w \neq 0$ and $r > 0$. Assuming $u = \frac{w}{s}$ then one can have,

$$P_Z(\mathfrak{T}(w), s) = P_Z(s\mathfrak{T}(u), s) = P_Z(\mathfrak{T}(u), 1) \geq P_W(u, \nu) = P_W(\frac{w}{s}, \nu) = P_W(wr, s), \text{where } \frac{1}{\nu} = r$$

In a parallel way we can show that

$$\begin{aligned} Q_Z(\mathfrak{T}(w), r) \leq Q_W(wr, s) \\ R_Z(\mathfrak{T}(w), r) \leq R_W(wr, s). \end{aligned}$$

Hence, the result follows. ∎

Theorem 3.6. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. If $\mathfrak{T}$ is weakly PF continuous at a point $w_0 \in W$, then $\mathfrak{T}$ is weakly PF bounded.*

We have omitted the proof, since it is quite similar to the proof of the Theorem 4.5.

Theorem 3.7. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *are two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is a linear operator. If* $\mathfrak{T}$ *is PFN strongly bounded, then it is PFN weakly bounded.*

Proof. Suppose $\mathfrak{T}$ is PFN strongly bounded. Then there exists a nonzero constant real number r such that for each $w \in W$ and $\forall\ s > 0$,

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s), \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s), \\ R_Z(\mathfrak{T}(w), s) &\leq R_W(rw, s). \end{aligned}$$

Since $P_W(rw, s), Q_W(rw, s), R_W(rw, s) \in [0, 1]$, we obtain that for any $0 < \mu < 1$ s.t.

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s) \geq 1 - \mu, \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s) \leq \mu, \\ R_Z(\mathfrak{T}(w), s) &\leq R_W(rw, s) \leq \mu. \end{aligned}$$

Clearly $\mathfrak{T}$ is PFN weakly bounded. ∎

Theorem 3.8. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *are two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is a linear operator. If* $\mathfrak{T}$ *is PFN strongly bounded if and only if* $\mathfrak{T}$ *is continuous.*

Proof. Suppose $\mathfrak{T}$ is PFN strongly bounded. Then for a nonzero real constant r such that for every $s > 0, \forall\ w \in W$, we have

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s) = P_W(w, \frac{s}{|r|}) = P_W(w, t), \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s) = Q_W(w, \frac{s}{|r|}) = Q_W(w, t), \\ R_Z(\mathfrak{T}(w), s) &\leq R_W(rw, s) = R_W(w, \frac{s}{|r|}) = R_W(w, t), \end{aligned}$$

where $t = \frac{r}{k} > 0$. Suppose $w_0 \in W, \mu \in (0, 1), s > 0$ and $\mu = \nu$. Also suppose that

$$P_W(w - w_0) \geq (1 - \mu),\ Q_W(w - w_0) \leq \mu,\ R_W(w - w_0) \leq \mu.$$

Then clearly we have,

$$\begin{aligned} P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0)) &\geq 1 - \mu, \\ Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0)) &\leq \mu, \\ R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0)) &\leq \mu. \end{aligned}$$

Hence $\mathfrak{T}$ is continuous. On the other hand, let us assume that $\mathfrak{T}$ is continuous on $(W, A_W, \odot, \circ)$. Then from the definition of continuity, $\mathfrak{T}$ is continuous at $\overline{0}$ and we have $\forall\ w \in W, 0 < \mu < 1$ and $r > 0, \exists\ 0 < \nu < 1$ and $t > 0$ we have,

$$\begin{aligned} P_W(w - \overline{0}, t) \geq (1 - \nu) \Rightarrow P_Z((\mathfrak{T}(w) - (\mathfrak{T}(\overline{0}), r) &\geq 1 - \mu \\ Q_W(w - \overline{0}, t) \leq \nu \Rightarrow Q_Z((\mathfrak{T}(w) - (\mathfrak{T}(\overline{0}), r) &\leq \mu \\ R_W(w - \overline{0}, t) \leq \nu \Rightarrow R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) &\leq \mu \end{aligned}$$

Now for any $\nu \in (0, 1)$ such that,

$$P_W(kw, r) \geq (1 - \nu), Q_W(kw, r) \leq \nu, R_W(kw, r) \leq \nu.$$

Thus,

$$P_W(w, \frac{r}{|k|}) = P_W(w, r) \geq (1-\nu),$$
$$Q_W(w, \frac{r}{|k|}) = Q_W(w, r) \leq \nu$$
$$R_W(w, \frac{r}{|k|}) = R_W(w, r) \leq \nu$$

By assuming $t = \frac{r}{|k|}$, we obtain that

$$\begin{aligned} P_W(w,t) \geq (1-\nu) \Rightarrow P_Z((\mathfrak{T}(w), r) &\geq 1-\mu \\ Q_W(w,t) \leq \nu \Rightarrow Q_Z(\mathfrak{T}(w), r) &\leq \mu \\ R_W(w,t) \leq \nu \Rightarrow R_Z((\mathfrak{T}(w), r) &\leq \mu. \end{aligned}$$

Hence, the result follows. ∎

Consider the set of all strongly PF bounded linear operators from a PFNLS $(W, A_W, \odot, \circ)$ to $(Z, A_Z, \odot, \circ)$ as $\mathfrak{F}(W, Z)$. We now show that $\mathfrak{F}(W, Z)$ is a PFL space.

Theorem 3.9. *$\mathfrak{F}(W, Z)$ is a PFL space.*

Proof. Suppose $\mathfrak{T}_1, \mathfrak{T}_2 \in \mathfrak{F}(W, Z)$ and $w \in W$. Since $\mathfrak{T}_1, \mathfrak{T}_2$ are strongly PF bounded, we have

$$\begin{aligned} P_Z(\mathfrak{T}_i(w), s) &\geq P_W(r_i w, s), \\ Q_Z(\mathfrak{T}_i(w), s) &\leq Q_W(r_i w, s), \\ R_Z(\mathfrak{T}_i(w), s) &\leq R_W(r_i w, s), \ i = 1, 2. \end{aligned}$$

Now for any two scaler γ, δ and $\forall\ w \in W$ we have,

$$\begin{aligned} P_Z((\gamma\mathfrak{T}_1 + \delta\mathfrak{T}_2)(w), s) &= P_Z((\gamma\mathfrak{T}_1(w)) + \delta\mathfrak{T}_2(w), s) \\ &\geq \min\left\{P_Z(\mathfrak{T}_1(\gamma w), \frac{s}{2}), P_Z(\mathfrak{T}_2(\delta w), \frac{s}{2})\right\} \\ &\geq \min\left\{P_Z(\mathfrak{T}_1(\gamma w), \frac{s}{2}), P_Z(\mathfrak{T}_2(\delta w), \frac{s}{2})\right\} \\ &\geq \min\left\{P_W(\mathfrak{T}_1(r_1\gamma w), \frac{s}{2}), P_W(\mathfrak{T}_2(r_2\delta w), \frac{s}{2})\right\} \\ &\geq \min\left\{P_W(\mathfrak{T}_1(w), \frac{s}{2|r_1\gamma|}), P_W(\mathfrak{T}_2(w), \frac{s}{2|r_2\delta|})\right\} \end{aligned}$$

Suppose $r = \max\{2|r_1\gamma|, 2r_2|\delta|\} + 1$. Then we have,

$$\min\left\{P_W(\mathfrak{T}_1(w), \frac{s}{2r_1|\gamma|}), P_W(\mathfrak{T}_2(w), \frac{s}{2r_2|\delta|})\right\} \geq P_W(w, \frac{s}{r}).$$

Considering all above we have $\forall\ w \in W, \forall s \in \mathbb{R}, \exists\ r > 0$ such that

$$P_Z((\gamma\mathfrak{T}_1 + \delta\mathfrak{T}_2)(w), s) \geq P_W(rw, s)$$

Similarly we can show that

$$Q_Z((\gamma\mathfrak{T}_1 + \delta\mathfrak{T}_2)(w), s) \leq Q_W(rw, s)$$
$$R_Z((\gamma\mathfrak{T}_1 + \delta\mathfrak{T}_2)(w), s) \leq R_W(rw, s).$$

Hence, the result follows. ∎

Now we consider the set of all weakly PF bounded linear operators from a PFNLS $(W, A_W, \odot, \circ)$ to $(Z, A_Z, \odot, \circ)$ as $\mathfrak{F}'(W, Z)$. Likewise, as in a previous manner, one can easily show that $\mathfrak{F}'(W, Z)$ is a PF linear space.

4. Results on Fixed Point Theory

Definition 4.1. *A function* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is said to be PFN Lipschitzian on* W *if* $\exists\, k > 0$ *such that*

$$\begin{aligned} P_Z(\mathfrak{T}(w_1) - \mathfrak{T}(w_2), s) &\geq P_W(w_1 - w_2, \frac{s}{k}), \\ Q_Z(\mathfrak{T}(w_1) - \mathfrak{T}(w_2), s) &\leq Q_W(w_1 - w_2, \frac{s}{k}), \\ R_Z(\mathfrak{T}(w_1) - \mathfrak{T}(w_2), s) &\leq R_W(w_1 - w_2, \frac{s}{k}). \end{aligned}$$

$\forall\, s > 0, \forall w_i \in W, i = 1, 2.$ $\mathfrak{T}$ *is said to be PFN contraction if* $k < 1$.

It is clear that a PFN contraction mapping $\mathfrak{T}$ is always PFN continuous.

Definition 4.2. *A PFN Banach Space (PFNBS) is a complete PFN linear space.*

Theorem 4.3. *Suppose* $(W, A_W, \odot, \circ)$ *is a PFNBS and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *be a PFN contraction, then* $\mathfrak{T}$ *has a unique fixed point.*

Proof. Suppose $w \in W$, then $\{\mathfrak{T}^n(w)\}$ is a Cauchy sequence. Now for $s > 0$ and $m \in \mathbb{N} \setminus \{0\}$, we have,

$$\begin{aligned} P_Z(\mathfrak{T}^{n+m}(w) - \mathfrak{T}^n(w), s) \geq P_Z(\mathfrak{T}^{n+m-1}(w) - \mathfrak{T}^{n-1}(w), \frac{s}{k}) \geq \ldots \geq P_Z(\mathfrak{T}^m(w) - w, \frac{s}{k^n}) \\ Q_Z(\mathfrak{T}^{n+m}(w) - \mathfrak{T}^n(w), s) \leq Q_Z(\mathfrak{T}^{n+m-1}(w) - \mathfrak{T}^{n-1}(w), \frac{s}{k}) \leq \ldots \leq Q_Z(\mathfrak{T}^m(w) - w, \frac{s}{k^n}) \\ R_Z(\mathfrak{T}^{n+m}(w) - \mathfrak{T}^n(w), s) \leq R_Z(\mathfrak{T}^{n+m-1}(w) - \mathfrak{T}^{n-1}(w), \frac{s}{k}) \leq \ldots \leq R_Z(\mathfrak{T}^m(w) - w, \frac{s}{k^n}) \end{aligned}$$

As $0 < k < 1$, we have $\lim_{n\to\infty} \frac{s}{k^n} = \infty$. Then for $n \to \infty$

$$\begin{aligned} P_Z(\mathfrak{T}^{n+m}(w) - \mathfrak{T}^n(w), s) = P_Z(\mathfrak{T}^m(w) - w, \frac{s}{k^n}) = 1 \\ Q_Z(\mathfrak{T}^{n+m}(w) - \mathfrak{T}^n(w), s) = Q_Z(\mathfrak{T}^m(w) - w, \frac{s}{k^n}) = 0 \\ R_Z(\mathfrak{T}^{n+m}(w) - \mathfrak{T}^n(w), s) = R_Z(\mathfrak{T}^m(w) - w, \frac{s}{k^n}) = 0 \end{aligned}$$

Since W is complete, thus, $\{\mathfrak{T}^n(w)\}$ is a convergent sequence. So, there exists $z \in W$ such that $\lim_{n\to\infty} \mathfrak{T}^n(w) = z$. Hence, we have, $z = \lim_{n\to\infty} \mathfrak{T}^{n+1}(w) = \mathfrak{T}(z)$. To examine the uniqueness, consider $w_1, w_2 \in W, w_1 \neq w_2$ such that $w_i = \mathfrak{T}(w_i), i = 1, 2$. Then $\exists\, c > 0$

$$P_Z(w_1 - w_2, c) = p < 1, Q_Z(w_1 - w_2, c) = q > 0, R_Z(w_1 - w_2, c) = r > 0.$$

Then $\forall\, n \in \mathbb{N} \setminus \{0\}$ we obtain

$$\begin{aligned} p = P_Z(w_1 - w_2, c) = P_Z(\mathfrak{T}^{\mathrm{n}}(w_1) - \mathfrak{T}^{\mathrm{n}}(w_2), c) \geq P_Z(w_1 - w_2, \frac{c}{k^n}) \to 1, \\ q = Q_Z(w_1 - w_2, c) = Q_Z(\mathfrak{T}^{\mathrm{n}}(w_1) - \mathfrak{T}^{\mathrm{n}}(w_2), c) \leq Q_Z(w_1 - w_2, \frac{c}{k^n}) \to 0, \\ r = R_Z(w_1 - w_2, c) = R_Z(\mathfrak{T}^{\mathrm{n}}(w_1) - \mathfrak{T}^{\mathrm{n}}(w_2), c) \leq R_Z(w_1 - w_2, \frac{c}{k^n}) \to 0, \end{aligned}$$

which gives a contradiction. Hence we are done. ∎

5. Conclusion

In this article, the continuity and boundedness properties of the PFNL operators are discussed. In addition, some properties related to continuous and bounded operators in PFNL space are discussed. All these concepts are illustrated with examples. Also we have studied some interesting relationship theorem between the PF continuity and PF boundedness of an operator on PFNLS. An application regarding fixed point theory has been studied in this article. In the future, we will study the concept of PF n-NLS and PF inner product space.

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