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Local and 2-local anti-derivations on filiform associative algebras

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ABSTRACT: This paper is devoted to the study of local and 2-local anti-derivations of null-filiform, filiform and naturally graded quasi-filiform associative algebras. We prove that these algebras as a rule admit local anti-derivations which are not anti-derivations. We show that filiform and naturally graded quasi-filiform associative algebras admit 2-local anti-derivations which are not anti-derivations and any 2-local anti-derivation of null-filiform associative algebras is an anti-derivation.

Key Words: Associative algebras, filiform associative algebras, quasi-filiform associative algebras, anti-derivation, local anti-derivation, 2-local anti-derivation.

Contents

1. Introduction

The notion of δ-derivations was introduced by V.Filippov for Lie algebras in [16,17]. The space of δ-derivations includes usual derivations ($\delta = 1$), anti-derivations ($\delta = -1$) and elements from the centroid. In [17] it was proved that prime Lie algebras, as a rule, do not have nonzero δ-derivations (provided $\delta \neq 1, -1, 0, \frac{1}{2}$), and that every primary Lie Φ-algebra $A\left(\frac{1}{6} \in \Phi\right)$ with a nonzero anti-derivation satisfies the identity $[(yz)(tx)]x + [(yx)(zx)]t = 0$ and is a three-dimensional central simple algebra over a field of quotients of the center $Z_R(A)$ of its right multiplication algebra $R(A)$. Moreover, all $\frac{1}{2}$-derivations of an arbitrary prime Lie algebra A over the field $\mathbb{F}$ of characteristic $p \neq 2, 3$ with a non-degenerate symmetric invariant bilinear form were described. It was proved that if A is a central simple Lie algebra over a field of characteristic $p \neq 2, 3$ with a non-degenerate symmetric invariant bilinear form, then any $\frac{1}{2}$-derivation φ has the form $\varphi(x) = \lambda x$ for some $\lambda \in \mathbb{F}$. In [18], δ-derivations were investigated for prime alternative and non-Lie Mal'tsev algebras, and it was proved that alternative and non-Lie Mal'tsev algebras with certain restrictions on the ring of operators F have no non-trivial δ-derivation.

Local derivations are useful tools in studying the structure of rings and algebras, where there are still many related unsolved problems. R.V.Kadison, D.R.Larson and A.R.Sourour first introduced the notion of local derivations on algebras in their remarkable paper [20,27]. Since then, many researchers have been studying local derivations of different types of algebras (e.g., see [2,3,6,7,10,24]). In [6] the authors proved that every local derivation on a finite-dimensional semisimple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic zero is a derivation. In [10] local derivations of solvable Lie algebras are investigated, and it is shown that in the class of solvable Lie algebras, there exist algebras which admit local derivations which are not ordinary derivations, and also algebras for which every local derivation is a derivation. Moreover, it is proved that every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and maximal dimension of complementary space is a derivation. In [24],

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the authors proved that every local derivation on solvable Lie algebras whose nilradical has maximal rank is a derivation. In [3], the authors proved that every local derivation on the conformal Galilei algebra is a derivation. The results of the paper [7] show that p-filiform Leibniz algebras as a rule admit local derivations which are not derivations. In [2] the authors proved proved that the direct sum of null-filiform nilpotent Leibniz algebras as a rule admits local derivations which are not derivations.

We note that the aforementioned algebras are finite-dimensional algebras. In the infinite-dimensional case, the authors of [8,14,30] proved that every local derivation on some class of the locally simple Lie algebras, generalized Witt algebras, Witt algebras, and the Witt algebras over a field of prime characteristic is a derivation.

A similar notion, which characterizes non-linear generalizations of automorphisms and derivations, was introduced by P.Šemrl in [28] as 2-local automorphisms (respectively, 2-local derivations).

Several papers have been devoted to similar notions and corresponding problems for 2-local derivations and automorphisms of Lie algebras [5,8,9,12,13,19,29,31]. Namely, in [5] it is proved that every 2-local derivation on the semi-simple Lie algebras is a derivation and that each finite-dimensional nilpotent Lie algebra, with dimension larger than two admits 2-local derivation, which is not a derivation. Let us present a list of finite or infinite-dimensional Lie algebras for which all 2-local derivations are derivations: finite-dimensional semi-simple Lie algebras over an algebraically closed field of characteristic zero; infinite-dimensional Witt algebras over an algebraically closed field of characteristic zero; locally finite split simple Lie algebras over a field of characteristic zero; Virasoro algebras; Virasoro-like algebra; the Schrodinger-Virasoro algebra; Jacobson-Witt algebras; planar Galilean conformal algebras.

Investigation of derivations, local derivations and 2-local derivations on filiform associative algebras was initiated in [1] by Sh.Ayupov, A.Abduraulov and B.Yusupov. Namely, in [1] it is proved we introduce the notion of local and 2-local derivations and describe local and 2-local derivation of filiform associative algebras.

Investigation of local and 2-local δ-derivations on Lie algebras was initiated in [25] by A. Khudoyberdiyev and B.Yusupov. Namely, in [25] we introduced the notion of local and 2-local δ-derivations and described local and 2-local $\frac{1}{2}$-derivation of finite-dimensional solvable Lie algebras with filiform, Heisenberg, and abelian nilradicals. Moreover, we gave the description of local $\frac{1}{2}$-derivation of oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras. B.Yusupov, V.Vaisova and T.Madrakhimov proved similar results concerning local $\frac{1}{2}$-derivations of naturally graded quasi-filiform Leibniz algebras of type I in their recent paper [32]. They proved that quasi-filiform Leibniz algebras of type I, as a rule, admit local $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations. Similar problem [26] U.Mamadaliyev, A.Sattarov and B.Yusupov investigated local and 2-local $\frac{1}{2}$-derivations on Leibniz algebras. They proved that any local $\frac{1}{2}$-derivation on the solvable Leibniz algebras with model or abelian nilradicals, whose the dimension of complementary space is maximal is a $\frac{1}{2}$-derivation. They proved that solvable Leibniz algebras with abelian nilradicals, which have 1-dimension complementary space is a $\frac{1}{2}$-derivation. Moreover, similar problem concerning 2-local $\frac{1}{2}$-derivations of such algebras are investigated and an example of solvable Leibniz algebra given such that any 2-local $\frac{1}{2}$-derivation on it is a $\frac{1}{2}$-derivation, but which admit 2-local $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations.

In the present paper we study anti-derivations, local anti-derivations and 2-local anti-derivations on filiform associative algebras. In Section 3 we describe the anti-derivations of null-filiform, filiform and naturally graded quasi-filiform associative algebras. In Section 4 we consider local and 2-local anti-derivations on arbitrary finite-dimensional null-filiform, filiform and naturally graded quasi-filiform associative algebras. In subsection 4.1 we show that these algebras as a rule admit local anti-derivations which are not anti-derivations. In subsection 4.2 we show that filiform and naturally graded quasi-filiform associative algebras admit 2-local anti-derivations which are not anti-derivations and any 2-local anti-derivation of null-filiform associative algebras is an anti-derivation.

2. Preliminaries

For an algebra $\mathbf{A}$ of an arbitrary variety, we consider the series

$$\mathbf{A}^1 = \mathbf{A}, \qquad \mathbf{A}^{i+1} = \sum_{k=1}^{i} \mathbf{A}^k \mathbf{A}^{i+1-k}, \qquad i \geq 1.$$

We say that an algebra $\mathbf{A}$ is nilpotent if $\mathbf{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer satisfying $\mathbf{A}^i = 0$ is called the index of nilpotency of or nilindex $\mathbf{A}$.

Definition 2.1 *An n-dimensional algebra $\mathbf{A}$ is called null-filiform if* $\dim \mathbf{A}^i = (n+1) - i,\ 1 \leq i \leq n+1$.

It is easy to see that an algebra has a maximum nilpotency index if and only if it is null-filiform. For a nilpotent algebra, the condition of null-filiformity is equivalent to the condition that the algebra is one-generated.

All null-filiform associative algebras were described in [23,21]:

Theorem 2.1 *An arbitrary n-dimensional null-filiform associative algebra is isomorphic to the algebra:*

$$\mu_0^n : \quad e_i e_j = e_{i+j}, \quad 2 \leq i+j \leq n,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra μ_0^n.

The following classes of nilpotent algebras are filiform and quasi-filiform algebras whose nilindex equals n and $n-1$, respectively.

Definition 2.2 *An n-dimensional algebra is called filiform if* $\dim(\mathbf{A}^i) = n - i,\ 2 \leq i \leq n$.

Definition 2.3 *An n-dimensional associative algebra $\mathbf{A}$ is called quasi-filiform algebra if $\mathbf{A}^{n-2} \neq 0$ and $\mathbf{A}^{n-1} = 0$.*

Definition 2.4 *Given a nilpotent associative algebra $\mathbf{A}$, put $\mathbf{A}_i = \mathbf{A}^i/\mathbf{A}^{i+1}$, $1 \leq i \leq k-1$, and $\mathfrak{gr}\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_k$. Then $\mathbf{A}_i\mathbf{A}_j \subseteq \mathbf{A}_{i+j}$ and we obtain the graded algebra $\mathfrak{gr}\mathbf{A}$. If the algebras $\mathfrak{gr}\mathbf{A}$ and $\mathbf{A}$ are isomorphic, denoted by $\mathfrak{gr}\mathbf{A} \cong \mathbf{A}$, then we say that the algebra $\mathbf{A}$ is naturally graded.*

All filiform and naturally graded quasi-filiform associative algebras were classified in [22].

Theorem 2.2 ([22]) *Every n-dimensional $(n > 3)$ complex filiform associative algebra is isomorphic to one of the following pairwise non-isomorphic algebras with basis $\{e_1, e_2, \ldots, e_n\}$:*

$$\begin{array}{lcllll}
\mu_{1,1}^n(\mu_0^{n-1} \oplus \mathbb{C}) & : & e_i e_j = e_{i+j}, & & & \\
\mu_{1,2}^n & : & e_i e_j = e_{i+j}, & e_n e_n = e_{n-1}, & & \\
\mu_{1,3}^n & : & e_i e_j = e_{i+j}, & e_1 e_n = e_{n-1}, & & \\
\mu_{1,4}^n & : & e_i e_j = e_{i+j}, & e_1 e_n = e_{n-1}, & e_n e_n = e_{n-1} &
\end{array}$$

where $2 \leq i+j \leq n-1$.

Theorem 2.3 ([22]) *Let $\mathbf{A}$ be n-dimensional $(n \geq 6)$ complex naturally graded quasi-filiform non-split associative algebra. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{lclll}
\mu_{2,1}^n & : & e_i e_j = e_{i+j}, & e_{n-1} e_1 = e_n & \\
\mu_{2,2}^n(\alpha) & : & e_i e_j = e_{i+j}, & e_1 e_{n-1} = e_n, & e_{n-1} e_1 = \alpha e_n \\
\mu_{2,3}^n & : & e_i e_j = e_{i+j}, & e_{n-1} e_{n-1} = e_n & \\
\mu_{2,4}^n & : & e_i e_j = e_{i+j}, & e_1 e_{n-1} = e_n, & e_{n-1} e_{n-1} = e_n
\end{array}$$

where $\alpha \in \mathbb{C}$ and $2 \leq i+j \leq n-2$.

Definition 2.5 *A δ-derivation on an algebra $\mathbf{A}$ is a linear map $D : \mathbf{A} \to \mathbf{A}$ which satisfies the Leibniz rule:*

$$D(x, y) = \delta\big(D(x), y + x, D(y)\big) \quad \textit{for any} \quad x, y \in \mathbf{A}.$$

Note that 1-derivation is a usual derivation and (-1)-derivation is called anti-derivation. If D_1 and D_2 are δ_1 and δ_2-derivations, respectively, then their commutator $[D_1, D_2] = D_1 D_2 - D_2 D_1$ ia a $\delta_1\delta_2$ derivation. Thus, the set of all δ-derivations of a Lie algebra $\mathcal{L}$ is a Lie algebra with respect to the commutator. The set of all δ-derivations, we denote by $Der_\delta(\mathcal{L})$.

Definition 2.6 *A linear operator Δ is called a local δ-derivation, if for any $x \in \mathcal{L}$, there exists a δ-derivation $D_x : \mathcal{L} \to \mathcal{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local δ-derivations on $\mathcal{L}$ we denote by* $\mathrm{LocDer}_\delta(\mathcal{L})$.

Definition 2.7 *A map $\nabla : \mathcal{L} \to \mathcal{L}$ (not necessary linear) is called a 2-local δ-derivation, if for any $x, y \in \mathcal{L}$, there exists a δ-derivation $D_{x,y} \in \mathrm{Der}_\delta(\mathcal{L})$ such that*

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

It should be noted that 2-local δ-derivation is not necessary linear, but for any $x \in \mathfrak{L}$ and for any scalar λ, we have that

$$\nabla(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \nabla(x).$$

3. Anti-derivations on null-filiform and filiform associative algebras

Now we study anti-derivations on n-dimensional null-filiform, n-dimensional ($n > 2$) complex filiform and n-dimensional ($n \geq 5$) complex naturally graded quasi-filiform non-split associative algebras.

Proposition 3.1 *The anti-derivations of the algebra μ_0^n are given as follows:*

$$\begin{array}{rcl} D(e_1) & = & \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n, \\ D(e_2) & = & -2\alpha_{n-2}e_{n-1} - 2\alpha_{n-1}e_n, \\ D(e_3) & = & \alpha_{n-2}e_n, \\ D(e_i) & = & 0, \quad 4 \leq i \leq n. \end{array}$$

Proof: Since the algebra μ_0^n has one generator e_1, any anti-derivation D on μ_0^n is completely determined by $D(e_1)$.

Let

$$D(e_1) = \sum_{i=1}^{n} \alpha_i e_i.$$

Applying the anti-derivation rule we have

$$D(e_2) = D(e_1e_1) = -(D(e_1)e_1 + e_1D(e_1)) = -\sum_{i=1}^{n} \alpha_i e_i e_1 - \sum_{i=1}^{n} \alpha_i e_1 e_i =$$

$$= -\sum_{i=2}^{n} \alpha_{i-1}e_i - \sum_{i=2}^{n} \alpha_{i-1}e_i = -2\sum_{i=2}^{n} \alpha_{i-1}e_i.$$

$$D(e_3) = D(e_2e_1) = -(D(e_2)e_1 + e_2D(e_1)) = 2\sum_{i=2}^{n} \alpha_{i-1}e_ie_1 - \sum_{i=1}^{n} \alpha_i e_2 e_i =$$

$$= 2\sum_{i=3}^{n} \alpha_{i-2}e_i - \sum_{i=3}^{n} \alpha_{i-2}e_i = \sum_{i=3}^{n} \alpha_{i-2}e_i.$$

Generally, one can prove by induction

$$D(e_i) = (-1)^{i+1}\frac{3 + (-1)^i}{2}\sum_{k=i}^{n} \alpha_{k-i+1}e_k, \qquad 2 \leq i \leq n-1.$$

Suppose the above is true for some i. Now we can determine for $i+1$.

$$D(e_{i+1}) = D(e_ie_1) = -(D(e_i)e_1 + e_iD(e_1)) = -((-1)^{i+1}\frac{3 + (-1)^i}{2}\sum_{k=i}^{n} \alpha_{k-i+1}e_ke_1 + \sum_{k=1}^{n} \alpha_k e_i e_k) =$$

$$= -\big((-1)^{i+1}\frac{3+(-1)^i}{2}\sum_{k=i+1}^{n}\alpha_{k-i}e_k + \sum_{k=i+1}^{n}\alpha_{k-i}e_k\big) = (-1)^{i+2}\frac{3+(-1)^{i+1}}{2}\sum_{k=i+1}^{n}\alpha_{k-i}e_k.$$

Let us determine the properties of anti-derivations:

$$D(e_2e_2) = -(D(e_2)e_2 + e_2D(e_2)) = 2\sum_{i=2}^{n}\alpha_{i-1}e_ie_2 + 2\sum_{i=2}^{n}\alpha_{i-1}e_2e_i = 4\sum_{j=4}^{n}\alpha_{j-3}e_j.$$

On the other hand $D(e_2e_2) = D(e_4) = -2\sum\limits_{i=4}^{n}\alpha_{j-3}e_j$.

Therefore we have $\alpha_j = 0,\ 1 \leq j \leq n-3$. This completes the proof of the proposition. □

Proposition 3.2 *Any anti-derivations of the n-dimensional ($n > 3$) complex filiform associative algebras are given as follows:*

- *for algebra* $\mu_{1,1}^n$:

$$\begin{aligned} D(e_1) &= \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n,\\ D(e_2) &= -2\alpha_{n-3}e_{n-2} - 2\alpha_{n-2}e_{n-1},\\ D(e_3) &= \alpha_{n-3}e_{n-1},\\ D(e_i) &= 0, \quad 4 \leq i \leq n-1,\\ D(e_n) &= \beta_{n-1}e_{n-1} + \beta_n e_n. \end{aligned}$$

- *for algebra* $\mu_{1,2}^n$:

$$\begin{aligned} D(e_1) &= \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n,\\ D(e_2) &= -2\alpha_{n-3}e_{n-2} - 2\alpha_{n-2}e_{n-1},\\ D(e_3) &= \alpha_{n-3}e_{n-1},\\ D(e_i) &= 0, \quad 4 \leq i \leq n-1,\\ D(e_n) &= -\alpha_n e_{n-2} + \beta_{n-1}e_{n-1}. \end{aligned}$$

- *for algebra* $\mu_{1,3}^n$:

$$\begin{aligned} D(e_1) &= \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n,\\ D(e_2) &= -2\alpha_{n-3}e_{n-2} - (2\alpha_{n-2} + \alpha_n)e_{n-1},\\ D(e_3) &= \alpha_{n-3}e_{n-1},\\ D(e_i) &= 0, \quad 4 \leq i \leq n-1,\\ D(e_n) &= \beta_{n-1}e_{n-1}. \end{aligned}$$

- *for algebra* $\mu_{1,4}^n$:

$$\begin{aligned} D(e_1) &= \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n,\\ D(e_2) &= -2\alpha_{n-3}e_{n-2} - (2\alpha_{n-2} + \alpha_n)e_{n-1},\\ D(e_3) &= \alpha_{n-3}e_{n-1},\\ D(e_i) &= 0, \quad 4 \leq i \leq n-1,\\ D(e_n) &= -\alpha_n e_{n-2} + \beta_{n-1}e_{n-1}. \end{aligned}$$

Proof: Let us prove Proposition for the algebra $\mu_{1,1}^n$, and for the algebras $\mu_{1,2}^n,\ \mu_{1,3}^n,\ \mu_{1,4}^n$ the proofs are similar. Since the algebra $\mu_{1,1}^n$ has two generators $\{e_1, e_n\}$, any derivation D on $\mu_{1,1}^n$ is completely determined by $D(e_1)$ and $D(e_n)$.

Let

$$D(e_1)=\sum_{i=1}^{n}\alpha_i e_i, \text{ and } D(e_n)=\sum_{j=1}^{n}\beta_j e_j.$$

Applying the anti-derivation rule we have

$$D(e_2)=D(e_1e_1)=-(D(e_1)e_1+e_1D(e_1))=-\sum_{i=1}^{n}\alpha_i e_ie_1-\sum_{i=1}^{n}\alpha_i e_1e_i$$
$$=-\sum_{i=2}^{n-1}\alpha_{i-1}e_i-\sum_{i=2}^{n-1}\alpha_{i-1}e_i=-2\sum_{i=2}^{n-1}\alpha_{i-1}e_i.$$

$$D(e_3)=D(e_2e_1)=-(D(e_2)e_1+e_2D(e_1))=2\sum_{i=2}^{n-1}\alpha_{i-1}e_ie_1-\sum_{i=1}^{n}\alpha_i e_2e_i$$
$$=2\sum_{i=3}^{n-1}\alpha_{i-2}e_i-\sum_{i=3}^{n-1}\alpha_{i-2}e_i=\sum_{i=3}^{n-1}\alpha_{i-2}e_i.$$

Generally, one can prove by induction

$$D(e_i)=(-1)^{i+1}\frac{3+(-1)^i}{2}\sum_{k=i}^{n-1}\alpha_{k-i+1}e_k, \qquad 2\leq i\leq n-1.$$

Suppose the above is true for some i. Now we can determine for $i+1$.

$$D(e_{i+1})=D(e_ie_1)=-(D(e_i)e_1+e_iD(e_1))$$
$$=-((-1)^{i+1}\frac{3+(-1)^i}{2}\sum_{k=i}^{n-1}\alpha_{k-i+1}e_ke_1+\sum_{k=1}^{n}\alpha_k e_ie_k)$$
$$=-\big((-1)^{i+1}\frac{3+(-1)^i}{2}\sum_{k=i+1}^{n-1}\alpha_{k-i}e_k+\sum_{k=i+1}^{n-1}\alpha_{k-i}e_k\big)$$
$$=(-1)^{i+2}\frac{3+(-1)^{i+1}}{2}\sum_{k=i+1}^{n-1}\alpha_{k-i}e_k.$$

Let us determine the properties of anti-derivations:

$$0=D(e_1e_n)=-(D(e_1)e_n+e_1D(e_n))=-\sum_{i=1}^{n}\alpha_i e_ie_n-\sum_{j=1}^{n}\beta_j e_1e_j=-\sum_{j=2}^{n-1}\beta_{j-1}e_j.$$

Therefore we have $\beta_j=0,\ 1\leq j\leq n-2$.

Let us determine the properties of anti-derivations:

$$D(e_2e_2)=-(D(e_2)e_2+e_2D(e_2))=2\sum_{i=2}^{n-1}\alpha_{i-1}e_ie_2+2\sum_{i=2}^{n-1}\alpha_{i-1}e_2e_i=4\sum_{j=4}^{n-1}\alpha_{j-3}e_j.$$

On the other hand $D(e_2e_2)=D(e_4)=-2\sum\limits_{i=4}^{n-1}\alpha_{j-3}e_j$.

Therefore we have $\alpha_j=0,\ 1\leq j\leq n-4$.

We complete the proof of Proposition. □

Proposition 3.3 *The anti-derivations of the n-dimensional $(n\geq 6)$ complex naturally graded quasi-filiform non-split associative algebras are given as follows:*

- *for algebra* $\mu_{2,1}^n$:

$$\begin{aligned}
D(e_1) &= \alpha_{n-4}e_{n-4} + \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n,\\
D(e_2) &= -2\alpha_{n-4}e_{n-3} - 2\alpha_{n-3}e_{n-2} - \alpha_{n-1}e_n,\\
D(e_3) &= \alpha_{n-4}e_{n-2},\\
D(e_i) &= 0, \quad 4 \leq i \leq n-2,\\
D(e_{n-1}) &= \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n,\\
D(e_n) &= -\beta_{n-1}e_n.
\end{aligned}$$

- *for algebra* $\mu_{2,2}^n(\alpha)$:

$$\begin{aligned}
D(e_1) &= \alpha_{n-4}e_{n-4} + \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_n e_n,\\
D(e_2) &= -2\alpha_{n-4}e_{n-3} - 2\alpha_{n-3}e_{n-2} - (1+\alpha)\alpha_{n-1}e_n,\\
D(e_3) &= \alpha_{n-4}e_{n-2},\\
D(e_i) &= 0, \quad 4 \leq i \leq n-2,\\
D(e_{n-1}) &= \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n,\\
D(e_n) &= -\beta_{n-1}e_n.
\end{aligned}$$

- *for algebra* $\mu_{2,3}^n$:

$$\begin{aligned}
D(e_1) &= \alpha_{n-4}e_{n-4} + \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_n e_n,\\
D(e_2) &= -2\alpha_{n-4}e_{n-3} - 2\alpha_{n-3}e_{n-2},\\
D(e_3) &= \alpha_{n-4}e_{n-2},\\
D(e_i) &= 0, \quad 4 \leq i \leq n-2,\\
D(e_{n-1}) &= \beta_{n-2}e_{n-2} + \beta_{n-1}e_{n-1} + \beta_n e_n,\\
D(e_n) &= -2\beta_{n-1}e_n.
\end{aligned}$$

- *for algebra* $\mu_{2,4}^n$:

$$\begin{aligned}
D(e_1) &= \alpha_{n-4}e_{n-4} + \alpha_{n-3}e_{n-3} + \alpha_{n-2}e_{n-2} + \alpha_n e_n,\\
D(e_2) &= -2\alpha_{n-4}e_{n-3} - 2\alpha_{n-3}e_{n-2},\\
D(e_3) &= \alpha_{n-4}e_{n-2},\\
D(e_i) &= 0, \quad 4 \leq i \leq n-2,\\
D(e_{n-1}) &= \beta_{n-2}e_{n-2} + \beta_n e_n,\\
D(e_n) &= 0.
\end{aligned}$$

Proof: The proof is similar to the proof of Proposition 3.2. □

4. Local and 2-Local anti-derivations on null-filiform and filiform associative algebras

In this section, we consider local and 2-local anti-derivations of null-filiform ($n > 2$), filiform ($n > 3$), and quasi-filiform ($n > 5$) algebras. All these algebras admit a local anti-derivation which is not an anti-derivation. At the same time we show that every 2-local anti-derivation of a null-filiform algebra is an anti-derivation. An example of a 2-local anti-derivation which is not an anti-derivation is given for filiform and quasi-filiform algebras.

4.1. Local anti-derivations on null-filiform and filiform associative algebras

Now we study local anti-derivations on n-dimensional null-filiform, n-dimensional $(n > 3)$ complex filiform and n-dimensional $(n \geq 6)$ complex naturally graded quasi-filiform non-split associative algebras.

Theorem 4.1 *Let Δ be a linear map from a null-filiform associative algebras into itself. Then Δ is a local anti-derivation, if and only if:*

$$\begin{array}{lcl} \Delta(e_1) & = & c_1 e_{n-2} + c_2 e_{n-1} + c_3 e_n, \\ \Delta(e_2) & = & -2c_4 e_{n-1} - 2c_5 e_n, \\ \Delta(e_3) & = & c_6 e_n, \\ \Delta(e_i) & = & 0, \quad 4 \leq i \leq n. \end{array}$$

Proof: Let $\mathfrak{C}$ be the matrix of a local anti-derivation Δ on μ_0^n :

$$\mathfrak{C} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n-1} & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n-1} & c_{2,n} \\ c_{3,1} & c_{3,2} & \cdots & c_{3,n-1} & c_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1,1} & c_{n-1,2} & \cdots & c_{n-1,n-1} & c_{n-1,n} \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n-1} & c_{n,n} \end{pmatrix}.$$

By the definition for every $x = \sum\limits_{i=1}^{n} x_i e_i \in \mu_0^n$ there exists an anti-derivation D_x on μ_0^n such that

$$\Delta(x) = D_x(x).$$

By Proposition 3.1, the anti-derivarion D_x has the following matrix form:

$$\mathfrak{C}_x = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-2}^x & 0 & 0 & \cdots & 0 & 0 \\ \alpha_{n-1}^x & -2\alpha_{n-2}^x & 0 & \cdots & 0 & 0 \\ \alpha_n^x & -2\alpha_{n-1}^x & \alpha_{n-2}^x & \cdots & 0 & \end{pmatrix}.$$

For the matrix $\mathfrak{C}$ of Δ by choosing subsequently $x = e_1, \ldots, x = e_n$ and using $\Delta(x) = D_x(x)$, i.e. $\mathfrak{C}\overline{x} = D_x(\overline{x})$, where $\overline{x}$ is the vector corresponding to x, we obtain

$$\mathfrak{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-2,1} & 0 & 0 & \cdots & 0 & 0 \\ c_{n-1,1} & c_{n-1,2} & 0 & \cdots & 0 & 0 \\ c_{n,1} & c_{n,2} & c_{n,3} & \cdots & 0 & 0 \end{pmatrix}.$$

Conversely, suppose that the matrix of Δ has the above form and let us show that Δ is a local anti-derivation. For each x we must find an anti-derivation D_x such that $\Delta(x) = D_x(x)$. We have $\mathfrak{C}\overline{x} = \mathfrak{C}_x(\overline{x})$, where $\overline{x}$ is the vector corresponding to $x = \sum\limits_{i=1}^{n} x_i e_i$,. This implies the following system of equalities

$$\begin{array}{ll} c_{n-2,1}x_1 & = \alpha_{n-2}^x x_1, \\ c_{n-1,1}x_1 + c_{n-1,2}x_2 & = \alpha_{n-1}^x x_1 - 2\alpha_{n-2}^x x_2, \\ c_{n,1}x_1 + c_{n,2}x_2 + c_{n,3}x_3 & = \alpha_n^x x_1 - 2\alpha_{n-1}^x x_2 + \alpha_{n-2}^x x_3. \end{array}$$

Let us consider two cases separately:

Case 1: If $x_1 \neq 0$, then

$$\begin{aligned}
\alpha_{n-2}^x &= c_{n-2,1}, \\
\alpha_{n-1}^x &= \frac{c_{n-1,1}x_1 + c_{n-1,2}x_2 + 2\alpha_{n-2}^x x_2}{x_1}, \\
\alpha_n^x &= \frac{c_{n,1}x_1 + c_{n,2}x_2 + c_{n,3}x_3 + 2\alpha_{n-1}^x x_2 - \alpha_{n-2}^x x_3}{x_1}.
\end{aligned}$$

Case 2: If $x_1 = 0$ and $x_2 \neq 0$, then

$$\begin{aligned}
\alpha_{n-2}^x &= -\frac{c_{n-1,2}}{2}, \\
\alpha_{n-1}^x &= -\frac{c_{n,2}x_2 + c_{n,3}x_3 - \alpha_{n-2}^x}{x_2},
\end{aligned}$$

where α_n^x defined arbitrary.

Case 3: If $x_1 = x_2 = 0$ and $x_3 \neq 0$, then $\alpha_{n-2}^x = c_{n,3}$, where $\alpha_{n-1}^x,\ \alpha_n^x$ defined arbitrary.

We find $\alpha_i^x,\ n-2 \leq i \leq n$. Thus, the matrix of the anti-derivation D_x is determined such that $\Delta(x) = D_x(x)$. The proof is complete.

□

Theorem 4.2 *Let Δ be a linear map on an n-dimensional ($n > 3$) complex filiform associative algebra. Then Δ is a local anti-derivation, if and only if:*

- *for the algebras $\mu_{1,1}^n,\ \mu_{1,2}^n,\ \mu_{1,4}^n$*

$$\begin{aligned}
\Delta(e_1) &= c_1e_{n-3} + c_2e_{n-2} + c_3e_{n-1} + c_4e_n, \\
\Delta(e_2) &= c_5e_{n-2} + c_6e_{n-1}, \\
\Delta(e_3) &= c_7e_{n-1}, \\
\Delta(e_i) &= 0,\ 4 \leq i \leq n-1, \\
\Delta(e_n) &= c_8e_{n-1} + c_9e_n.
\end{aligned}$$

- *for the algebra $\mu_{1,3}^n$*

$$\begin{aligned}
\Delta(e_1) &= c_1e_{n-3} + c_2e_{n-2} + c_3e_{n-1} + c_4e_n, \\
\Delta(e_2) &= c_5e_{n-2} + c_6e_{n-1}, \\
\Delta(e_3) &= c_7e_{n-1}, \\
\Delta(e_i) &= 0,\ 4 \leq i \leq n-1, \\
\Delta(e_n) &= c_8e_{n-1}.
\end{aligned}$$

Proof: The proof is obtained by straightforward calculations similarly to the proof of Theorem 4.1. □

Theorem 4.3 *Let Δ be a linear map on an n-dimensional ($n \geq 6$) complex naturally graded quasi-filiform non-split associative algebra. Then Δ is a local anti-derivation, if and only if:*

- *for the algebra $\mu_{2,1}^n$,*

$$\begin{aligned}
\Delta(e_1) &= c_1e_{n-4} + c_2e_{n-3} + c_3e_{n-2} + c_4e_{n-1} + c_5e_n, \\
\Delta(e_2) &= c_6e_{n-3} + c_7e_{n-2} + c_8e_n, \\
\Delta(e_3) &= c_9e_{n-2}, \\
\Delta(e_i) &= 0, \quad 4 \leq i \leq n-2, \\
\Delta(e_{n-1}) &= c_{10}e_{n-2} + c_{11}e_{n-1} + c_{12}e_n, \\
\Delta(e_n) &= c_{13}e_n.
\end{aligned}$$

- *for the algebra* $\mu^n_{2,2}(\alpha = 1)$,

$$\begin{aligned}
\Delta(e_1) &= c_1 e_{n-4} + c_2 e_{n-3} + c_3 e_{n-2} + c_4 e_{n-1} + c_5 e_n, \\
\Delta(e_2) &= c_6 e_{n-3} + c_7 e_{n-2} + (1+\alpha) c_8 e_n, \\
\Delta(e_3) &= c_9 e_{n-2}, \\
\Delta(e_i) &= 0, \quad 4 \le i \le n-2, \\
\Delta(e_{n-1}) &= c_{10} e_{n-2} + c_{11} e_{n-1} + c_{12} e_n, \\
\Delta(e_n) &= c_{13} e_n.
\end{aligned}$$

for the algebra $\mu^n_{2,3}$,

$$\begin{aligned}
\Delta(e_1) &= c_1 e_{n-4} + c_2 e_{n-3} + c_3 e_{n-2} + c_4 e_n, \\
\Delta(e_2) &= c_5 e_{n-3} + c_6 e_{n-2}, \\
\Delta(e_3) &= c_7 e_{n-2}, \\
\Delta(e_i) &= 0, \quad 4 \le i \le n-2, \\
\Delta(e_{n-1}) &= c_8 e_{n-2} + c_9 e_{n-1} + c_{10} e_n, \\
\Delta(e_n) &= c_{11} e_n.
\end{aligned}$$

for the algebras $\mu^n_{2,4}$

$$\begin{aligned}
\Delta(e_1) &= c_1 e_{n-4} + c_2 e_{n-3} + c_3 e_{n-2} + c_4 e_n, \\
\Delta(e_2) &= c_5 e_{n-3} + c_6 e_{n-2}, \\
\Delta(e_3) &= c_7 e_{n-2}, \\
\Delta(e_i) &= 0, \quad 4 \le i \le n-2, \\
\Delta(e_{n-1}) &= c_8 e_{n-2} + c_9 e_n, \\
\Delta(e_n) &= 0.
\end{aligned}$$

Proof: The proof is obtained by straightforward calculations similarly to the proof of Theorem 4.1. □

As above, one can calculate dimensions of the spaces of anti-derivation and local anti-derivations of null-filiform, filiform and naturally graded quasi-filiform non-split associative algebras.

Algebra	The dimensions of the space of anti-derivations	The dimensions of the space of local anti-derivations
μ^n_0	3	6
$\mu^n_{1,1}$	6	9
$\mu^n_{1,2}$	5	9
$\mu^n_{1,3}$	5	8
$\mu^n_{1,4}$	5	8
$\mu^n_{2,1}$	8	13
$\mu^n_{2,2}(\alpha)\ \alpha = -1$	8	12
$\mu^n_{2,2}(\alpha),\ \alpha \neq -1$	8	13
$\mu^n_{2,3}$	7	11
$\mu^n_{2,4}$	6	9

Corollary 4.1 *The null-filiform associative algebras, filiform associative algebras, and naturally graded quasi-filiform non-split associative algebras admit local anti-derivations which are not anti-derivations.*

Remark 4.1 *Also, note that local anti-derivations of an arbitrary low-dimension algebra can be similarly described using a common form of the matrix of anti-derivations on this algebra. A technique for constructing a local anti-derivation, which is not an anti-derivation, developed by us, can be applied to an arbitrary low-dimension algebra, anti-derivations of which have a matrix of a common form.*

5. 2-Local anti-derivations on null-filiform and filiform associative algebras

Now we study 2-local anti-derivations on n-dimensional null-filiform, n-dimensional $(n > 3)$ complex filiform and n-dimensional $(n \geq 6)$ complex naturally graded quasi-filiform non-split associative algebras.

Theorem 5.1 *Any 2-local anti-derivation on the null-filiform associative algebra μ_0^n is an anti-derivation.*

Proof: Consider a 2-local anti-derivation ∇ on μ_0^n such that $\nabla(e_1) = 0$.

For the element $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \in \mu_0^n, \quad x_i \in \mathbb{C}, \quad 1 \leq i \leq n$, there is an anti-derivation $D_{e_1,x}$ such that

$$\nabla(e_1) = D_{e_1,x}(e_1), \quad \nabla(x) = D_{e_1,x}(x).$$

Then we have

$$0 = \nabla(e_1) = D_{e_1,x}(e_1) = \alpha_{n-2}e_{n-2} + \alpha_{n-1}e_{n-1} + \alpha_ne_n.$$

Hence $\alpha_{n-2} = \alpha_{n-1} = \alpha_n = 0$ and therefore $D_{e_1,x} = 0$. Thus, $\nabla = 0$.

Now let ∇ be an arbitrary 2-local anti-derivation of the algebra μ_0^n. Then there is an anti-derivation D such that $\nabla(e_1) = D(e_1)$. An operator $\nabla - D$ is a 2-local anti-derivation and $(\nabla - D)(e_1) = 0$, which implies $\nabla \equiv D$. Hence ∇ is an anti-derivation. □

Theorem 5.2 *The n-dimensional $(n > 3)$ complex filiform associative and n-dimensional $(n \geq 6)$ complex naturally graded quasi-filiform non-split associative algebras admit 2-local anti-derivations which are not anti-derivations.*

Proof: We proof the theorem for the algebra $\mu_{1,1}^n$; for the algebras $\mu_{1,2}^n, \ \mu_{1,3}^n, \ \mu_{1,4}^n, \mu_{2,1}^n, \ \mu_{2,2}^n(\alpha), \ \mu_{2,3}^n, \ \mu_{2,4}^n$ the proofs are similar.

Let us define a homogeneous non additive function f on $\mathbb{C}^2$ as follows

$$f(z_1, z_n) = \begin{cases} \frac{z_1^2}{z_n}, & \text{if } z_n \neq 0, \\ 0, & \text{if } z_n = 0. \end{cases}$$

where $(z_1, z_n) \in \mathbb{C}^2$. Consider the map $\nabla : \mu_{1,1}^n \to \mu_{1,1}^n$ defined by the rule

$$\nabla(x) = f(x_1, x_n)e_n, \quad \text{where} \quad x = \sum_{i=1}^{n} x_ie_i \in \mu_{1,1}^n.$$

Since f is not additive, ∇ is not an anti-derivation.

Let us show that ∇ is a 2-local anti-derivation. For the elements

$$x = \sum_{i=1}^{n} x_ie_i, \quad y = \sum_{i=1}^{n} y_ie_i,$$

we search an anti-derivation D in the form:

$$D(e_1) = \alpha_ne_n, \quad D(e_n) = \beta_ne_n, \quad D(e_i) = 0, \quad 3 \leq i \leq n.$$

Assume that $\nabla(x) = D(x)$ and $\nabla(y) = D(y)$. Then we obtain the following system of equations for α_n and β :

$$\begin{cases} x_1\alpha_n + x_n\beta_n = f(x_1, x_n), \\ y_1\alpha_n + y_n\beta_n = f(y_1, y_n). \end{cases} \tag{5.1}$$

Case 1. Let $x_1y_n - x_ny_1 = 0$, then the system has infinitely many solutions, because of the right-hand side of this system is homogeneous.

Case 2. Let $x_1y_n - x_ny_1 \neq 0$, then the system has a unique solution. The proof is complete. □

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