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PMC-Graphs Derived from Cycles

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ABSTRACT: The graph $G = (V, E)$ consists of p vertices and q edges. Let

$$\rho = \begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma = \{\pm 1, \pm 2, \ldots, \pm \rho\}$. Consider a function $\Lambda : V \to \Gamma$ that allocates unique labels from Γ to the various vertices of V when p is even and allocates a unique labels in Γ to $p-1$ vertices of V, repeating a label for the remaining one vertex when p is odd. Then the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge uv of G, there is a labeling $\frac{\Lambda(u)+\Lambda(v)}{2}$ if $\Lambda(u) + \Lambda(v)$ is even and $\frac{\Lambda(u)+\Lambda(v)+1}{2}$ if $\Lambda(u) + \Lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\Lambda_1} - \bar{\mathbb{S}}_{\Lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\Lambda_1}$ and $\bar{\mathbb{S}}_{\Lambda_1^c}$ are denoted the number of edges labelled with 1 and the number of edges not labelled with 1, respectively. A graph G that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-Graph). This research paper examines the PMC-labeling behaviour of some graphs, like the pagoda graph, antiweb-gear graph, spherical graph, alternate triangular cycle, alternate quadrilateral cycle, balloon of triangular snake and balloon of quadrilateral snake.

Key Words: pagoda graph, antiweb-gear graph, spherical graph, balloon of triangular snake and balloon of quadrilateral snake.

Contents

1. Introduction

In this research paper, we examine the finite, simple and undirected graph G only. The process of assigning labels or integers to the vertices, edges or both based on certain conditions is known as graph labeling. The first paper on graph labeling was introduced by Rosa in 1967 [20]. On a family of the super edge-graceful trees were examined in [11]. Zeen El Deen [24] have investigated the edge δ-graceful labeling for some cyclic-related graphs. Chartrand et al. [4] have studied the radio labelings of graphs. On the radio antipodal geometric mean number of ladder related graphs were explored in [7]. Selvam Avadayapppan and Sinthu [21] have investigated the new families of super mean graphs. Sugumaran and Vishnu Prakash [22] worked on some new results of prime cordial labeling. Prime labelings on graphs have investigated by Baskar Babujee [3]. On the total rainbow connection of the wheel related graphs were examined in [10]. Melina and Salman [12] have computed the rainbow connection number of spectrum graphs. Giridaran et al. [7] have studied on the radio antipodal geometric mean number of ladder related graphs. H. U. Afzal [2] have worked on super edge-magicness of two special families of graphs. Edge-magic labelings of wheel graphs have investigated by Y. Fukuchi [6]. Daoud and Mohamed [5] have exmined the complexity of some families of cycle-related graphs.

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Rathod and Kanani [19] have investigated the V_4-cordial labeling of quadrilateral snakes. Lucas divisor cordial labeling have derived by Sugumaran and Rajesh [23]. SD-Prime cordial labeling of alternate k-polygonal snake of various types were studied in [15]. Patel et al. [14] have investigated the product cordial labeling of extensions of barbell graph. Double divisor cordial labeling of graphs have examined by Parthiban and Vishally Sharma [13]. Sum divisor cordial labeling in the context of duplication of graph elements were explored in [1]. We adhere to different graph theoretic notations and terminology [8]. We have introduced a new kind of labeling called PMC-labeling in [16] and the PMC-labeling behavior of more kinds of snake related graphs have been explored in [17,18]. Terms that are not defined in this paper are derived from Harary [9]. This research paper examines the PMC-labeling behaviour of some graphs, like the pagoda graph, antiweb-gear graph, spherical graph, alternate triangular cycle, alternate quadrilateral cycle, balloon of triangular snake and balloon of quadrilateral snake.

2. PMC Graph

Definition 2.1 *The graph $G = (V, E)$ consists of p vertices and q edges. Let*

$$\rho = \begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma = \{\pm 1, \pm 2, \ldots, \pm \rho\}$. Consider a function $\Lambda : V \to \Gamma$ that allocates unique labels from Γ to the various vertices of V when p is even and allocates a unique labels in Γ to $p-1$ vertices of V, repeating a label for the remaining one vertex when p is odd. Then the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge uv of G, there is a labeling $\frac{\Lambda(u)+\Lambda(v)}{2}$ if $\Lambda(u)+\Lambda(v)$ is even and $\frac{\Lambda(u)+\Lambda(v)+1}{2}$ if $\Lambda(u)+\Lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\Lambda_1} - \bar{\mathbb{S}}_{\Lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\Lambda_1}$ and $\bar{\mathbb{S}}_{\Lambda_1^c}$ are denoted the number of edges labelled with 1 and the number of edges not labelled with 1, respectively. A graph G that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-Graph).
A simple example of PMC-graph is shown in figure 1.

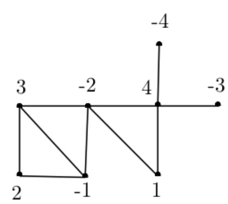

Figure 1: PMC-Graph

3. Preliminaries

In this section, we present a few fundamental definitions that are essential for the upcoming section.

Definition 3.1 *[7] The pagoda graph PG_n is a ladder graph formed by adding a vertex v_0 in such a way that it is adjacent to u_n and v_n.*

Definition 3.2 *[21] The balloon of triangular snake $BT_{n,m}$ is the graph obtained from C_n by identifying an end vertex of the path in triangular snake T_m at a vertex of C_n.*

Definition 3.3 *[21] The balloon of quadrilateral snake $BQ_{n,m}$ is the graph obtained from C_n by identifying an end vertex of the path in quadrilateral snake Q_m at a vertex of C_n.*

Definition 3.4 *[5] The spherical graph SP_n, $n \geq 1$ is a connected graph $C_{2^n} + \bar{K}_2$ with $2(2^{n-1}+1)$ vertices and $3(2^n)$ edges.*

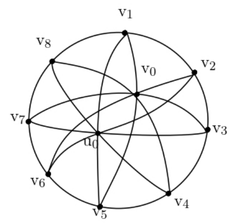

Figure 2: The spherical graph SP_3

Definition 3.5 *[10] The antiweb-gear graph AWG_n, $n \geq 3$ is a connected graph with the vertex set and edge set, respectively, as follows $V(AWG_n) = \{v_0, v_i, u_i \mid 1 \leq i \leq n\}$ and $E(AWG_n) = \{v_0v_i, v_iu_i \mid 1 \leq i \leq n\} \cup \{v_iv_{i+1}, v_nv_1, u_iu_{i+1}, u_nu_1, u_iv_{i+1}, u_nv_1 \mid 1 \leq i \leq n-1\}$.*

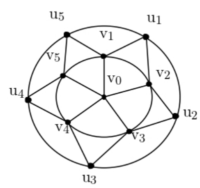

Figure 3: The antiweb-gear graph AWG_5

Definition 3.6 *[24] An alternate triangular cycle ATC_n is the graph obtained from an even cycle $C_{2n} = \{v_1, u_1, v_2, u_2, \ldots v_n, u_n\}$ by joining v_i and u_i to a new vertex w_i. That is, every alternate edge of a cycle is replaced by C_3.*

Definition 3.7 *[24] An alternate quadrilateral cycle AQC_n is the graph obtained from an even cycle $C_{2n} = \{v_1, u_1, v_2, u_2, \ldots v_n, u_n\}$ by joining v_i and u_i to a new vertex w_i. That is, every alternate edge of a cycle is replaced by C_4.*

4. Main Theorems

Theorem 4.1 *The pagoda graph PG_n is a PMC-graph for all $n \geq 2$.*

Proof: Let us consider the pagoda graph PG_n, $n \geq 2$. Denote by $V(PG_n) = \{v_0, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(PG_n) = \{u_iv_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1}, v_nv_0, u_nv_0 \mid 1 \leq i \leq n-1\}$ respectively, the vertex set and edge set of the pagoda graph PG_n. Thus, PG_n has $3n$ edges and $2n+1$ vertices. Let $\Lambda(V_0) = 1$. We have consider two cases:

Case $(i) : n \equiv 0 \pmod 4$

We assign the vertices $u_1, u_3, \ldots, u_{\frac{n+2}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n}{2}}$ with labels $2, 4, \ldots, \frac{n+4}{2}$ and $-2, -4, \ldots, \frac{-n}{2}$ respectively. Then, designate the vertices $u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \ldots, u_{n-1}$ with labels $\frac{n+6}{2}, \frac{n+8}{2}, \ldots, n$ respectively. Fix the vertex u_n with label 1. Further, we designate the vertices $v_1, v_3, \ldots, v_{\frac{n+2}{2}}$ and $v_2, v_4, \ldots, v_{\frac{n}{2}}$ with labels $-1, -3, \ldots, \frac{-n-2}{2}$ and $1, 3, \ldots, \frac{n+2}{2}$ respectively. Next, designate the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, \ldots, v_n$ with labels $\frac{-n-4}{2}, \frac{-n-6}{2}, \ldots, -n$ respectively.

Case $(ii) : n \equiv 1 \pmod 4$

Designate the vertices $u_1, u_3, \ldots, u_{\frac{n+1}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n-1}{2}}$ with labels $2, 4, \ldots, \frac{n+3}{2}$ and $-2, -4, \ldots, \frac{-n+1}{2}$ respectively. Further, designate the vertices $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \ldots, u_{n-1}$ with labels $\frac{n+5}{2}, \frac{n+7}{2}, \ldots, n$ respectively. Fix the vertex u_n with label 1. More over, we designate the vertices $v_1, v_3, \ldots, v_{\frac{n+1}{2}}$ and

$v_2, v_4, \ldots, v_{\frac{n-1}{2}}$ with labels $-1, -3, \ldots, \frac{-n-1}{2}$ and $3, 5, \ldots, \frac{n+1}{2}$ respectively. Thus, designate the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \ldots, v_n$ with labels $\frac{-n-3}{2}, \frac{-n-5}{2}, \ldots, -n$ respectively.

Case $(iii) : n \equiv 2 \pmod 4$

Also assign the vertices $u_1, u_3, \ldots, u_{\frac{n}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n+2}{2}}$ with labels $2, 4, \ldots, \frac{n+2}{2}$ and $-2, -4, \ldots, \frac{-n-2}{2}$ respectively. So, designate the vertices $u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \ldots, u_n$ with labels $\frac{-n-4}{2}, \frac{-n-6}{2}, \ldots, -n$ respectively. Also, we designate the vertices $v_1, v_3, \ldots, v_{\frac{n}{2}}$ and $v_2, v_4, \ldots, v_{\frac{n+2}{2}}$ with labels $-1, -3, \ldots, \frac{-n}{2}$ and $3, 5, \ldots, \frac{n+4}{2}$ respectively. Hence, designate the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, \ldots, v_{n-1}$ with labels $\frac{n+6}{2}, \frac{n+8}{2}, \ldots, n$ respectively. Fix the vertex v_n with label 1.

Case $(iv) : n \equiv 3 \pmod 4$

Designate the vertices $u_1, u_3, \ldots, u_{\frac{n-1}{2}}$ and $u_2, u_4, \ldots, u_{\frac{n+1}{2}}$ with labels $2, 4, \ldots, \frac{n+1}{2}$ and $-2, -4, \ldots, \frac{-n-1}{2}$ respectively. Then, designate the vertices $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \ldots, u_n$ with labels $\frac{-n-3}{2}, \frac{-n-5}{2}, \ldots, -n$ respectively. So designate the vertices $v_1, v_3, \ldots, v_{\frac{n-1}{2}}$ and $v_2, v_4, \ldots, v_{\frac{n+1}{2}}$ with labels $-1, -3, \ldots, \frac{-n+1}{2}$ and $3, 5, \ldots, \frac{n+3}{2}$ respectively. Therefore, designate the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \ldots, v_{n-1}$ with labels $\frac{n+5}{2}, \frac{n+7}{2}, \ldots, n$ respectively. Fix the vertex v_n with label 1.

□

Table 1: PMC-labeling of the pagoda graph PG_n, $n \geq 3$.

Value of n	$\mathbb{S}_{\Lambda_1^c}$	$\mathbb{S}_{\Lambda_1}$
$n \equiv 0 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

Example 4.1 *Figure 4 illustrates the PMC-labeling of the pagoda graph PG_6.*

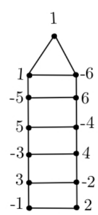

Figure 4: PMC-labeling of the pagoda graph PG_6

Theorem 4.2 *The antiweb-gear graph AWG_n is not PMC-graph for all $n \geq 3$.*

Proof: Consider the antiweb-gear graph AWG_n. Let $V(AWG_n) = \{v_0, v_i, u_i \mid 1 \leq i \leq n\}$ and $E(AWG_n) = \{v_0v_i, v_iu_i \mid 1 \leq i \leq n\} \cup \{v_iv_{i+1}, v_nv_1, u_iu_{i+1}, u_nu_1, u_iv_{i+1},$ $u_nv_1 \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the antiweb-gear graph AWG_n. Then, AWG_n has $5n$ edges and $2n+1$ vertices. Suppose that the antiweb-gear graph AWG_n is a PMC-graph. If the edge uv is given the label 1, the possible results are either $\Lambda(u) + \Lambda(v) = 1$ or $\Lambda(u) + \Lambda(v) = 2$. Thus, the maximum possible number of edges designated with a label 1 is $2n-1$. Subsequently, the minimum number of edges that are not designated with a label 1 is $3n+1$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \geq n+2 \geq 5 > 1$, we get a contradiction.

□

Theorem 4.3 *The spherical graph SP_n is not PMC-graph for all $n \geq 1$.*

Proof: Consider the spherical graph SP_n, $n \geq 1$. Let $V(SP_n) = \{u_0, v_0, v_i \mid 1 \leq i \leq 2^n\}$ and $E(SP_n) = \{v_0v_i, u_0v_i \mid 1 \leq i \leq 2^n\} \cup \{v_iv_{i+1}, v_nv_1, \mid 1 \leq i \leq 2^n - 1\}$ denote, respectively, the vertex set and edge set of the spherical graph SP_n. Thus, SP_n has $3(2^n)$ edges and $2(2^{n-1} + 1)$ vertices. Suppose that the spherical graph SP_n is a PMC-graph. Thus, the maximum possible number of edges designated with a label 1 is $2n - 1$. Subsequently, the minimum number of edges that are not designated with a label 1 is $3(2^n) - 2n + 1$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \geq 3(2^n) - 4n + 2 \geq 4 > 1$, we get a contradiction.

□

Theorem 4.4 *The alternate triangular cycle ATC_n is a PMC-graph for all $n \geq 3$.*

Proof: Consider the alternate triangular cycle ATC_n, $n \geq 3$. Let $V(ATC_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(ATC_n) = \{v_iw_i, v_iu_i, w_iu_i, u_iv_{i+1}, u_nv_1 \mid 1 \leq i \leq n\}$ denote, respectively, the vertex set and edge set of the alternate triangular cycle ATC_n. Then, ATC_n has $4n$ edges and $3n$ vertices. Define $\Lambda(w_{n-1}) = 1$. We designate the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ with labels $-1, -2, \ldots, -n$ and $2, 3, \ldots, n+1$ respectively. We have consider two cases:

Case (i) : n is odd
We designate the vertices $w_1, w_3, \ldots, w_{n-2}$ and $w_2, w_4, \ldots, w_{n-3}$ with labels $-n-1, -n-2, \ldots, \frac{-3n+1}{2}$ and $n+2, n+3, \ldots, \frac{3n-1}{2}$ respectively. Fix the vertex w_n with label 1.

Case (ii) : n is even
Let $\Lambda(w_1) = -n-1$ Also, designate the vertices $w_3, w_5, \ldots, w_{n-3}$ and $w_2, w_4, \ldots, w_{n-3}$ with labels $-n-2, -n-3, \ldots, \frac{-3n}{2}$ and $n+3, n+4, \ldots, \frac{3n}{2}$ respectively. Fix the vertex w_n with label $n+2$. In all cases, $\bar{\mathbb{S}}_{\Lambda_1^c} = 2n = \bar{\mathbb{S}}_{\Lambda_1}$. □

Example 4.2 *Figure 5 illustrates the PMC-labeling of the alternate triangular cycle ATC_5.*

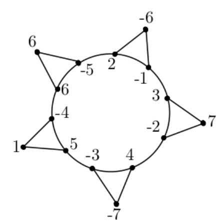

Figure 5: PMC-labeling of the alternate triangular cycle ATC_5

Theorem 4.5 *The alternate quadrilateral cycle AQC_n is a PMC-graph for all $n \geq 3$.*

Proof: Consider the alternate quadrilateral cycle AQC_n, $n \geq 3$. Let $V(AQC_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(AQC_n) = \{v_iw_i, v_iu_i, w_iu_i, u_iv_{i+1}, u_nv_1 \mid 1 \leq i \leq n\}$ denote, respectively, the vertex set and edge set of the alternate quadrilateral cycle AQC_n. Then, AQC_n has $5n$ edges and $4n$ vertices. We designate the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ with labels $-1, -2, \ldots, -n$ and $2, 3, \ldots, n+1$ respectively. We have consider two cases:
Case (i) : n is odd
Now, we designate the vertices $x_1, x_2, \ldots, x_{\frac{n+1}{2}}$ and $y_1, y_2, \ldots, y_{\frac{n+1}{2}}$ with labels $-n-1, -n-2, \ldots, \frac{-3n-1}{2}$ and $n+2, n+3, \ldots, \frac{3n+3}{2}$ respectively. Then, designate the vertices $x_{\frac{n+3}{2}}, y_{\frac{n+3}{2}}$ and $x_{\frac{n+5}{2}}, y_{\frac{n+5}{2}}$ with labels $\frac{-3n-3}{2}$, $\frac{-3n-5}{2}$ and $\frac{3n+5}{2}$, $\frac{3n+7}{2}$ respectively. Thus, designate the vertices $x_{\frac{n+7}{2}}, y_{\frac{n+7}{2}}$ and $x_{\frac{n+9}{2}}, y_{\frac{n+9}{2}}$ with labels $\frac{-3n-7}{2}$, $\frac{-3n-9}{2}$ and $\frac{3n+5}{2}$, $\frac{3n+7}{2}$ respectively. Proceeding like this, until designate the vertices x_{n-1}, y_{n-1} and x_n, y_n with labels $-2n+1, -2n$ and $2n, 1$ respectively when $n \equiv 1 \pmod 4$ and designate the vertices x_{n-1}, y_{n-1} and x_n, y_n with labels $2n-1, 2n$ and $-2n, 1$ respectively when $n \equiv 3 \pmod 4$.

Case (ii) : n is even
Now, we designate the vertices $x_1, x_2, \ldots, x_{\frac{n+2}{2}}$ and $y_1, y_2, \ldots, y_{\frac{n+2}{2}}$ with labels $-n-1, -n-2, \ldots, \frac{-3n-2}{2}$ and $n+2, n+3, \ldots, \frac{3n+4}{2}$ respectively. Then, designate the vertices $x_{\frac{n+4}{2}}, y_{\frac{n+4}{2}}$ and $x_{\frac{n+6}{2}}, y_{\frac{n+6}{2}}$ with labels $\frac{-3n-4}{2}, \frac{-3n-6}{2}$ and $\frac{3n+6}{2}, \frac{3n+8}{2}$ respectively. Thus, designate the vertices $x_{\frac{n+8}{2}}, y_{\frac{n+8}{2}}$ and $x_{\frac{n+10}{2}}, y_{\frac{n+10}{2}}$ with labels $\frac{-3n-8}{2}, \frac{-3n-10}{2}$ and $\frac{3n+10}{2}, \frac{3n+12}{2}$ respectively. Proceeding like this, until designate the vertices x_{n-1}, y_{n-1} and x_n, y_n with labels $2n-1, 2n$ and $-2n, 1$ respectively when $n \equiv 0 \pmod 4$ and designate the vertices x_{n-1}, y_{n-1} and x_n, y_n with labels $-2n+1, -2n$ and $2n, 1$ respectively when $n \equiv 2 \pmod 4$. □

Table 2: PMC-labeling of the alternate quadrilateral cycle AQC_n, $n \geq 3$.

Value of n	$\mathbb{S}_{\Lambda_1^c}$	$\mathbb{S}_{\Lambda_1}$
n is odd	$\frac{5n+1}{2}$	$\frac{5n-1}{2}$
n is even	$\frac{5n}{2}$	$\frac{5n}{2}$

Example 4.3 *Figure 6 illustrates the PMC-labeling of the alternate quadrilateral cycle AQC_5.*

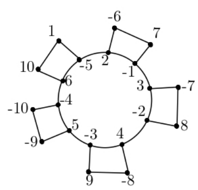

Figure 6: PMC-labeling of the alternate quadrilateral cycle AQC_5

Theorem 4.6 *The balloon of triangular snake $BT_{n,m}$ is a PMC-graph for all $n \geq 4$ and $m \geq 2$.*

Proof: Consider the balloon of triangular snake $BT_{n,m}$. Let $V(BT_{n,m}) = \{v_i, u_j, w_j \mid 1 \leq i \leq n \,\&\, 1 \leq j \leq m-1\}$ and $E(BT_{n,m}) = \{v_iv_{i+1}, v_nv_1, v_nu_1, v_nw_1 \mid 1 \leq i \leq n-1\} \cup \{u_ju_{j+1}, u_jw_{j+1} \mid 1 \leq j \leq m-2\} \cup \{w_ju_j \mid 1 \leq i \leq m-1\}$ denote, respectively, the vertex set and edge set of the balloon of triangular snake $BT_{n,m}$. Then, $BT_{n,m}$ has $n+3m-3$ edges and $n+2m-2$ vertices. We have consider three cases:
Case (i) : $n = 4$
Subcase (i) : m is odd
Note that $BT_{4,m}$ has $3m+1$ edges and $2m+2$ vertices. We designate the vertices v_1, v_2, v_3, v_4 and $u_1, u_2, \ldots, u_{m-1}$ with labels $2, -1, 3, -2$ and $-3, -4, \ldots, -m-1$ respectively. Then, designate the vertices $w_1, w_2, \ldots, w_{\frac{m-1}{2}}$ and $w_{\frac{m+3}{2}}, w_{\frac{m+5}{2}}, \ldots, w_{m-1}$ with labels $4, 5, \ldots, \frac{m+5}{2}$ and $\frac{m+7}{2}, \frac{m+9}{2}, \ldots, m+1$ respectively. Fix the vertex $u_{\frac{m+1}{2}}$ with label 1. Hence, $\bar{\mathbb{S}}_{\Lambda_1^c} = \frac{3m+1}{2} = \bar{\mathbb{S}}_{\Lambda_1}$.
Subcase (ii) : m is even
We designate labels to the vertices v_i, u_j, $1 \leq i \leq 4 \,\&\, 1 \leq j \leq m-1$ as in subcase (i). Then, designate the vertices $w_1, w_2, \ldots, w_{\frac{m-2}{2}}$ and $w_{\frac{m+2}{2}}, w_{\frac{m+4}{2}}, \ldots, w_{m-1}$ with labels $4, 5, \ldots, \frac{m+4}{2}$ and $\frac{m+6}{2}, \frac{m+8}{2}, \ldots, m+1$ respectively. Fix the vertex $u_{\frac{m}{2}}$ with label 1. Hence, $\bar{\mathbb{S}}_{\Lambda_1^c} = \frac{3m+@}{2}$ and $\bar{\mathbb{S}}_{\Lambda_1} = \frac{3m}{2}$.
Case (ii) : $n \geq 5$ and n is odd
We also designate the vertices v_1, v_2, v_3, v_4 and $v_5, v_7, \ldots, v_n$ with labels $2, -1, 3, -2$ and $-3, -4, \ldots, \frac{-n-1}{2}$ respectively. Next, designate the vertices $v_6, v_8, \ldots, v_{n-1}$ with labels $4, 5, \ldots, \frac{n+1}{2}$ respectively. Fix the vertices u_{m-1}, w_1, w_{m-1} with labels $1, \frac{n+3}{2}, 1$.
Subcase (i) : m is odd
Further, designate the vertices $u_1, u_2, \ldots, u_{\frac{m-3}{2}}$ and $u_{\frac{m-1}{2}}, u_{\frac{m+1}{2}}, \ldots, u_{m-2}$ with labels $\frac{n+5}{2}, \frac{n+7}{2}, \ldots, \frac{n+m}{2}$ and $\frac{-n-m}{2}, \frac{-n-m-2}{2}, \ldots, \frac{-n-2m+3}{2}$ respectively. Then designate the vertices $w_2, w_3, \ldots, w_{\frac{m-1}{2}}$ and $w_{\frac{m+1}{2}}$,

$w_{\frac{m+3}{2}}, \ldots, w_{m-2}$ with labels $\frac{-n-3}{2}, \frac{-n-5}{2}, \ldots, \frac{-n-m+2}{2}$ and $\frac{n+m+2}{2}, \frac{n+m+4}{2}, \ldots, \frac{n+2m-3}{2}$ respectively.

Subcase (ii) : m is even

Let us now designate the vertices $u_1, u_2, \ldots, u_{\frac{m-2}{2}}$ and $u_{\frac{m}{2}}, u_{\frac{m+2}{2}}, \ldots, u_{m-2}$ with labels $\frac{n+5}{2}, \frac{n+7}{2}, \ldots, \frac{n+m+1}{2}$ and $\frac{-n-m-1}{2}, \frac{-n-m-3}{2}, \ldots, \frac{-n-2m+3}{2}$ respectively. Designate the vertices $w_2, w_3, \ldots, w_{\frac{m}{2}}$ and $w_{\frac{m+2}{2}}, w_{\frac{m+4}{2}}$,
$\ldots, w_{m-2}$ with labels $\frac{-n-3}{2}, \frac{-n-5}{2}, \ldots, \frac{-n-m+1}{2}$ and $\frac{n+m+3}{2}, \frac{n+m+5}{2}, \ldots, \frac{n+2m-3}{2}$ respectively.

Case (iii) : $n \geq 6$ and n is even

We designate the vertices v_1, v_2, v_3, v_4 and $v_5, v_7, \ldots, v_{n-1}$ with labels $2, -1, 3, -2$ and $-3, -4, \ldots, \frac{-n}{2}$ respectively. Next, designate the vertices $v_6, v_8, \ldots, v_n$ with labels $4, 5, \ldots, \frac{n+2}{2}$ respectively. Fix the vertices u_{m-1}, w_1, w_{m-1} with labels $1, \frac{n+3}{2}, 1$.

Subcase (i) : m is odd

Next, designate the vertices $u_1, u_2, \ldots, u_{m-1}$ with labels $\frac{-n-2}{2}, \frac{-n-4}{2}, \ldots, \frac{-n-2m+2}{2}$ respectively. So, designate the vertices $w_1, w_2, \ldots, w_{\frac{m+1}{2}}$ and $w_{\frac{m+5}{2}}, w_{\frac{m+7}{2}}, \ldots, w_{m-1}$ with labels $\frac{n+4}{2}, \frac{n+6}{2}, \ldots, \frac{n+m+3}{2}$ and $\frac{n+m+5}{2}, \frac{n+m+7}{2}, \ldots, \frac{n+2m-2}{2}$ respectively. Fix the vertex $w_{\frac{m+3}{2}}$ with label 1.

Subcase (ii) : m is even

Now, designate the labels to the vertices u_i, $1 \leq i \leq m-1$ as in subcase (i) of case (iii). Thus, designate the vertices $w_1, w_2, \ldots, w_{\frac{m}{2}}$ and $w_{\frac{m+4}{2}}, w_{\frac{m+6}{2}}, \ldots, w_{m-1}$ with labels $\frac{n+4}{2}, \frac{n+6}{2}, \ldots, \frac{n+m+2}{2}$ and $\frac{n+m+4}{2}, \frac{n+m+6}{2}, \ldots, \frac{n+2m-2}{2}$ respectively. Fix the vertex $w_{\frac{m+2}{2}}$ with label 1. □

Table 3: PMC-labeling of the balloon of triangular snake $BT_{n,m}$, $n \geq 4$ & $m \geq 2$.

Value of n and m	$\mathbb{S}_{\Lambda_1^c}$	$\mathbb{S}_{\Lambda_1}$
n & m are odd	$\frac{n+3m-2}{2}$	$\frac{n+3m-4}{2}$
n is odd & m is even	$\frac{n+3m-3}{2}$	$\frac{n+3m-3}{2}$
n is even & m is odd	$\frac{n+3m-3}{2}$	$\frac{n+3m-3}{2}$
n & m are even	$\frac{n+3m-2}{2}$	$\frac{n+3m-4}{2}$

Example 4.4 *Figure* 7 *illustrates the PMC-labeling of the balloon of triangular snake* $BT_{9,5}$.

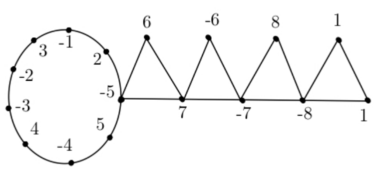

Figure 7: PMC-labeling of the balloon of triangular snake $BT_{9,5}$

Theorem 4.7 *The balloon of triangular snake* $BT_{3,m}$ *is a PMC-graph only for all* $m \geq 3$.

Proof: Consider the balloon of triangular snake $BT_{3,m}$, $m \geq 3$. Then, $BT_{3,m}$ has $3m$ edges and $2m+1$ vertices. We have consider two cases:

Case (i) : $m = 2$

Suppose that the balloon of triangular snake $BT_{3,m}$ is a PMC-graph. Hence, the maximum possible number of edges designated with a label 1 is 2. Subsequently, the minimum number of edges that are not designated with a label 1 is 4. Therefore, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \geq 2 > 1$, this is a contradiction.

Case (ii) : $m \geq 3$

We designate the vertices v_1, v_2, v_3 with labels $2, -1, 3$ respectively. Let $\Lambda(u_{m-1}) = 1$ and $\Lambda(v_{m-1}) = 1$. Consider the following two subcases arises

Subcase (i) : m is odd
Let us now designate the vertices $u_1, u_2, \ldots, u_{\frac{m-3}{2}}$ and $u_{\frac{m-1}{2}}, v_{\frac{m+1}{2}}, \ldots, u_{m-2}$ with labels $4, 5, \ldots, \frac{m+3}{2}$ and $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, -m$ respectively. Then designate the vertices $w_1, w_2, \ldots, w_{\frac{m-1}{2}}$ and $w_{\frac{m+1}{2}}, w_{\frac{m+3}{2}}, \ldots, w_{m-2}$ with labels $-2, -3, \ldots, \frac{-m-1}{2}$ and $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, m$ respectively.
Subcase (ii) : m is even
Also we designate the vertices $u_1, u_2, \ldots, u_{\frac{m-2}{2}}$ and $u_{\frac{m}{2}}, v_{\frac{m+2}{2}}, \ldots, u_{m-2}$ with labels $4, 5, \ldots, \frac{m+2}{2}$ and $\frac{-m-2}{2}, \frac{-m-4}{2}, \ldots, -m$ respectively. Moreover, designate the vertices $w_1, w_2, \ldots, w_{\frac{m}{2}}$ and $w_{\frac{m+2}{2}}, w_{\frac{m+4}{2}}, \ldots, w_{m-2}$ with labels $-2, -3, \ldots, \frac{-m-2}{2}$ and $\frac{m+4}{2}, \frac{m+6}{2}, \ldots, m$ respectively. □

Table 4: PMC-labeling of the balloon of triangular snake $BT_{3,m}$, $m \geq 2$.

Value of m	$\mathbb{S}_{\Lambda_1^c}$	$\mathbb{S}_{\Lambda_1}$
m is odd	$\frac{3m+1}{2}$	$\frac{3m-1}{2}$
m is even	$\frac{3m}{2}$	$\frac{3m}{2}$

Theorem 4.8 *The balloon of quadrilateral snake $BQ_{n,m}$ is a PMC-graph for all $n \geq 3$ and $m \geq 2$.*

Proof: Consider the balloon of quadrilateral snake $BQ_{n,m}$, $n \geq 3$ and $m \geq 2$. Let $V(BQ_{n,m}) = \{v_i, u_j, x_j, y_j \mid 1 \leq i \leq n \,\&\, 1 \leq j \leq m-1\}$ and $E(BQ_{n,m}) = \{v_iv_{i+1}, v_nv_1, v_nu_1, v_nx_1 \mid 1 \leq i \leq n-1\} \cup \{u_ju_{j+1}, u_jy_{j+1} \mid 1 \leq j \leq m-2\} \cup \{x_jy_j, y_ju_j \mid 1 \leq i \leq m-1\}$ denote, respectively, the vertex set and edge set of the balloon of quadrilateral snake $BQ_{n,m}$. Thus, $BQ_{n,m}$ has $n+4m-4$ edges and $n+3m-3$ vertices. We have consider four cases:
Case (i) : $n = 3$
We designate the vertices v_1, v_2, v_3 with labels $2, -1, 3$ respectively. Let $\Lambda(u_{m-1}) = 1$. Consider the following two subcases arises
Subcase (i) : m is odd
In this case, designate the vertices $u_1, u_3, \ldots, u_{m-2}$ and $u_2, u_4, \ldots, u_{m-3}$ with labels $\frac{n+5}{2}, \frac{n+11}{2}, \ldots, \frac{n+3m-4}{2}$ and $\frac{n+9}{2}, \frac{n+15}{2}, \ldots, \frac{n+3m-6}{2}$ respectively. Then, we designate the vertices $x_1, x_3, \ldots, x_{m-2}$ and $x_2, x_4, \ldots,$
x_{m-3} with labels $\frac{-n-1}{2}, \frac{-n-7}{2}, \ldots, \frac{-n-3m+8}{2}$ and $\frac{n+7}{2}, \frac{n+13}{2}, \ldots, \frac{n+3m-8}{2}$ respectively. Fix the vertex x_{m-1} with label $\frac{-n-3m+6}{2}$. Thereafter we designate the vertices $y_1, y_3, \ldots, y_{m-2}$ and $y_2, y_4, \ldots, y_{m-1}$ with labels $\frac{-n-3}{2}, \frac{-n-9}{2}, \ldots, \frac{-n-3m+6}{2}$ and $\frac{-n-5}{2}, \frac{-n-11}{2}, \ldots, \frac{-n-3m+4}{2}$ respectively.
Subcase (ii) : m is even
More over, designate the vertices $u_1, u_3, \ldots, u_{m-3}$ and $u_2, u_4, \ldots, u_{m-2}$ with labels $\frac{n+5}{2}, \frac{n+11}{2}, \ldots, \frac{n+3m-7}{2}$ and $\frac{n+9}{2}, \frac{n+15}{2}, \ldots, \frac{n+3m-3}{2}$ respectively. Also, we designate the vertices $x_1, x_3, \ldots, x_{m-1}$ and $x_2, x_4, \ldots, x_{m-2}$ with labels $\frac{-n-1}{2}, \frac{-n-7}{2}, \ldots, \frac{-n-3m+5}{2}$ and $\frac{n+7}{2}, \frac{n+13}{2}, \ldots, \frac{n+3m-5}{2}$ respectively. So designate the vertices $y_1, y_3, \ldots, y_{m-1}$ and $y_2, y_4, \ldots, y_{m-2}$ with labels $\frac{-n-3}{2}, \frac{-n-9}{2}, \ldots, \frac{-n-3m+3}{2}$ and $\frac{-n-5}{2}, \frac{-n-11}{2}, \ldots, \frac{-n-3m+7}{2}$ respectively.

Case (ii) : $n = 4$
Let us now designate the vertices v_1, v_2, v_3, v_4 with labels $2, -1, 3, -2$ respectively. Let $\Lambda(y_{m-1}) = 1$. Consider the following two subcases arises
Subcase (i) : m is odd
Let $\Lambda(u_1) = \frac{n+6}{2}$. We also designate the vertices $u_2, u_4, \ldots, u_{m-1}$ and $u_3, u_5, \ldots, u_{m-2}$ with labels $\frac{-n-6}{2}, \frac{-n-12}{2}, \ldots, \frac{-n-3m+3}{2}$ and $\frac{-n-8}{2}, \frac{-n-14}{2}, \ldots, \frac{-n-3m+7}{2}$ respectively. Then designate the vertices $x_1, x_3, \ldots, x_{m-2}$ and $x_2, x_4, \ldots, x_{m-1}$ with labels $\frac{n+4}{2}, \frac{n+10}{2}, \ldots,$
$\frac{n+3m-5}{2}$ and $\frac{-n-4}{2}, \frac{-n-10}{2}, \ldots, \frac{-n-3m+5}{2}$ respectively. Fix the vertex y_1 with label $\frac{-n-2}{2}$. Therefore, designate the vertices $y_2, y_4, \ldots, y_{m-3}$ and $y_3, y_5, \ldots, y_{m-2}$ with labels $\frac{n+8}{2}, \frac{n+14}{2}, \ldots, \frac{n+3m-7}{2}$ and $\frac{n+12}{2}, \frac{n+18}{2}, \ldots, \frac{n+3m-3}{2}$ respectively.
Subcase (ii) : m is even
More over, we designate the vertices $u_1, u_3, \ldots, u_{m-1}$ and $u_2, u_4, \ldots, u_{m-2}$ with labels $\frac{-n-2}{2}, \frac{-n-8}{2}, \ldots,$

$\frac{-n-3m+4}{2}$ and $\frac{-n-6}{2}, \frac{-n-12}{2}, \ldots, \frac{-n-3m+6}{2}$ respectively. Then, designate the vertices $x_1, x_3, \ldots, x_{m-3}$ and $x_2, x_4, \ldots, x_{m-2}$ with labels $\frac{n+4}{2}, \frac{n+10}{2}, \ldots, \frac{n+3m-8}{2}$ and $\frac{-n-4}{2}, \frac{-n-10}{2}, \ldots, \frac{-n-3m+8}{2}$ respectively. Fix the vertex x_{m-1} with label $\frac{n+3m-4}{2}$. Assign the vertices $y_1, y_3, \ldots, y_{m-3}$ and $y_2, y_4, \ldots, y_{m-2}$ with labels $\frac{n+6}{2}, \frac{n+12}{2}, \ldots, \frac{n+3m-6}{2}$ and $\frac{n+8}{2}, \frac{n+14}{2}, \ldots, \frac{n+3m-4}{2}$ respectively.

Case (iii) : $n \geq 5$ and n is odd
We designate the vertices v_1, v_2, v_3, v_4 with labels $2, -1, 3, -2$ respectively. Then, designate the vertices $v_5, v_7, \ldots, v_n$ and $v_6, v_8, \ldots, v_{n-1}$ with labels $-3, -4, \ldots, \frac{-n-1}{2}$ and $4, 5, \ldots, \frac{n+1}{2}$ respectively. Let $\Lambda(u_{m-1}) = 1$ and $\Lambda(x_1) = \frac{n+3}{2}$. Consider the following two subcases arises
Subcase (i) : m is odd
Furthermore, we designate the vertices $u_1, u_3, \ldots, u_{m-2}$ and $u_2, u_4, \ldots, u_{m-3}$ with labels $\frac{n+5}{2}, \frac{n+11}{2}, \ldots, \frac{n+3m-4}{2}$ and $\frac{n+9}{2}, \frac{n+15}{2}, \ldots, \frac{n+3m-6}{2}$ respectively. Hence designate the labels to the vertices x_j, $2 \leq j \leq m-1$ y_j, $1 \leq j \leq m-1$, as in case(i) of subcase (i).
Subcase (ii) : m is even
In this case, we designate the vertices $u_1, u_3, \ldots, u_{m-3}$ and $u_2, u_4, \ldots, u_{m-2}$ with labels $\frac{n+5}{2}, \frac{n+11}{2}, \ldots, \frac{n+3m-7}{2}$ and $\frac{n+9}{2}, \frac{n+15}{2}, \ldots, \frac{n+3m-3}{2}$ respectively. Hence designate the labels to the vertices x_j, $2 \leq j \leq m-1$ y_j, $1 \leq j \leq m-1$, as in case(i) of subcase (ii).

Case (iv) : $n \geq 5$ and n is even
Also, designate the vertices v_1, v_2, v_3, v_4 with labels $2, -1, 3, -2$ respectively. Then, designate the vertices $v_5, v_7, \ldots, v_{n-1}$ and $v_6, v_8, \ldots, v_{n-2}$ with labels $-3, -4, \ldots, \frac{-n}{2}$ and $4, 5, \ldots, \frac{n+2}{2}$ respectively. Let $\Lambda(u_{m-1}) = 1$. Consider the following two subcases arises
Subcase (i) : m is odd
More over, designate the vertices $u_1, u_3, \ldots, u_{m-2}$ and $u_2, u_4, \ldots, u_{m-3}$ with labels $\frac{n+6}{2}, \frac{n+12}{2}, \ldots, \frac{n+3m-3}{2}$ and $\frac{n+8}{2}, \frac{n+14}{2}, \ldots, \frac{n+3m-7}{2}$ respectively. Hence designate the labels to the vertices x_j, $1 \leq j \leq m-1$, as in case(ii) of subcase (i). Thus, designate the vertices $y_1, y_3, \ldots, y_{m-2}$ and $y_2, y_4, \ldots, y_{m-1}$ with labels $\frac{-n-2}{2}, \frac{-n-8}{2}, \ldots, \frac{-n-3m+7}{2}$ and $\frac{-n-6}{2}, \frac{-n-12}{2}, \ldots, \frac{-n-3m+3}{2}$ respectively.

Subcase (ii) : m is even
Then, designate the vertices $u_1, u_3, \ldots, u_{m-3}$ and $u_2, u_4, \ldots, u_{m-2}$ with labels $\frac{n+6}{2}, \frac{n+12}{2}, \ldots, \frac{n+3m-6}{2}$ and $\frac{n+8}{2}, \frac{n+14}{2}, \ldots, \frac{n+3m-4}{2}$ respectively. Hence, designate the labels to the vertices x_j, $1 \leq j \leq m-2$, as in case(ii) of subcase (ii). Fix the vertex x_{m-1} with label $\frac{-n-3m+6}{2}$. Therefore, designate the vertices $y_1, y_3, \ldots, y_{m-1}$ and $y_2, y_4, \ldots, y_{m-2}$ with labels $\frac{-n-2}{2}, \frac{-n-8}{2}, \ldots, \frac{-n-3m+4}{2}$ and $\frac{-n-6}{2}, \frac{-n-12}{2}, \ldots, \frac{-n-3m+6}{2}$ respectively. □

Example 4.5 *Figure* 8 *illustrates the PMC-labeling of the balloon of quadrilateral snake* $BQ_{6,4}$.

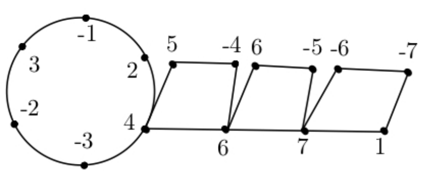

Figure 8: PMC-labeling of the balloon of balloon of quadrilateral snake $BQ_{6,4}$

5. Conclusion

In this research paper, the PMC-labeling behaviour of some graphs, like the pagoda graph, antiwebgear graph, alternate triangular cycle, alternate quadrilateral cycle, balloon of triangular snake and balloon of quadrilateral snake has been investigated. In the future, the determination of PMC-labeling of different graph families is an open problem.

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