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# Prime Submodules (Modules) relative to an Ideal

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ABSTRACT: In this study, we present the concepts of prime submodules relative to an ideal and prime modules relative to an ideal. Let $W$ be a unital left $R$-module.

A proper submodule $H$ of $W$ is called prime submodule relative to a proper ideal $I$ of $R$, if whenever $a \in R$, $x \in W$, $ax \in H$, then either $x \in H + IW$ or $a \in (H : W) + I$. $W$ is named a prime module relative to $I$ provided (0) is a prime submodule relative to $I$. Many properties of prime submodules (modules) relative to an ideal are given in this paper.

Key Words: prime submodule, prime submodule relative to an ideal, prime module relative to an ideal.

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## 1. Introduction

In this work, $R$ denotes a commutative ring with unity and $W$ is a left $R$-module. It is well known that the notion of primer ideals is generalized to prime submodules, also a module W is named prime module if the zero submodule is prime. They played an important rote to in ring theory and module theory. These concepts are introduced by J.Dauns, G.Desal and C.P.Lu,see [1,2,3].
Since the emergence of These concepts was studied by many reasearchers see [4,5]. Also many generalizations presented see [6]. Recently several authors gave numerous concepts related with that pervious notions [7]– [25].
Moreover as dual notions of prime submodules and prime modules considered by various authors, see [26,27,28,29]. For the sake of completeness, some notations will be listed in this paper: $A \leq W (A < W)$ stands for $A$ is a submodule of $W$ ($A$ is a proper submodule of $W$). For any two submodules $A$ and $B$ of $W$, and $J$ is an ideal of $R$ $(A :_R B) = \{r \in R : rB \subseteq A\}$ and $(A :_W J) = \{x \in W : Jx \subseteq A\}$, $J \leq$
As special case, $(0 :_R W)$ denoted by $ann_R W$, and $(A :_W a)$, $a \in R$ means $(A :_W Ra)$, i.e., $Ra = \langle a \rangle$. Also, $ann_W(a)$ means $(0 :_W Ra)$, $ann_R(x)$ means $(0 :_R Rx)$ where $x \in W$, and $Rx = \langle x \rangle$. Remember from [3,4] for $N < W$, $N$ is named prime submodule if $ax \in N$ with $a \in R$, $x \in W$, then either $x \in N$ or $a \in (N :_R W)$. A module $W$ is known as a prime module if (0) is a prime submodule of $W$.

In this study, the notions of prime submodule (prime module) relative to an ideal are introduced where if $W$ is an $R$-module, $H < W$, $I < R$, $H$ is named a prime submodule relative to $I$ if whenever $a \in R$, $x \in W$, $ax \in H$ then $x \in H + IW$ or $a \in (H :_R W)$. $W$ is termed a prime module relative to $I$ if (0) is a prime submodule relative to $I$. In section 2, many properties and characterizations of prime submodules relative to an ideal are introduced, see Theorem 2.3 and Proposition 2.4. The image and inverse image of prime submodules relative to an ideal are considered, see 2.8 and 2.9. Furthermore, the direct sum of prime submodules relative to an ideal is discussed in section 3, several properties of prime modules relative to an ideal are established.

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## 2. Prime Submodules relative to an Ideal

**Definition 2.1** *Let $W$ be an $R$-module, $H < W$, $I < R$. $H$ is designated a prime submodule relative to $I$ (shortly p. s. r. to $I$) if for each $a \in R$, $m \in W$, $am \in N$, then either $m \in H + IW$ or $a \in (H :_R W) + I$. An ideal $J$ of a ring is $(H : W) + I$ named a prime ideal relative to $I$ (briefly p. i. r. to $I$) if $J$ is a p.s.r. to $I$ of $R$ as $R$-module, that is $J$ is a p.i.r to $I$ if whenver $a, b \in R, ab \in I$ implies $a \in I + J$ or $b \in I + J$.*

**Remark 2.1**

1. *obviously every prime submodule is a p.s.r. to $I$ ,for any $I < R$.*
2. *The contrary of (1) may be not hold, for instance consider $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module, $H = \langle \overline{4} \rangle$, $I = 2\mathbb{Z}$, $H$ is not a prime submodule of $\mathbb{Z}_{12}$. Now, $H + I\mathbb{Z}_{12} = \langle \overline{2} \rangle$, $(H :_Z \mathbb{Z}_{12}) + I = 2\mathbb{Z}$.*

   *$\overline{4} = 2 \cdot \overline{2} \in H$ and $\overline{2} \in \langle \overline{2} \rangle$, $\overline{0} = 3 \cdot \overline{4} \in H$ and $\overline{4} \in \langle \overline{2} \rangle$, $\overline{0} = 4 \cdot \overline{3} \in H$ and $\overline{3} \notin \langle \overline{2} \rangle$, but $4 \in (H : \mathbb{Z}_{12}) + I = 2\mathbb{Z}$.*

   *$\overline{0} = 8 \cdot \overline{3} \in H$ and $\overline{3} \notin \langle \overline{2} \rangle$, but $\overline{8} \in 2\mathbb{Z}$, $\overline{8} = 4 \cdot 2 \in H$ and $\overline{2} \in \langle \overline{2} \rangle$, $\overline{8} = 2 \cdot \overline{4} \in H$ and $\overline{4} \in \langle \overline{2} \rangle$,*

   *$\overline{0} = 2 \cdot \overline{6} \in H$ and $\overline{6} \in \langle \overline{2} \rangle$, $\overline{0} = 6 \cdot \overline{2} \in H$ and $\overline{2} \in \langle \overline{2} \rangle$. Thus $H$ is a p.s.r to $I$.*
3. *If $I = (0)$, then a submodule $H$ is prime if and only if $H$ is p.s.r to $I$.*
4. *If $I, J < R$ with $J \subseteq I$ and $H < W$, then $H$ is a p.s.r to $J$ implies $H$ is a p.s.r to $I$.*
5. *The converse of part (4) may not be true, for instance: consider $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module, $H = \langle \overline{4} \rangle$, $I = 2\mathbb{Z}$, $J = 4\mathbb{Z}$, $H$ is a p.s.r to $I$ but it is not a p.s.r to $J$, since $\overline{4} = 2 \cdot \overline{2} \in H$, $\overline{2} \notin J\mathbb{Z}_{12} = \langle \overline{4} \rangle$ and $2 \notin (H :_Z \mathbb{Z}_{12}) + J = 4\mathbb{Z}$+$4\mathbb{Z}$ =$4\mathbb{Z}$.*
6. *Let $H < W$, $I < R$ with $I \subseteq (H :_R W)$ then $H$ is a prime submodule of $W$ is identical with $H$ is a p.s.r to $I$.*
7. *Consider the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$, for any $H < \mathbb{Z}_{p^\infty}$, $I$< $\mathbb{Z} \cdot (H :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) = 0$ and $I\mathbb{Z}_{p^\infty} = \mathbb{Z}_{p^\infty}$. Hence for any $am \in H$, $m \in H + I\mathbb{Z}_{p^\infty} = \mathbb{Z}_{p^\infty}$. So that $H$ is a p.s.r to any $0 \neq I < R$.*

**Theorem 2.1** *Suppose $W$ is an $R$-module, $H < W$ and $I < R$. then $H$ is a p.s.r to $I$ is selfsame ( tanlamount), for each $J < R$, $K < W$, $J \cdot K \leq H$ implies $K \leq H + IW$ or $J \subseteq (H : W) + I$.*

**Proof:** $\Rightarrow$ *Assume $JK \leq H$ and there exists $x \in K$ with $x \notin H + IW$. As $jx \in H$ for any $j \in J$, then $j \in (H : W) + I$ (because $H$ is a p.s.r to $I$), thus $J \subseteq (H : W) + I$*

$\Leftarrow$ *Let $am \in H$ where $a \in R$, $m \in W$. Then $\langle a \rangle \langle m \rangle \subseteq H$ and so by hypothesis, either $\langle m \rangle \subseteq H + IW$ or $\langle a \rangle \subseteq (H : W) + I$. Thus $m \in H + IW$ or $a \in (H : W) + I$.* □

The following result give more characterization for a p.s.r to $I$.

**Proposition 2.1** *Let $W$ be an $R$-module, $H < W$, and $I < R$. Accordingly the next asserations are selfsame tanlamount carhun:*

1. *$H$ is a p.s.r to $I$.*
2. *$(H :_W r) + IW = H + IW, \forall r \notin (H :_R W) + I$.*
3. *$(H :_R W) + I = (H :_R (c)) + I, \forall c \in w, c \notin H + IW$.*

**Proof:** **(1)** $\Rightarrow$ **(2)** *Let $m + x \in (H :_W r) + IW, r \notin (H :_R W) + I, m \in (H :_W r)$ and $x \in IW$. Hence $rm \in H$ and as $H$ is a p.s.r to $I$, either $m \in H + IW$ or $r \in (H :_R W) + I$. For that resason $m \in H + IW$ since $r \notin (H :_R W)$. Thus $(H :_W r) + IW \subseteq H + IW$. Let $h + x \in H + IW$, $h \in H$, $x \in IW$. Then $rh \in H$ and so $h \in (H :_W r)$. Thus $h + x \in (H :_W r) + IW$; that $H + IW \subseteq (H :_W r) + IW$.*

(2) ⇒ (1) *Let* $rm \in H$ *with* $m \in W$, $r \in R$ *such that* $r \notin (H :_R W) + I$. *Then* $m \in (H :_W r)$ *and so* $m + x \in (H :_W r) + IW$ *for any* $x \in IW$. *It follows that* $m + x \in H + IW$. *Which implies that* $m \in H + IW$.

(1) ⇒ (3) *Let* $r + i \in (H :_R (c)) + I$, *where* $r \in (H :_R (c))$, $i \in I$ *and* $c \notin H + IW$. *Then* $rc \in H$. *As* $H$ *is a s.p.r to* $I$, *then* $r \in (H :_R W) + I$. *Thus* $(H :_R (c)) + I \subseteq (H :_R W) + I$. *If* $r + i \in (H :_R W) + i$, *where* $r \in (H :_R W), i \in I$, *then* $rW \subseteq H$. *Since* $rc \in rW$ *for any* $c \in W$, *so that* $rc \in H$; *that is* $r \in (H :_R (c))$. *As a result* $r + i \subseteq (H :_R (c)) + I$. *Thus* $(H :_R W) + I \subseteq (H :_R (c)) + I$

(3) ⇒ (1) *Let* $rm \in H$, *where* $r \in R$, $m \in W$. *Suppose that* $m \notin H + IW$. *Then* $r \in (H :_R (m))$ *and so* $r \in (H :_R (m)) + I$. *By condition (3),* $r \in (H :_R W) + I$. □

**Theorem 2.2** *Let* $W$ *be a finitely generated **(f. g)** multiplication R-module and* $H < W$. *Then* $H$ *is a p.s.r to* $I$ *selfsame* $(H :_R W)$ *is a p.i.r to* $I$.

**Proof:** ⇒ *Assume* $ab \in (H :_R W)$, *where* $a$ *and* $b$ *in* $R$, *which ensure guarantee* $\langle a\rangle \ \langle b\rangle \ W \subseteq H$. *Since* $H$ *is a p.s.r to* $I$, *so either* $\langle b\rangle \ W \subseteq H + IW$, *or* $\langle a\rangle \subseteq (H : W) + I$, *by Theorem 2.3. If* $\langle a\rangle \subseteq (H : W) + I$, *then* $a \in (H :_R W) + I$, *and nothing to prove. If* $\langle b\rangle W \subseteq H + IW = (H : W)W + IW = ((H : W) + I) W$. *But* $W$ *is a multiplication f. g R-module, hence* $\langle b\rangle \subseteq ((H :_R W) + I) + ann_R W$ *by[14]. This implies* $\langle b\rangle \subseteq (H : W) + I$, *since* $annW \subseteq (H :_R W)$. *Thus* $b \in (H : W) + I$.

⇐ *To prove* $H$ *is a p.s.r to* $I$. *Let* $a \in R$, $m \in W$ *and* $am \in H$. *Hence* $\langle a\rangle \langle m\rangle \subseteq H$. *Since* $W$ *is a multiplication R-module there exists* $J < R$ *with* $\langle m\rangle = JW$, *also* $H = (H :_R W) W$. *Hence* $\langle a\rangle JW \subseteq (H : W)W$. *As* $W$ *is a f. g multiplication R-module, so by [31]* $\langle a\rangle J \subseteq (H :_R W) + annW$. *Hence* $\langle a\rangle J \subseteq (H :_R W)$ *since* $annW \subseteq (H :_R W)$. *Beside this* $(H :_R W)$ *is a p.i.r to* $I$ *by hypothesis, we conclude that either* $\langle a\rangle \subseteq (H :_R W) + I$ *or* $J \subseteq (H : W) + I$; *which means either* $a \in (H :_R W) + I$ *or* $\langle m\rangle = JW \subseteq (H : W)W + IW = H + IW$. *Thus either* $a \in (H :_R W) + I$ *or* $m \in H + IW$.

*As an application of Theorem 2.5, for the* $\mathbb{Z}$*-module* $\mathbb{Z}_n$ $(n > 1)$, *every submodule* $H$ *of* $\mathbb{Z}_n$, *and* $I < \mathbb{Z}$. $H$ *is a p.s.r to* $I$ *if and only if* $(H :_{\mathbb{Z}} Z_n)$ *is a p.i.r to* $I$. □

**Remark 2.2** *The condition* $W$ *is f. g multiplication R-module is necessary in Theorem 2.5.*

**Example 2.1** *Let* $W = \mathbb{Z} \oplus \mathbb{Z}_2$ *as* $Z$*-module,* $I = 6\mathbb{Z}$, $H = (0) + (\overline{0})$. *Then*

$$(H :_{\mathbb{Z}} W) = \left((0) + (\overline{0}) :_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z}_2\right) = (0).$$

*So* $(H : W)$ *is a p.i.r to* $I$. *Now,* $8\left(0, \overline{1}\right) = \left(0, \overline{0}\right) \in H$, *but* $\left(0, \overline{1}\right) \notin H + IW = 6\mathbb{Z} \oplus \left(\overline{0}\right)$, *also* $8 \notin (H :_{\mathbb{Z}} W) + I = (0) + 6\mathbb{Z}$. *Thus* $H$ *is not a p.s.r to* $I$.

**Proposition 2.2** *Let* $f : W \to L$ *be an R-epimorphism,* $H < W$ *and* $I < R$. *If* $A$ *is a p.s.r to* $I$ *and* $ker\ f \subseteq H$ *then* $f(H)$ *is a p.s.r to* $I$.

**Proof:** *Let* $a \in R$, $y \in L$ *with* $ay \in f(H)$. *Then* $ay = f(h)$ *for some* $h \in H$. *As* $f$ *is an epimorphism,* $y = f(x)$ *for some* $x \in W$. *Thus* $af(x) = f(h)$; *that* $ax - h \in ker\ f \subseteq H$, *so that* $ax \in H$. *But* $H$ *is a p.s.r to* $I$, *hence either* $x \in H + IW$ *or* $a \in (H : W) + I$. *Consequently* $f(x) \in f(H) + IL$ *or* $a \in (f(H) : L) + I$ *since* $((H :_R W) : (f(H) :_R f(W)) = (f(H) : L)$. *Thus* $f(H)$ *is a p.s.r to* $I$. □

**Proposition 2.3** *Let* $f : W \to L$ *be an R-epimorphism,* $I < R$. *If* $K$ *is a p.s.r to* $I$ *in* $L$ *and* $kerf \subseteq f^{-1}(K)$, *then* $f^{-1}(K)$ *is a p.s.r to* $I$ *in* $W$.

**Proof:** *Let* $a \in R$, $m \in W$ *and* $am \in f^{-1}(K)$. *Then* $af(m) \in K$. *Since* $K$ *is a p.s.r to* $I$ *in* $W$, *so either* $f(m) \in K + IL$ *or* $a \in (K : W) + I$. *If* $f(m) \in K + IL$. *Since* $f$ *is an epimorphism,* $K = ff^{-1}(K)$ *and* $L = f(W)$. *Thus* $f(m) \in ff^{-1}(K) + If(W)$; *that is* $f(m) \in f\left(f^{-1}(K) + IW\right)$ *and so* $f(m) = f(x)$ *for some* $x \in f^{-1}(K) + IW$ *and hence* $m - x \in kerf \subseteq f^{-1}(K) \subseteq f^{-1}(K) + IW$. *As consequently* $m \in f^{-1}(K) + IW = f^{-1}(K) + If^{-1}(L)$. *If* $a \in (K : W) + I$, *then* $a \in \left(f^{-1}(K) : f^{-1}(L)\right) + I$. *But* $(K :_R W) = \left(f^{-1}(K) :_R f^{-1}(L)\right)$ *(since* $f$ *is an epimorphism and* $kerf \subseteq f^{-1}(K)$ *thus,* $a \in (K : W) + I$ *and* $f^{-1}(K)$ *is a p.s.r to* $I$. □

**Theorem 2.3** *If $W$ is a f. g $R$-module, $S$ is a multiplicative closed subset of $R$ (m.c.s), $H < W$ and $I < R$. If $H$ is a p.s.r to $I$ in $W$, then $S^{-1}H$ is a p.s.r to $S^{-1}I$ in $S^{-1}R$-module $S^{-1}W$.*

**Proof:** *Let $\frac{a}{s} \in S^{-1}R$, $\frac{m}{t} \in S^{-1}W, \frac{a}{s}\frac{m}{t} \in S^{-1}H$. Then $\exists h \in H, t_1 \in S$ such that $\frac{am}{st} = \frac{h}{t_1}$ and so there exists $s_1 \in S$ and $at_1s_1m = ss_1th \in H$, i.e., $at_1s_1m \in H$. As $H$ is a p.s.r to $I$ in $W$, either $m \in H + IW$ or $at_1s_1 \in (H :_R W) + I$. As a deduction either $\frac{m}{t} \in S^{-1}(H + IW) = S^{-1}(H) + S^{-1}I \cdot S^{-1}W$ or $\frac{a}{s} = \frac{at_1s_1}{at_1s} \in S^{-1}[(H :_R W) + I] = S^{-1}(H :_R W) + S^{-1}I$, but $W$ is a f. g $R$-module implies $S^{-1}(H :_R W) = (S^{-1}(H) :_{S^{-1}R} S^{-1}W)$ so $\frac{a}{s} \in \left(S^{-1}H :_{S^{-1}R} S^{-1}W\right) + S^{-1}I$. Thus $S^{-1}H$ is a p.s.r to $S^{-1}I$ in $S^{-1}W$.* □

**Proposition 2.4** *Let $W_1$ and $W_2$ be $R$-module, $H < W_1$, $I < R$. Then $H$ is a p.s.r to $I$ if and only if $H \oplus W_2$ is a p.s.r to $I$ in $W_1 \oplus W_2$.*

**Proof:** $\Rightarrow$ *Let $a \in R$, $(m_1, m_2) \in W_1 \oplus W_2$ with $a\,(m_1, m_2) \in H \oplus W_2$. Then $am_1 \in H$ and $am_2 \in W_2$. Since $H$ is a p.s.r to $I$ either $m_1 \in H + IW_1$ or $a \in (H :_R W_1) + I$. It follows that $(m_1, m_2) \in (H + IW_1) \oplus (W_2 + IW_2)$ or $a \in (H \oplus W_2 :_R W_1 \oplus W_2)$. Thus $(m_1, m_2) \in (H \oplus W_2) + I\,(W_1 \oplus W_2)$. Therefore $H \oplus W_2$ is a p.s.r to $I$ in $W_1 \oplus W_2$.*
$\Leftarrow$ *Let $\rho : W_1 \oplus W_2 \to W_1$ be the natural projection. Then $\rho\,(H \oplus W_2) = H$ and so $H$ is a p.s.r to $I$ by Proposition 2.7.* □

**Proposition 2.5** *Let $N_1$ and $N_2$ be $R$-modules such that $ann_R N_1 + ann_R N_2 = R$ and $H$ is a p.s.r to $I < R$ in $N_1 \oplus N_2$. Then either:*

1. *$H = L \oplus K$, for some $L < N_1, K < N_2$ which are p.s.r to $I$ in $W_1, W_2$ respectively.*
2. *$H = L \oplus N_2$ , for some p.s.r to $L$ in $N_1$.*
3. *$H = N_1 \oplus K$, for some p.s.r to $K$ in $N_2$.*

**Proof:** *Since $ann_R N_1 + ann_R N_2 = R$, we conclude that there exist $L \leq N_1$ and $K \leq N_2$, with $H = L \oplus K$. But $H < N_1 \oplus N_2$ yeilds 3 cases.*

1. *$L < N_1$ and $K < N_2$,*
2. *$L < N_1$ and $K = N_2$,*
3. *$L = N_1$ and $K < N_2$.*

***Case 1.*** *$H = L \oplus K$, $L < N_1$ and $K < N_2$. First to prove $L$ is a p.s.r to $I$ in $W$, let $am \in L$, where $a \in R$ and $m \in W$. then $a(m, 0) \in L \oplus N_2 = H$. since $H$ is a p.s.r to $I$, hence either $(m, 0) \in (L \oplus K) + I\,(N_1 \oplus N_2)$ or $a \in (L \oplus K :_R N_1 \oplus N_2) + I$; that is either $(m, 0) \in (L + IN_1) \oplus (K + IN_2)$ or $a \in (\,(L :_R N_1) \cap (K :_R N_2)\,) + I$. Thus, either $m \in L + IW$ or $a \in (L :_R N_1) + I$ and so $L$ is a p.s.r to $I$ in $N_1$.*
*Similarly, one can show that $K$ is a p.s.r to $I$.*
***Case 2, and Case 3:*** *By Proposition 2.7, $L$ is a p.s.r to $I$ and $K$ is a p.s.r to $I$.*
*It is well known a ring $R$ is a Noetherian whenever every ideal is f. g.* □

From [32], $R$ is Noetherian ring if every prime ideal is f. g. We obtained that :

**Proposition 2.6** *Let $R$ be a ring, $I < R$. Then $R$ is Noetherian identical with every p.i. r to $I$ is f. g.*

**Proof:** $\Rightarrow$ *Since $R$ is Noetherian implies every ideal is f. g, hence every p.i.r to $I$.*
$\Leftarrow$ *Assume $P$ be a prime ideal. Then $P$ is a p.i.r to $I$ and so $P$ is f. g and so $R$ is Noetherian.* □

Remember that for a f. g module $W$, $W$ is Noetherian if and only if every prime submodule of $W$ is f. g [3].

**Proposition 2.7** *Let $W$ be a f. g module, $I < R, W$ is Noetherian identical with every p.s.r to $I$ be f. g.*

**Proof:** $\Rightarrow$ *It is clear, since very submodule of Noetherian module is f. g.*
$\Leftarrow$ *Since every prime submodule is a p.s.r to $I$, hence every prime is f. g. Thus $W$ is Noetherian.* □

## 3. Prime Modules relative to an Ideal

**Definition 3.1** *Let $W$ be an $R$-module, $I < R$. $W$ is nominated a prime module relative to $I$ (p.m.r to $I$) if the zero submodule is a p.s.r to $I$ in $W$.*
*If $(0)$ is a p.i.r to $I$ in $R$. A ring $R$ is known as a prime ring relative to $I$*

**Remark 3.1**

1. A module $W$ is a p.m.r to $I$ if whenever $a \in R$, $m \in W$, $am = 0$, then either $m \in IW$ or $a \in annW + I$
2. If $I = (0)$ then an $R$-module $W$ is a p.m.r to $I$ selfsame $W$ is a prime module.
3. $\mathbb{Z}_{\mathbf{12}}$ as $\mathbb{Z}$-module is not a p.m.r to $I = 4\mathbb{Z}$. since $0 = 2 \cdot \overline{6} \in \left(\overline{0}\right)$, $\overline{6} \notin (0) + I\mathbb{Z}_{12} = \langle\overline{4}\rangle$ and $2 \notin \left(\overline{0} :_Z \mathbb{Z}_{12}\right) + 4\mathbb{Z} = 4\mathbb{Z}$.
4. $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module is a p.m.r to $I = 2\mathbb{Z}$, since $\overline{0} = 3 \cdot \overline{4}$, $\overline{4} \in I\mathbb{Z}_{12} = \langle\overline{2}\rangle$. $\overline{0} = 4 \cdot \overline{3}$, $4 \in ann\mathbb{Z}_{12} + 2\mathbb{Z} = 2\mathbb{Z}$. $\overline{0} = 2 \cdot \overline{6}$, $\overline{6} \in I\mathbb{Z}_{12} = \langle\overline{2}\rangle$. $\overline{0} = 6 \cdot \overline{2}, 6 \in ann\mathbb{Z}_{12} + 2\mathbb{Z} = 2\mathbb{Z}$. $\overline{0} = 3 \cdot \overline{8}, \overline{8} \in I\mathbb{Z}_{12} = \langle\overline{2}\rangle$. $\overline{0} = 8 \cdot \overline{3}, 8 \in ann\mathbb{Z}_{12} + 2\mathbb{Z} = 2\mathbb{Z}$.
5. Every prime module is p.m.r to $I$ $(I < R)$, but not conversely, see part 4.

**Proposition 3.1** *For an $R$- module $W$, $K < W$, $J < R$ and $I < R$. $W$ is a p.m.r to $I$ commensurate with $JK = (0)$ designate $K \subseteq IW$ or $J \subseteq ann\ W + I$.*

**Proof:** *As a result directly by Theorem 2.3, by considering $H = (0)$.* □

**Proposition 3.2** *Let $W$ be an $R$-module, $I < R$. The next assertions are identical :*

1. *$W$ is a p.m.r to $I$.*
2. *$ann_W(r) \subseteq IW, \forall r \notin ann_R(m) + I$*

**Proof:** *It concludes by Proposition 2.4.* □

It is famous that W a prime module selfsame with $ann_R(m) = ann_RW, \forall m \in W - (0)$ if and only if $ann_RN = ann_RW, \forall 0 \neq N \leq M$. We have the following:

**Theorem 3.1** *Suppose $W$ is an $R$-module, $I < R$, the next assertions are identical:*

1. *$W$ is a p.m.r to $I$.*
2. *$ann_R(m) + I = ann_RW + I$, $\forall m \notin IW$.*
3. *$ann_RH + I = ann_RW + I, \forall H \nsubseteq IW$.*

**Proof:** $(\mathbf{1}) \Rightarrow (\mathbf{2})$ *It is clear that $ann(M) \supseteq ann(W)$, $\forall m \notin IW$. Hence $ann_R(m)+I \supseteq ann_RW+I, \forall m \notin IW$. Now, let $r + i \in ann_R(m) + I$ where $r \in ann_R(m), i \in I$, such that $rm = 0$. As $W$ is a p.m.r to $I$, and $m \notin IW$, $r \in ann_RW$ and hence $r + i \in ann\ W + I$.*
$(2) \Rightarrow (3)$ *Since $ann\ W \subseteq ann\ H$, $\forall H \leq W$ and $H \neq 0$. Then $ann\ W \subseteq ann\ H$, $\forall H \nsubseteq IW$ and clearly $ann\ W + I \subseteq ann\ H + I$. Let $a \in ann_RH + I$, so $a = r + i$ for some $r \in ann_RH$, $i \in I$. hence $rh = 0$ for each $h \in H$. As $H \nsubseteq IW, \exists h_1 \in H$ and $h_1 \notin IW$. It follows that $rh_1 = 0$ ans so $r + i \in ann_R(h_1) + I$. but by condition 2, $ann_R(h_1) + I = ann_RW + I$, hence $a = r + i \in ann\ W + I$. Thus $ann_RH + I \subseteq ann_RW + I$.*
$(3) \Rightarrow (1)$ *Let $r \in R$, $m \in W$, and $rm = 0$. Suppose that $m \notin IW$, hence $r \in ann_R(m) \subseteq ann(m)+I$. but by condition 3, $ann_R(m)+I = ann_R(W)+I$. Thus $r \in ann_RW+I$ and so $W$ is a p.m.r to $I$. Remember that when $W$ is a prime $R$-module, yields $ann_RH$ is a prime ideal in $R$, with a certain condition $I$, we prove analogous result for p.m.r. to $I$.* □

**Proposition 3.3** *Let $W$ be an $R$-module, $I$ a maximal ideal in $R$, $H \leq W$ and $H \nleq IW \neq W$ If $W$ is a p.m.r to $I$, then $ann_R H$ is a p.i.r to $I$.*

**Proof:** *Let $a, b \in R$ with $ab \in ann_R H, H \nleq IW$. Then $abh = 0$, for any $h \notin IW$. As $W$ is a p.m.r to $I$, either $bh \in IW$ or $a \in ann\ W + I$. if $a \in ann_R W + I$, then there is nothing to prove. If $bh \in IW$, but $I$ is a maxiimal ideal and $IW \neq W$ indicates $IW$ is a prime submodule [5] So that either $h \in IW$ or $b \in (IW :_R W)$. Thus $b \in (IW :_R W)$ since $h \notin IW$. On the other hand $(IW :_R W) = I$, hence $b \in I$ which implies $b \in ann_R H + I$. therefore $ann_R H$ is a p.s.r to $I$.* □

**Proposition 3.4** *Suppose that $M$ is a f. g $R$-module, $S$ is a multiplicative closed subset of $R$ and $I < R$. if $M$ is a p.m.r to $I$, implies the $S^{-1}R$ module $S^{-1}M$ is a p.m.r to $S^{-1}I$.*

**Proof:** *It follows by Proposition 2.9, by taking $H = (0)$.* □

## 4. Conclusion

Most properties of prime submodules and prime modules generalized to prime submodules relative to an ideal, and prime modules relative to an ideal. However, some properties verified with extra condition.


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