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Analysis of bifurcation solutions to nonlinear wave equations with Corner singularities

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ABSTRACT: We investigate bifurcation solutions of a wave ODE with new nonlinear parts that involves elastic beams on elastic bases with the new nonlinear part: $-z^2\left(zz'' + \frac{3}{2}(z')^2\right)$ by employing the local LyapunovSchmidt method. The key function that corresponds to the functional of the ODE is identified. Subsequently, we demonstrate that the key function is equivalent to a smooth function of fifth degree. This study focuses on the singularities at the corners of the fifth-degree function to conduct a bifurcation analysis of ordinary differential equation solutions, employing real analysis, functional analysis, and catastrophe theory. This study aims to establish the parametric equation of the bifurcation set (caustic) and provide a geometric interpretation, along with an analysis of the bifurcation propagation of critical points (singularities).

Key Words: Corner singularities, Lyapunov-Schmidt approach, Bifurcation solutions, Caustic.

Contents

1. Introduction

The Lyapunov-Schmidt (LS) method is used to research nonlinear problem solutions that come up in physics, geometry, economics, and mathematics when the solutions do not satisfy theorem of implicit function. The method (LS) converts nonlinear equations in the infinite dimension of the type;

$$g(x, \lambda) = b, x \in O, b \in Y, \lambda \in \mathbb{R}^n. \tag{1.1}$$

to equations of the type

$$\Theta(\xi, \lambda) = \eta, \xi \in \mathcal{A}, \eta \in \mathcal{B}, \tag{1.2}$$

in the finite dimension, so that g is an index-zero smooth Fredholm map, the spaces of Banach are $X, Y, O \subseteq X$ is open in which $\mathcal{A}$ and $\mathcal{B}$ are finite dimensions smooth manifolds [15,21]. All of the analytical and topological properties of equation (1.1), such as Multiplicity and bifurcation diagrams are presented in equation (1.2). (See [7,9,12,16,19,23]).

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The following ODE can be used to describe the elastic beams' wave motion and oscillations on elastic foundations;

$$\delta\frac{d^4z}{dx^4}+\alpha\frac{d^2z}{dx^2}+\beta z+G(\tilde{z},\lambda)=\psi. \tag{1.3}$$
$$\tilde{z}=(z,z',z'',z''',z'''')$$
$$\lambda=(\delta,\alpha,\beta),$$

In this context, z represents the deflection of the beam. The relationship is defined as $\delta=1+\sigma$, in which σ is a small parameter and $\delta,\alpha,\beta\in\mathbb{R}$ are parameters. Additionally, ψ is expressed as $\tilde{\varepsilon}\varphi(x)$ and $\varphi(x)$ is a continuous function. Furthermore, $G(\tilde{z},\lambda)$ denotes a generic nonlinearity. Equation (1.3) has been analyzed by numerous authors [8,13,18,22,24]. For instance, when $G(\tilde{z},\lambda)=-kz^3$, (in which k is a parameter) [13], with $\sigma=0$ and $\psi=0$, this equation has been examined by Thompson and Stewart (1986) [24], as well as Furta and Piccione (2002) [8]. In the case in which $G(\tilde{z},\lambda)$ is defined as z^2+z^3, the subsequent boundary conditions have been implemented for the analysis of equation (1.3).
$z(0)=z(\pi)=z''(0)=z''(\pi)=0$, and with small perturbation by Shanan A. K. (2010) [22], such that he used the variational method of Lyapunov -Schmidt.

The research of smooth map singularities is necessary for the investigation of the bifurcation solutions of BVPs [10]. The Sapronov group's early interest was piqued by the research of smooth map singularities and its applications to BVP in numerous of their investigations., for example, in (2001) of this site [6] Danilova applied the boundaries singularities' method to study boundaries singularities of the function;

$$\widetilde{W}(\eta,\gamma)=\eta_1^4+\eta_2^4+a\eta_1^2\eta_2^2+\frac{1}{2}\left(\beta_1(\lambda)\eta_1^2+\beta_2(\lambda)\eta_2^2\right)+o|\eta|^4+O(\lambda)\left(O|\eta|^4\right)$$
$$\eta=(\eta_1,\eta_2),\gamma=(\beta_1(\lambda),\beta_2(\lambda)),\eta_1\geq 0$$

and took into consideration the functional:

$$V(u,\lambda)=\int_0^\pi\left(\frac{(u'')^2}{2}-\alpha\frac{(u')^2}{2}+\beta\frac{u^2}{2}+\frac{u^4}{4}\right)dx$$

with the additional condition,

$$\langle u,\omega\rangle=\int_0^\pi u(x)\omega(x)dx\geq 0$$

as an implementation of her outcomes where $\lambda\in\mathbb{R}$ is a parameter..

Recently, comparable works to those mentioned above have been published [14,25] for example, in [14] Kadhim and Abdul Hussain (2020) have studied the border singularities of the following function,

$$W(s,\rho)=\frac{\eta_1^3}{3}+\frac{\eta_2^3}{3}+\eta_2\eta_1^2+\eta_2^2\eta_1+\lambda_1\eta_2\eta_1+\lambda_2\left(\eta_1^2+\eta_2^2\right)$$

in which $\eta=(\eta_1,\eta_2),\eta_1,\eta_2\geq 0,\rho=(\lambda_1,\lambda_2)$ and consider the functional,

$$V(z,\lambda)=\frac{1}{2}\int_0^1\left(-\alpha\left(z'\right)^2+\beta z^2+z^3+z\left(z'\right)^2\right)dx$$

with the additional conditions, $\langle e_1,z\rangle+a\langle e_2,z\rangle\geq 0$ and $\langle e_1,z\rangle-a\langle e_2,z\rangle\geq 0$,$\lambda_1,\lambda_2\in\mathbb{R}$ are parameters.

The method (LS) assumes that a Fredholm map $g:\Omega\subset E\to F$ is smooth and nonlinear. When $V:\Omega\subset E\to\mathbb{R}$ exists as a functional such that $\frac{\partial V}{\partial x}(x,\lambda)h=\langle g(x,\lambda),h\rangle_H$, for all $x\in\Omega,h\in E$ in which $E\subset F\subset H$, then the map g has a variational property. The own critical points of functional

$V(x,\lambda)$ are the solutions to the equation $g(x,\lambda)=0$. The problem $V(x,\lambda)\to$ extr, $x\in E, \lambda\in R^n$ can be reduced by the method (LS) to an equivalent problem $W(\xi,\lambda)\to$ extr, $\xi\in R^n$, in which $W(\xi,\lambda)$ is designated as a key function. The function W exhibits identical topological and analytical characteristics to the functional V , including aspects such as multiplicity and the bifurcation diagram, among others [19].

The process is similar to analyzing key function W in relation to bifurcating solutions for research functional V and its corresponding bifurcating solutions.

2. Methodology of the Study

In this work, we study Equation (1.3) with $\sigma=0,\psi=0$ and $G(\tilde{z},\lambda)=-z^2\left(zz''+\frac{3}{2}(z')^2\right)$. So, we interest investigating the following type of Equation (1.3):

$$z^{(4)}+\alpha z''+\beta z-z^2\left(zz''+\frac{3}{2}(z')^2\right)=0$$

with the following boundary circumstances:
$z(0)=z(1)=z''(0)=z''(1)=0$, in which $z=z(x), x\in[0,1]$ and $\alpha,\beta\in\mathbb{R}$ are parameters.

We define $g:E\to F$ as a zero-index nonlinear Fredholm's operator, in which the space of all continuous functions with derivatives of order no more than four is represented by the formula $E=C^4([0,1],\mathbb{R})$, and $F=C^0([0,1],\mathbb{R})$ is represented as the space of all continuous functions, while the operator equation g is defined by,

$$g(z,\lambda)=z^{(4)}+\alpha z''+\beta z-z^2\left(zz''+\frac{3}{2}(z')^2\right)=0. \tag{2.1}$$

Also, we consider the functional,

$$V(z,\lambda)=\frac{1}{2}\int_0^1\left((z'')^2-\alpha(z')^2+\beta z^2+z^3(z')^2\right)dx \tag{2.2}$$

in which $z=z(x),\lambda=(\alpha,\beta)$ and $\alpha,\beta\in\mathbb{R}$ are parameters. The functional (2.2) has the variational property for the equation (2.1).
It has been demonstrated that the key function matching functional (2.2) is the same as the following real smooth function,

$$W(y,\lambda)=\frac{1}{5}t_1^5+t_1t_2^4+t_1^3t_2^2+\beta_1t_1^2+\beta_2t_2^2 \tag{2.3}$$
$$,y=(t_1,t_2),\lambda=(\beta_1,\beta_2)$$

and $\beta_1,\beta_2\in\mathbb{R}$ are parameters. Hence, we study the corner singularities of the function (2.3). Finding the bifurcating solution areas of equation is the aim of our problem, in which every equation (2.1)'s bifurcating solution equals a functional (2.2) critical point, and every functional (2.2) critical point correlates with a critical point of the functional (2.2) key function [7]. Therefore, we show that the key function corresponding to the functional (2.2) is equivalent to the function (2.3). That is, analyzing the bifurcating solutions of equation (2.1) corresponds to analyzing the bifurcating solutions of function (2.3). Hence, we have interested studying the bifurcating solutions of the function (2.3).

3. The key function

In this part, we discover the normal form of the key function that corresponds to the functional (2.2), in which this form is equivalent to the function (2.3).

Theorem 3.1 *The key function's normal form W_1, which corresponds to the functional (2.2), is*

$$W_1(y,\lambda)=\frac{1}{5}t_1^5+t_1t_2^4+t_1^3t_2^2+\beta_1t_1^2+\beta_2t_2^2$$

in which $y=(t_1,t_2)\,,\lambda=(\beta_1,\beta_2)$ and $\beta_1,\beta_2\in\mathbb{R}$ are parameters.

Proof: Considering the Lyapunov-Schmidt scheme, the linearized formula that corresponds to equation (2.1) at the point $(0,\lambda)$ has the structure that is described below:

$$Dk=0,\quad k\in E$$
$$k(0)=k(1)=k''(0)=k''(1),$$

in which, $D=\frac{d^4}{dx^4}+\alpha\frac{d^2}{dx^2}+\beta$.

The solution to the linearized formula that satisfies the boundary conditions is expressed as: $e_q(x)=c_q\sin(q\pi x)$, in which $q=1,2,\ldots$ The corresponding standard formula associated with this solution is $(q\pi)^4-\ \alpha(q\pi)^2+\beta=0$. This formula provides the standard lines ℓ_q in a 2-dimensional space. The coordinates (α,β) define the standard lines ℓ_q, ensuring that the linearized formula yields non-zero solutions [20]. The bifurcation point represents the intersection of the standard lines within the $\alpha\beta$-plane. Therefore, the point $(\alpha,\beta)=(0,0)$ represents the bifurcation point of equation (2.1). Localized parameters α and β are defined as follows: $\alpha=0+\ \delta_1,\beta=0+\delta_2$, Parameters δ_1, and δ_2 are small and result in bifurcation along the modes $e_1(x)=c_1\sin(\pi x)$, and $e_2(x)=c_2\sin(2\pi x)$. Given that $\|e_1\|=\|e_2\|=1$, it follows that $c_1=c_2=\sqrt{2}$.

Let N = Ker(D) = span $\{e_1,e_2\}$, The space E can be expressed as a direct sum of two subspaces: N and the orthogonal complement of N .

$$E=N\oplus N^{\perp},N^{\perp}=\left\{v\in E:\int_0^1 ve_kdx=0,k=1,2\right\}$$

Two subspaces, N and N 's orthogonal complement, can be directly added to form the space F.

$$F=N\oplus\widetilde{N}^{\perp},\widetilde{N}^{\perp}=\left\{\omega\in F:\int_0^1\omega e_kdx=0,k=1,2\right\}.$$

There exist two projections $P:E\to N$ and $I-P:E\to N^{\perp}$ such that $Pm=\omega$ and $(I-P)m=v$, (I is the identity's operator). So, it is possible to express every vector $m\in E$ as $m=\omega+v$, in which $\omega=t_1e_1+t_2e_2\in N,v\in N^{\perp}$,
$t_i=\langle m,e_i\rangle$. A smooth map $\Theta:N\to N^{\perp}$ exists, according to the implicit function theorem.
such that
$\widetilde{W}(\zeta,\eta)=V(\Theta(\omega,\eta),\eta),\zeta=(t_1,t_2)\,,\eta=(\delta_1,\delta_2)$,
after which the key function $\widetilde{W}$ can be expressed as follows:

$$\begin{aligned}\widetilde{W}(\zeta,\eta)&=V\left(t_1e_1+t_2e_2+\Theta\left(t_1e_1+t_2e_2,\eta\right),\eta\right)\\&=W_2(\zeta,\eta)+o\left(|\zeta|^5\right)+O\left(|\zeta|^5\right)O(\eta),\end{aligned}$$

in which,

$$V(\omega,\eta)=\frac{1}{2}\int_0^1\left(\left(\omega''\right)^2-\alpha\left(\omega'\right)^2+\beta\omega^2+\omega^3\left(\omega'\right)^2\right)dx$$

hence,

$$\begin{aligned}W_2(\zeta,\eta)=&\frac{4\pi\sqrt{2}{t_1}^5}{15}+\frac{1088\pi\sqrt{2}t_1{t_2}^4}{315}+\frac{352\pi\sqrt{2}{t_1}^3{t_2}^2}{105}\\&+\left(1/2\pi^4-1/2\pi^2\alpha+\beta/2\right)t_1^2+\left(8\pi^4-2\pi^2\alpha+\beta/2\right){t_2}^2\end{aligned}$$

The first asymptotic of branches of bifurcating critical points and their geometrical form for the function $\widetilde{W}$ are completely determined by its principal part W_2. If, we replace t_1 by $\sqrt[5]{\frac{3}{4\pi\sqrt{2}}}t_1$ and t_2 by $\sqrt[4]{\frac{315\sqrt[5]{\frac{4\pi\sqrt{2}}{3}}}{1088\pi\sqrt{2}}}t_2$ in the function W_2, then since W_1 and W_2 in this instance have a similar germ $W_0 = \frac{t_1^5}{5} + t_1{t_2}^4$ (that is, the same principal portion), and the deformation, so they are contact equivalence.

As a result, the function W_2 's caustic and the function W_1 's caustic are identical [17]. Therefore, the function W_1 possesses every topological and analytical characteristic of the functional (2.2). Also, analyzing the bifurcation of equation (2.1) is equivalent to analyzing the bifurcation of the function W_1. This indicates that there are similarities between examining the bifurcating solutions of function (2.3) and the bifurcating solutions of equation (2.1). □

4. Fredholm functional's corner singularities

To study how a Fredholm functional behaves in a corner singular point's neighborhood, the reducing to an equivalent extremes problem is employed:

$$W(x) \rightarrow \text{ extr}$$

such that, $x \in D, D = \left\{x = (x_1, x_2)^\top \in \mathbb{R}^2 : x_2 \geq 0\right\}$.

A point a in D of the smooth function W is referred to as conditionally critical in $\mathbb{R}^2$, if gradW (a) (grad denotes gradient of W) is perpendicular to D's least face that contains a. The quotient algebra's dimension is called multiplicity of the conditionally critical point a (and is denoted by $\bar{\mu}$), in which the quotient algebra denotes by, $\bar{Q} = \frac{\Gamma_a(\mathbb{R}^2)}{I}$, such that $\Gamma_a\left(\mathbb{R}^2\right)$ is the ring containing the smooth function germs on $\mathbb{R}^2$ at point a and $I = \left(\frac{\partial W}{\partial x_1}, x_2\frac{\partial W}{\partial x_2}\right)$ which in $\Gamma_a\left(\mathbb{R}^2\right)$ is the corner Jacobi ideal. A conditionally critical point's multiplicity $\bar{\mu}$ is equal to the total of multiplicities $\mu + \mu_0$, in which μ represents W 's (ordinary) multiplicity on $\mathbb{R}^2$, and μ_0 denotes the restriction $W \mid \partial D$'s (ordinary) multiplicity (with ∂D denotes the set D 's boundary).

The row (r_0, r_1, r_2) represents the propagation of bifurcating extremes (bifurcation propagation) if the critical point is ordinary, wherein r_i is the Morse index i 's number of critical points. In the event that a corner (or boundary) critical point is under consideration, the bifurcation propagation can be represented by the next matrix of order 2×3 :

$$\begin{pmatrix} {r_0}^1 & {r_1}^1 & r_2^1 \\ r_0 & r_1 & r_2 \end{pmatrix}.$$

Here ${r_i}^j$ is the number of the corner critical points of index i (for $j = 1$), while r_i is the number of usual (situated inside D) critical points of index $i (i = 0, 1, 2)$ [10].

5. Singularities of the Codimension 15 function

The function (2.3), which was defined in the previous section, is taken into consideration in this section. Due to the fact that the function (2.3) has a codimension of fifteen at the origin, it possesses a multiplicity of sixteen. The primary objectives are to first ascertain the geometrical description of the caustic of the function (2.3), which is represented by a bifurcation diagram, and secondly to ascertain the spreading of the critical spots included within this function. When studying the function (2.3), it is important to avoid encountering some difficulties. We make the following assumptions, $t_1 = t, t_2^2 = u$. Therefore, studying of the function (2.3) is equivalent to the studying of the following function:

$$W(s,\lambda) = \frac{t^5}{5} + tu^2 + t^3u + \beta_1 t^2 + \beta_2 u \tag{5.1}$$
$$s = (t,u), \lambda = (\beta_1, \beta_2) \text{ and } u \geq 0 \ni \beta_1, \beta_2 \in \mathbb{R} \text{ are parameters.}$$

Since, the germ (the principal part) of the function (5.1) is: $W_0 = \frac{t^5}{5} + tu^2$, so, from the fourth section we have, $I = \left(\frac{\partial W_0}{\partial t}, u\frac{\partial W_0}{\partial u}\right) = \left(t^4 + u^2, 2tu^2\right) = \left(t^4 + u^2, tu^2\right)$, and $\bar{\mu} = 10$ in which $\mu = 6, \mu_0 = 4$. Given that multiplicity $\bar{\mu}$ equals the total number of critical points [5], the function (5.1) has ten critical points, four of which are on the boundary $u = 0$ and six of which are in the interior. Thus, the union of three sets, $\Xi^{\text{int}}_{1,0}, \Xi^{\text{ext}}_{1,0}$, and $\Xi_{1,1}$ constitutes the caustic of function (5.1), in which, respectively, $\Xi^{int}_{1,0}$ and $\Xi^{ext}_{1,0}$ are Caustic components represent the boundary and normal singularity degeneration, while $\Xi_{1,1}$ represents the non-border degeneration of the inner critical points. Both of these components are referred to as negative components.

5.1. Internal degradation approaching $\mathbf{u} = 0$

Lemma 5.1 *The set $\Xi^{\text{int}}_{1,0}$ is described by the parametric formula that has the following structure: $\beta_1 = 0$.*

Proof: All points $(t, 0, \beta_1, \beta_2)$ that satisfy the following relations are represented by the set $\Xi^{\text{int}}_{1,0}$: $\frac{\partial W(t,0,\beta_1,\beta_2)}{\partial t} = \frac{\partial^2 W(t,0,\beta_1,\beta_2)}{\partial t^2} = 0$ where $\beta_1, \beta_2 \in \mathbb{R}$ are parameters. From these relations, we have: $t^4 + 2t\beta_1 = 4t^3 + 2\beta_1 = 0$. We can rewrite the prior relations as the system:

$$t^4 + 2t\beta_1 = 0 \tag{5.2a}$$
$$4t^3 + 2\beta_1 = 0. \tag{5.2b}$$

From (5.2a), we get, $t = 0$ or $t^3 + 2\beta_1 = 0$. Put $t = 0$ in (5.2b) to get $\beta_1 = 0$. Let, $t^3 + 2\beta_1 = 0, t \neq 0$ and solving for t (when t is a real number), we get $t = \sqrt[3]{-2\beta_1}$. So, t $= \sqrt[3]{-2\beta_1}$ and $t \neq 0$. Substituting the value of t in (5.2b), we obtain: $\beta_1 = 0$, hence $t = \sqrt[3]{-2\beta_1} = 0$ this is contradiction. From this proof we get the wanted result. □

5.2. Degeneration along the normal of the boundary $u = 0$ (external degeneration)

Lemma 5.2 *The set $\Xi^{ext}_{1,0}$ is described by the parametric formula that has the following structure:*

$$\beta_1\beta_2^2(-2\beta_1 + \beta_2) = 0, \textit{ where } \beta_1, \beta_2 \in \mathbb{R} \textit{ are parameters.}$$

Proof: All points $(t, 0, \beta_1, \beta_2)$ that fulfill the following relations are represented by the set $\Xi^{ext}_{1,0}$: $\frac{\partial W(t,0,\beta_1,\beta_2)}{\partial t} = \frac{\partial W(t,0,\beta_1,\beta_2)}{\partial u} = 0$, this implies $t^4 + 2t\beta_1 = t^3 + \beta_2 = 0$. These the relations are equivalent to the following equations system:

$$t^4 + 2t\beta_1 = 0 \tag{5.3a}$$
$$t^3 + \beta_2 = 0 \tag{5.3b}$$

From (5.3a), we get, $t = 0$ or $t^3 + 2\beta_1 = 0$. If $t = 0, t^3 + 2\beta_1 \neq 0$, then, put $t = 0$ in (5.3b) to get,

$$\beta_2 = 0 \tag{5.4}$$

If, $t = t^3 + 2\beta_1 = 0$, from this and (5.3b) we have,

$$\beta_1\beta_2 = 0 \tag{5.5}$$

If $t^3 + 2\beta_1 = 0, t \neq 0$, and solving for t (when t is a real number), we get t $= \sqrt[3]{-2\beta_1}$ when $t \neq 0$. Substituting the value of t in (5.3b), we obtain:

$$-2\beta_1 + \beta_2 = 0 \tag{5.6}$$

Therefore, from the equations (5.4),(5.5) and (5.6) we have: $\beta_1 {\beta_2}^2 (-2\beta_1 + \beta_2) = 0$. □

5.3. Interior (non-boundary) Degeneration

Lemma 5.3 *A parametric equation that denotes the family $\Xi_{1,1}$ is as follows: $\beta_2^2 = 0$.*

Proof: The family $\Xi_{1,1}$ consists of all parameters β_1, β_2 that satisfy the following relations:

$$\frac{\partial W(t,u,\beta_1,\beta_2)}{\partial t} = \frac{\partial W(t,u,\beta_1,\beta_2)}{\partial u} = 0, u > 0 \text{ where } \beta_1, \beta_2 \in \mathbb{R} \text{ are parameters.}$$

These relations imply:

$$t^4 + 3t^2 u + 2t\beta_1 + u^2 = 0 \tag{5.7a}$$

$$t^3 + 2tu + \beta_2 = 0 \tag{5.7b}$$

To get degenerate critical points, the following is the result that we obtained by setting the determinate of the Hessian matrix of the function (5.1) to zero:

$$-t^4 + 4t\beta_1 - 4u^2 = 0 \tag{5.7c}$$

Adding the equation (5.7a) into the equation (5.7c), one gets: $3t^2u + 6t\beta_1 - 3u^2 = 0$, and solving this equation for u, one obtains, $u = \frac{1}{2}\left(t^2 \pm \sqrt{t^4 + 8t\beta_1}\right)$. We have two cases for u :

Case 1: if $u = \frac{1}{2}\left(t^2 + \sqrt{t^4 + 8t\beta_1}\right)$, then, put the value of u in the equation (5.7c) with solving for t (as t is a real variable), we have, $t = 0$. Putting $t = 0$ in the equation (5.7b), we find,

$$\beta_2 = 0 \tag{5.8}$$

Case 2: if $u = \frac{1}{2}\left(t^2 - \sqrt{t^4 + 8t\beta_1}\right)$, then, similarly with Case 1 , we have,

$$\beta_2 = 0 \tag{5.9}$$

From the equations (5.8) and (5.9) we have: ${\beta_2}^2 = 0$.

□

Theorem 5.1 *What follows is the parametric equation of the Caustic set of bifurcations for the function* (5.1).

$$\beta_1^2 \beta_2^2 (-2\beta_1 + \beta_2) = 0 \text{ where } \beta_1, \beta_2 \in \mathbb{R} \text{ are parameters .}$$

Proof: Given that the function (5.1)'s caustic is the union of the three sets shown below:

$$\Xi = \Xi_{1,0}^{\text{int}} \bigcup \Xi_{1,0}^{ext} \bigcup \Xi_{1,1}$$

Therefore, by multiplying all of the formulae for the caustic components' left sides together and putting them equal to zero, the parametric equation for the caustic will be constructed. We know that the caustic constituent equations have been discovered in Lemma 5.1, Lemma 5.2, and Lemma 5.3, consequently the following formula, $\beta_1^2\beta_2^2(-2\beta_1 + \beta_2) = 0$, gives a parametric method for the function's bifurcating set (caustic) (5.1). □

Proposition 5.1 *There are three real non-degenerate critical points (one inner point and two border points) in function (5.1) if either the condition $\beta_1 < 0$ and ($\beta_2 < 2\beta_1$ or $2\beta_1 < \beta_2 < 0$ or $\beta_2 > 0$)) or $\beta_1 > 0$ is met where $\beta_1, \beta_2 \in \mathbb{R}$ are parameters.*

Proof: The following equations system represents the function's critical points (5.1):

$$t^4 + 3t^2u + 2t\beta_1 + u^2 = 0 \tag{5.10a}$$

$$t^3 + 2tu + \beta_2 = 0 \tag{5.10b}$$

Multiplying the equation (5.10b) by $-\frac{3t}{2}$ and adding the result to the equation (5.10a) we have $-1/2t^4 + (2\beta_1 - 3/2\beta_2)\, t + u^2 = 0$, then we solve this equation for u to get $u = \pm\frac{1}{2}\sqrt{2t^4 - 8t\beta_1 + 6t\beta_2}$. Since $u > 0$, so $u = \frac{1}{2}\sqrt{2t^4 - 8t\beta_1 + 6t\beta_2}$. Additionally, by changing the value of u in the equation (5.10b) we have $t^3 + t\sqrt{2t^4 - 8t\beta_1 + 6t\beta_2} + \beta_2 = 0$, then we solve this equation for t to obtain:

$$t = \left(4\beta_1 - 2\beta_2 \pm \sqrt{(16\beta_1^2 - 16\beta_1\beta_2 + 5\beta_2^2)}\right)^{\frac{1}{3}} \tag{5.11a}$$

In order get a real inner critical point, we must set the following:

$$2t^4 - 8\beta_1 t + 6\beta_2 t > 0 \tag{5.11b}$$

$$16\beta_1^2 - 16\beta_2\beta_1 + 5\beta_2^2 \geq 0. \tag{(5.11c)}$$

The conditions that must be satisfy to ensure the realization of the relations (5.11a), (5.11b) and ((5.11c)) are:

$$(\beta_1 \leq 0 \text{ and } (\beta_2 < 2\beta_1 \text{ or } 2\beta_1 < \beta_2 < 0 \text{ or } \beta_2 > 0)) \text{ or } \beta_1 > 0$$

in which

$$t = \left(4\beta_1 - 2\beta_2 + \sqrt{(16\beta_1^2 - 16\beta_1\beta_2 + 5\beta_2^2}\right)\Bigg)^{\frac{1}{3}}$$

To get a nondegenerate critical point, we set the following: $\beta_1 \neq 0, \beta_2 \neq 0$ (see Theorem 5.1). Hence, we have one real inner nondegenerate critical point when the above conditions are satisfied.

The real boundary critical points can get by the equation: $t^4 + 2\beta_1 t = 0$, in which its solution is the set: $\left\{0, -2^{\frac{1}{3}}\sqrt[3]{\beta_1}\right\}$ such that $\beta_1 \geq 0$ or $\beta_1 \leq 0$. The nondegenerate boundary critical points can be obtained by setting $\beta_1 \neq 0, \beta_2 \neq 0$. Therefore, from the proof above we get the wanted result. □

Theorem 5.2 *The matrices of bifurcation spreading of the critical points of the function (5.1) are as follow:*

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdots \tag{5.12}$$

Proof: Theorem (5.1) has established the caustic equation, and from this equation, we may deduce its geometric representation, which is shown in Figure 1.

The parameters plane can be divided into six areas, G_i, in which $i = 1, 2, 3, 4, 5,$ and 6 . Each area has a predetermined number of real and critical points that are not degenerate. There are two categories for these points: boundary and interior points. The second derivative‘s test can be used to determine the interior and boundary point types (with the help of Mathematica program in classification the critical points). Thus, the following is a propagating of the critical points:

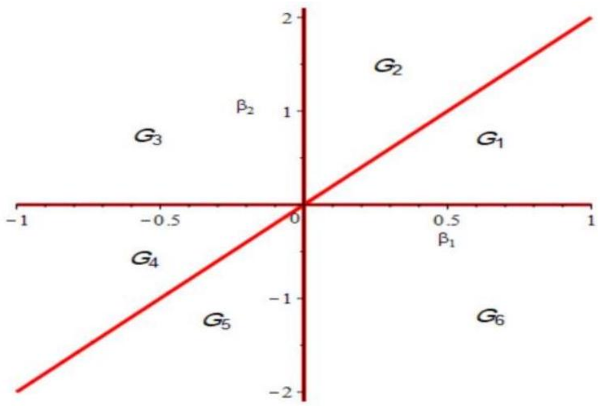
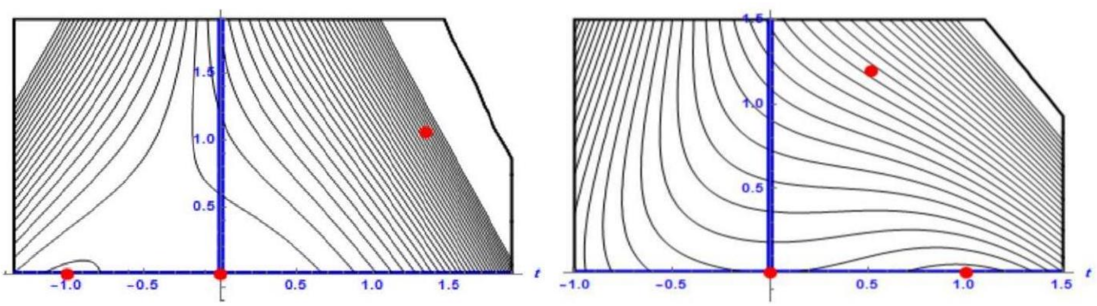


Figure 1: Pure graphimine structure

1. if the parameters pair (β_1, β_2) belongs to G_1 or G_2 or G_6, then there are three critical points (two boundary points (one minimum point and another saddle) on boundary $u = 0$, and one saddle point in the interior).
2. if the parameters pair (β_1, β_2) belongs to G_3 or G_4 or G_5, then there are three critical points (two saddle boundary points and one saddle point in the interior).

We construct the matrices from the two points above of bifurcation spreading as are described in (5.12).



(a) (b)

Figure 2:

In Figure 2, Sections (a) and (b) display the contour line locations with regard to the function (7)'s domain boundaries, as well as the number and type of critical points that correspond to each region in the function (5.1)'s caustic ,such that showing of (a) corresponds the regions G_1 or G_2 or G_6 and (b) corresponds the regions G_3 or G_4 or G_5. □

6. Conclusions and Works Ahead

6.1. Conclusions

We focused on the novel nonlinear term in the bifurcation solutions to a nonlinear wave ODE that models elastic beams on elastic foundations. Key study findings are:

Key Functions: A smooth fifth-degree function was the functional's key function. This prepared for the function behavior study under diverse scenarios. Corner singularities were essential to understanding

ODE bifurcations. These singularities exposed bifurcation point nature and stability.
Parametric Equation for the (Caustic) Bifurcation Set: To visualize bifurcation patterns and critical point propagation, we created a parametric equation. This caustic diagram shows the system's multifaceted behavior. Critical point propagation was methodically analyzed to create a bifurcation diagram showing how parameters affect solution stability. Results show many critical locations with boundary and interior solutions affecting behavior.

It is important to note that the study of bifurcation solutions can be applied to several types of issues, including well-posedness, equilibrium issues and nonlocal issues (see [1,2,3,4] and [11]).

6.2. Works Ahead

This work can inform future research in many ways:

Validating this study's analytical results may need extensive numerical simulations of the nonlinear wave ODE. Multiple parameter regimes and real-time solution behavior testing are needed. To expand the investigation, higher-dimensional systems can have more parameters. We would better grasp bifurcation phenomena and their impact in complex contexts. Experiments to observe physical systems predicted by our equations may provide insights. Theory and practice are linked by experimental validations. Future elastic beam equation research could address other nonlinearities. Understanding how nonlinear terms effect bifurcation and dynamic behavior would be easier. Finally, nonlinear differential equations are needed to represent complicated systems in materials science, engineering, and economics, thus the findings could be utilized there. These prospective fields can help researchers understand bifurcation theory and its applications and improve theoretical and practical mathematics.

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