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Advanced Structures in SuperHyper Algebra: Monoids and Ideals

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ABSTRACT: This paper delves into the intricate and innovative realm of superhyper monoids and their associated ideals, presenting a comprehensive study of their algebraic structures and properties. By integrating the principles of superhyper operations with the foundational aspects of algebra, we develop a generalized monoid framework that adeptly addresses associative and identity properties while managing indeterminate and conflicting information. The paper introduces and formalizes key concepts such as commutativity, idempotency, and homomorphisms within the context of superhyper monoids. We also explore the notion of superhyper submonoids and ideals, emphasizing their theoretical significance and practical applications. Through rigorous definitions, illustrative examples, and theorems, we highlight the structural coherence and potential of superhyper algebra to extend traditional algebraic boundaries. This work not only contributes to the theoretical advancement of algebraic hyperstructures but also paves the way for future research in handling complex and uncertain mathematical data.

Key Words: SuperHyper operations, SuperHyper algebras, monoids, ideals, maximal ideals.

Contents

1. Introduction

The study of algebraic structures has always been a central theme in mathematics, providing a foundation for numerous theoretical and applied disciplines. In recent years, there has been a growing interest in the exploration of generalized algebraic systems that can handle indeterminate and conflicting information. One such promising extension is the concept of superhyper algebra, which merges the principles of hyperstructures with those of super algebras. In classical algebra, structures such as groups, rings, and fields have well-defined operations that are associative, commutative, and possess identity elements. However, these structures often fall short when applied to complex systems where uncertainty, partial truth, or conflicting information play a significant role. superhyper algebra addresses these limitations by introducing operations that can map elements to sets of elements, thus providing a more flexible and comprehensive framework. In 2016, Smarandache [39] presented the ideas of superhyper algebra and neutrosophic superhyper algebra. This work extends traditional algebraic structures by incorporating hyperoperations and neutrosophic logic, providing a robust framework for modeling systems characterized by indeterminate and inconsistent data [42]. In the recent study by Smarandache [40], the concepts of superhyper structure and neutrosophic superhyper structure are thoroughly examined to address the complexities and uncertainties inherent in data analysis.

Let S be a non-empty set. A superhyper structure is a structure defined on the n-th powerset $P^n(S)$ of S, where n is an integer greater than or equal to 1. This structure features superhyper operators that are defined as follows:

$$\#SHS : \left(P^r(S)\right)^m \longrightarrow P^n(S),$$

where $P^r(S)$ represents the r-powerset of S, with $r \geq 1$, and $P^n(S)$ denotes the n-th powerset of S.

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Moreover, Smarandache [41] explored the concepts of the superhyper function and the neutrosophic superhyper function. This study extends the traditional function theory to encompass hyperoperations and neutrosophic logic, offering a novel approach to handling functions in systems with indeterminate and inconsistent data. In another study, Smarandache [38] introduced superhyper algebra and neutrosophic superhyper algebra. This work provides a comprehensive overview of these advanced algebraic structures, detailing their theoretical properties and potential applications across various fields dealing with uncertainty and complexity. Xin [43] explored the structure and properties of hyper BCI-algebras. This study provides foundational insights into the algebraic framework, enhancing the understanding of algebraic systems and their applications. In a related work, Hamidi [23] delved into superhyper BCK-algebras, presenting a detailed analysis published in neutrosophic sets and systems. This work extends the understanding of BCK-algebras within the framework of superhyper structures, offering a deeper exploration of algebraic systems capable of handling complex and uncertain data scenarios.

In 2023, Smarandache [37] laid the foundation for the superhypersoft set and its fuzzy extension, presenting a new vision in neutrosophic systems with applications. This work explores the integration of hyperoperations and fuzzy logic within the superhypersoft set framework, highlighting its potential applications in areas requiring sophisticated data analysis techniques. On the other hand, Muhiuddin et al. [25] applied hyperstructure theory to BF-algebras. This study explores the interactions between hyperstructures and BF-algebras, offering a novel perspective on the algebraic properties and potential applications of BF-algebras within the framework of hyperstructure theory. Moreover, Smarandache [36] extends the concept of soft sets to hypersoft sets and subsequently to plithogenic hypersoft sets.

This study introduces new algebraic structures designed to address MCDM processes involving complex and inconsistent data. Santhakumar, Sumathi, and Mahalakshmi [35] offer a novel approach to the algebraic framework of neutrosophic superhyper algebra. Their work explores the theoretical properties and potential applications of this advanced algebraic structure in managing complex and uncertain data. Rahmati and Hamidi [33] extend G algebras to superhyper G algebras. This research contributes to the development of superhyper G algebras, offering a deeper understanding of algebraic structures capable of handling complex and uncertain data scenarios.

Recently, Antokoletz and Heredia [2] explored the atomicity properties of power monoids derived from puiseux monoids, presenting significant insights into their structure and behavior. Their findings provide a foundation for broader algebraic theory and applications, addressing the complexities of atomicity in these unique algebraic systems. Baeth, Gotti, and O'Neill [3] focused on the atomicity of puiseux monoids, presenting comprehensive results that enhance the understanding of these algebraic structures. Their work examines the conditions under which these monoids are atomic, providing detailed proofs and theoretical underpinnings that are essential for researchers in the field. Masroor [24] extended classical algebraic structures to the hyperoctahedral group, focusing on the schur algebra and polynomial web category. Due to their ability to represent hierarchical concepts, superhyperstructures have attracted considerable attention in recent years, with numerous studies focusing on their theoretical development and applications [1,15,17–20,34]. Relatedly, Das and collaborators have made significant contributions in the areas of fuzzy soft sets, intuitionistic fuzzy rough relations, and decision-making models [4–12,26–32]. These works laid foundational insights into handling uncertainty, multisets, and algebraic generalizations, providing motivation for extending such ideas into the superhyper framework.

This paper focuses on a specific type of superhyper algebraic structure known as the superhyper monoid. A superhyper monoid is a set equipped with a binary superhyper operation that satisfies closure, associativity, and the existence of an identity element. This structure allows for the systematic study of properties and behaviors that are crucial for handling indeterminate and conflicting information. We begin by revisiting fundamental concepts pertinent to superhyper algebra and then delve into the formal definition of a superhyper monoid. Key properties such as closure, associativity, commutativity, and idempotency are explored in detail. We also introduce the notion of superhyper monoid homomorphisms, which are functions preserving the superhyper structure between monoids. Additionally, we examine the concept of superhyper submonoids, which are subsets of superhyper monoids that themselves form a monoid under the inherited operation. This is followed by an in-depth discussion of superhyper ideals, which are special subsets closed under the superhyper operation with any element from the monoid. Throughout this paper, we provide illustrative examples and theorems to elucidate the properties and interrelationships of these structures. Our aim is to contribute to the theoretical underpinnings of superhyper algebra, providing a robust mathematical framework that can address the complexities of modern scientific and engineering problems. The rest of this paper is structured as follows: Section 2 formally defines the superhyper monoid and explores its properties. Section 3 introduces superhyper ideals and investigates their characteristics. Finally, we conclude with potential applications and future research directions in the field of superhyper algebra in Section 4.

2. SuperHyper Monoid

Superhyper monoid is a generalized algebraic structure that combines the principles of superhyper operations and algebras into a monoid framework. This structure allows for the exploration of closure, associative, and the existence of identity properties, and the handling of indeterminate and conflicting information.
Let S be a non-empty set. Each element $x \in S$. Let $P^n(S)$ is the n-th power set of S, and $\Omega \subseteq P^n(S)$.

Definition 2.1 *A superhyper monoid is a set (Ω, Θ) equipped with a binary superhyper operation Θ satisfying the following conditions:*

1. ***Closure:*** *For all $x, y \in \Omega$, the result of the superhyper operation $x\Theta y$ is in Ω. Symbolically,*
$$x\Theta y \in \Omega.$$
2. ***Associativity:*** *For all $x, y, z \in \Omega$, there exists an element $w \in \Omega$ such that the superhyper operation is associative. Formally,*
$$(x\Theta y)\Theta z = x\Theta(y\Theta z) = w.$$
3. ***Identity:*** *There exists an element $\theta \in \Omega$ (called the identity element) such that for all $x \in \Omega$, the superhyper operation with the identity element results in x itself. Formally,*
$$x\Theta\theta = \theta\Theta x = x.$$

Let's construct an example of a superhyper monoid.

Example 2.1 *Consider the set $S = \{d, k\}$.*
Power set of S, $P(S)$:
$$P(S) = \{\emptyset, \{d\}, \{k\}, \{d, k\}\}.$$
Power set of $P(S)$, $P^2(S)$:
$$\begin{aligned} P^2(S) = \{&\emptyset, \{\emptyset\}, \{\{d\}\}, \{\{k\}\}, \{\{d,k\}\}, \{\emptyset, \{d\}\}, \{\emptyset, \{k\}\}, \{\emptyset, \{d,k\}\}, \\ &\{\{d\}, \{k\}\}, \{\{d\}, \{d,k\}\}, \{\{k\}, \{d,k\}\}, \{\emptyset, \{d\}, \{k\}\}, \{\emptyset, \{d\}, \{d,k\}\}, \\ &\{\emptyset, \{k\}, \{d,k\}\}, \{\{d\}, \{k\}, \{d,k\}\}, \{\emptyset, \{d\}, \{k\}, \{d,k\}\}\}. \end{aligned}$$
Let
$$\Omega = \big\{\{\emptyset\}, \{\{d\}\}, \{\{k\}\}, \{\{d,k\}\}\big\} \subseteq P^2(S).$$
The superhyper operation Θ is defined as follows: for any $X, Y \in \Omega$,
$$X\Theta Y = \{x \cup y \mid x \in X, y \in Y\}$$
and the superhyper operation Θ presented in Table 1.

Table 1: Superhyper Operation Θ on Ω

Θ	$\{\emptyset\}$	$\{\{d\}\}$	$\{\{k\}\}$	$\{\{d,k\}\}$
$\{\emptyset\}$	$\{\emptyset\}$	$\{\{d\}\}$	$\{\{k\}\}$	$\{\{d,k\}\}$
$\{\{d\}\}$	$\{\{d\}\}$	$\{\{d\}\}$	$\{\{d,k\}\}$	$\{\{d,k\}\}$
$\{\{k\}\}$	$\{\{k\}\}$	$\{\{d,k\}\}$	$\{\{k\}\}$	$\{\{d,k\}\}$
$\{\{d,k\}\}$	$\{\{d,k\}\}$	$\{\{d,k\}\}$	$\{\{d,k\}\}$	$\{\{d,k\}\}$

Let's verify that (Ω, Θ) is a superhyper monoid:

1. ***Closure:*** *For all $x, y \in \Omega$, the result of the superhyper operation $x\Theta y$ is in Ω. From the table, each result of $x\Theta y$ is either $\{\emptyset\}, \{\{d\}\}, \{\{k\}\}$, or $\{\{d,k\}\}$. All these subsets are contained within Ω, thus satisfying closure.*

2. ***Associativity:*** *For all $x, y, z \in \Omega$, there exists an element $w \in \Omega$ such that the superhyper operation is associative. For example, if $x = \{\{d\}\}, y = \{\{k\}\}$, and $z = \{\{d,k\}\}$, then:*

$$(x\Theta y)\Theta z = \{\{d,k\}\}\Theta\{\{d,k\}\} = \{\{d,k\}\}$$

and

$$x\Theta(y\Theta z) = \{\{d\}\}\Theta\{\{d,k\}\} = \{\{d,k\}\}$$

Hence, $(x\Theta y)\Theta z = x\Theta(y\Theta z) = w$. Similarly, we can show that for all $x, y, z \in \Omega$, there exists an element $w \in \Omega$ such that the superhyper operation is associative.

3. ***Identity:*** *For the Identity property, there should exist an element $\theta \in \Omega$ such that for all $x \in \Omega$, the operation $x\Theta\theta = \theta\Theta x = x$, where Θ is the superhyper operation. From the table:*

$$\{\emptyset\}\Theta\{\emptyset\} = \{\emptyset\}, \quad \{\emptyset\}\Theta\{\{d\}\} = \{\{d\}\}\Theta\{\emptyset\} = \{\{d\}\},$$

$$\{\emptyset\}\Theta\{\{k\}\} = \{\{k\}\}\Theta\{\emptyset\} = \{\{k\}\}, \quad \{\emptyset\}\Theta\{\{d,k\}\} = \{\{d,k\}\}\Theta\{\emptyset\} = \{\{d,k\}\}.$$

Therefore, from the given table, the element $\{\emptyset\}$ acts as the identity element θ.

The construction we've outlined demonstrates that the structure (Ω, Θ) satisfies all the necessary conditions to be considered a superhyper monoid with respect to the superhyper operation Θ. This includes closure, associativity, and the existence of an identity element θ. So, (Ω, Θ) indeed qualifies as a superhyper monoid with respect to Θ.

Definition 2.2 *A superhyper monoid (Ω, Θ) is said to be* commutative *if for all $x, y \in \Omega$,*

$$x\,\Theta\,y \;=\; y\,\Theta\,x.$$

Example 2.2 *Consider the superhyper monoid (Ω, Θ), where*

$$\Omega = \{\{\emptyset\}, \{\{d\}\}, \{\{k\}\}, \{\{d,k\}\}\} \subseteq P^2(\{d,k\}).$$

The superhyper operation Θ is defined as follows: for any $X, Y \in \Omega$,

$$X\Theta Y = \{x \cup y \mid x \in X, y \in Y\}.$$

Let us verify the commutative property by checking if $x\Theta y = y\Theta x$ for all $x, y \in \Omega$:

$$\begin{gathered}
\{\emptyset\}\Theta\{\emptyset\} = \{\emptyset\}\Theta\{\emptyset\} = \{\emptyset\}, \\
\{\emptyset\}\Theta\{\{d\}\} = \{\{d\}\}\Theta\{\emptyset\} = \{\{d\}\}, \\
\{\emptyset\}\Theta\{\{k\}\} = \{\{k\}\}\Theta\{\emptyset\} = \{\{k\}\}, \\
\{\emptyset\}\Theta\{\{d,k\}\} = \{\{d,k\}\}\Theta\{\emptyset\} = \{\{d,k\}\}, \\
\{\{d\}\}\Theta\{\{k\}\} = \{\{k\}\}\Theta\{\{d\}\} = \{\{d,k\}\}, \\
\{\{d\}\}\Theta\{\{d,k\}\} = \{\{d,k\}\}\Theta\{\{d\}\} = \{\{d,k\}\}, \\
\{\{k\}\}\Theta\{\{d,k\}\} = \{\{d,k\}\}\Theta\{\{k\}\} = \{\{d,k\}\}, \\
\{\{d,k\}\}\Theta\{\{d,k\}\} = \{\{d,k\}\}\Theta\{\{d,k\}\} = \{\{d,k\}\}.
\end{gathered}$$

From the above calculations, we see that $x\Theta y = y\Theta x$ for all $x, y \in \Omega$. Hence, the superhyper monoid (Ω, Θ) is commutative.

The study of algebraic generalizations in fuzzy soft sets and multisets [5, 7, 11, 26–29] shows similar extensions of classical structures, reinforcing the importance of superhyper monoids in capturing uncertain and indeterminate algebraic behavior.

Definition 2.3 *In a superhyper monoid* (Ω, Θ)*, an element* $x \in \Omega$ *is said to be idempotent if*

$$x\Theta x = x.$$

This means that applying the superhyper operation Θ *to* x *with itself results in* x *itself. In other words, if an element remains unchanged when combined with itself under the superhyper operation, it is idempotent.*

Example 2.3 *Consider the superhyper monoid* (Ω, Θ) *where*

$$\Omega = \{\{\emptyset\}, \{\{d\}\}, \{\{k\}\}, \{\{d,k\}\}\} \subseteq P^2(\{d,k\}).$$

The superhyper operation Θ *is defined as follows: for any* $X, Y \in \Omega$,

$$X\Theta Y = \{x \cup y \mid x \in X, y \in Y\}.$$

Let's verify if an element is idempotent(in this structure). Consider the set $\{\{d\}\}$ *in* $P^2(\{d,k\})$.

The superhyper operation Θ *is defined as the union of sets. If we take* $X = \{\{d\}\}$ *and compute* $X\Theta X$*, we have* $\{\{d\}\}\Theta\{\{d\}\} = \{\{d\} \cup \{d\}\} = \{\{d\}\}$.
Since $\{\{d\}\}\Theta\{\{d\}\} = \{\{d\}\}$*, the element* $\{\{d\}\}$ *is idempotent in this superhyper monoid structure.*

This verifies the example of an idempotent element in a superhyper monoid.

Proposition 2.1 *In a superhyper monoid* (Ω, Θ)*, the identity element* θ *is unique.*

Proof: Assume there are two identity elements θ and θ' in the superhyper monoid (Ω, Θ). By the definition of the identity, forall $x \in \Omega$, $\quad x\Theta\theta = x \quad$ and $\quad x\Theta\theta' = x$.

Now, consider $\theta\Theta\theta'$. Since θ is an identity element, $\theta\Theta\theta' = \theta'$. and similarly, since θ' is an identity element, $\theta\Theta\theta' = \theta$, therefore, we have $\theta\Theta\theta' = \theta' \quad$ and $\quad \theta\Theta\theta' = \theta$.

This implies $\theta = \theta'$.

Thus, the identity element is unique in the superhyper monoid (Ω, Θ).

□

Definition 2.4 *Let* $\Omega \subseteq P^n(S)$ *and* $\Psi \subseteq P^m(T)$*. Let* (Ω, Θ) *and* (Ψ, Φ) *be two superhyper monoids. A function* $f : \Omega \to \Psi$ *is a* superhyper monoid homomorphism *if it satisfies the following conditions:*

1. *For all* $x, y \in \Omega$, $f(x\Theta y) \subseteq f(x)\Phi f(y)$.
2. $f(\theta_\Omega) = \theta_\Psi$*, where* θ_Ω *is the identity element in* (Ω, Θ) *and* θ_Ψ *is the identity element in* (Ψ, Φ).

Example 2.4 *Consider two sets* $S = \{a, b\}$ *and* $T = \{1, 2, 3\}$.
Let's consider two superhyper monoids:

- (Ω, Θ)*, where* $\Omega = \{\{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}\} \subseteq P^2(\{a,b\})$ *with the superhyper operation* Θ *defined as the union of all pairs, i.e.,* $X\Theta Y = \{x \cup y \mid x \in X, y \in Y\} \quad$ *for any* $X, Y \in \Omega$.
- (Ψ, Φ)*, where*

$$\Psi = \left\{ \begin{array}{l} \{\emptyset\}, \\ \{\{2,3\}\}, \\ \{\{1\},\{2\}\}, \\ \{\{1\},\{3\}\}, \\ \{\{2,3\},\{1,2,3\}\}, \\ \{\{1\},\{1,2\},\{1,3\},\{2,3\}\}, \\ \{\{1\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\} \end{array} \right\} \subseteq P^2(\{1,2,3\}).$$

with the superhyper operation Φ *defined similarly as the union of all pairs.*

Now, let's define a function $f : \Omega \to \Psi$ as follows:

$$f(\{\emptyset\}) = \{\emptyset\}, \quad f(\{\{a\}\}) = \{\{1\}, \{2\}\}, \quad f(\{\{b\}\}) = \{\{1\}, \{3\}\}, \quad f(\{\{a, b\}\}) = \{\{2, 3\}\}.$$

This function f maps elements from the first superhyper monoid to the second one.
Now, let's verify if f is a homomorphism:

1. *For all $x, y \in \Omega$, we need to check if $f(x\Theta y) \subseteq f(x)\Phi f(y)$.*
   Let's take $x = \{\{a\}\}$ and $y = \{\{b\}\}$. Then, $x\Theta y = \{\{a\} \cup \{b\}\} = \{\{a, b\}\}$.
   Now, $f(x) = \{\{1\}, \{2\}\}, \quad f(y) = \{\{1\}, \{3\}\}$.
   Also, $f(x)\Phi f(y) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$.
   Therefore, $f(x\Theta y) = f(\{\{a, b\}\}) = \{\{2, 3\}\} \subseteq \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\} = f(x)\Phi f(y)$, and this holds for all $x, y \in \Omega$.

2. *$f(\theta_{\{a,b\}}) = \theta_{\{1,2,3\}}$, where $\theta_{\{a,b\}} = \{\emptyset\}$ is the identity element in (Ω, Θ) and $\theta_{\{1,2,3\}} = \{\emptyset\}$ is the identity element in (Ψ, Φ).*
   Since $f(\{\emptyset\}) = \{\emptyset\} = \theta_{\{1,2,3\}}$, this condition is satisfied.

Therefore, the function f is a homomorphism between the given superhyper monoids.

Theorem 2.1 *Let S_1, S_2, S_3 be three non-empty sets, and let $\Omega_1 \subseteq P^n(S_1)$, $\Omega_2 \subseteq P^m(S_2)$, and $\Omega_3 \subseteq P^r(S_3)$. If $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ are homomorphisms between the superhyper monoids (Ω_1, Θ_1), (Ω_2, Θ_2), and (Ω_3, Θ_3), then the composition $g \circ f : \Omega_1 \to \Omega_3$ is also a homomorphism.*

Proof: Let's prove the theorem step by step:

1. First, recall that a homomorphism $f : \Omega_1 \to \Omega_2$ between superhyper monoids satisfies:
   - For all $x, y \in \Omega_1$, $f(x\Theta_1 y) \subseteq f(x)\Theta_2 f(y)$, where Θ_1 and Θ_2 are the superhyper operations in (Ω_1, Θ_1) and (Ω_2, Θ_2) respectively.
   - $f(\theta_{\Omega_1}) = \theta_{\Omega_2}$, where θ_{Ω_1} and θ_{Ω_2} are the identity elements in (Ω_1, Θ_1) and (Ω_2, Θ_2) respectively.

2. Similarly, a homomorphism $g : \Omega_2 \to \Omega_3$ between superhyper monoids satisfies similar conditions with superhyper operations Θ_2 and Θ_3 and identity elements θ_{Ω_2} and θ_{Ω_3}.

3. Now, consider the composition $g \circ f : \Omega_1 \to \Omega_3$.
   We need to show that $g \circ f$ satisfies the conditions of being a homomorphism.
   - For all $x, y \in \Omega_1$, we have $(g \circ f)(x\Theta_1 y) = g(f(x\Theta_1 y)) \subseteq g(f(x)\Theta_2 f(y)) \subseteq (g \circ f)(x)\Theta_3(g \circ f)(y)$. Thus, $g \circ f$ preserves the superhyper operation Θ_3, which is the composition of Θ_1 and Θ_2.
   - Since f and g are homomorphisms, we have $f(\theta_{\Omega_1}) = \theta_{\Omega_2}, \quad g(\theta_{\Omega_2}) = \theta_{\Omega_3}$.

   Therefore, $(g \circ f)(\theta_{\Omega_1}) = \theta_{\Omega_3}$, preserving the identity element.

Therefore, since $g \circ f$ satisfies all the conditions of being a homomorphism, we conclude that $g \circ f$ is indeed a homomorphism between (Ω_1, Θ_1) and (Ω_3, Θ_3).

□

Definition 2.5 *Let (Ω, Θ) be a superhyper monoid with a binary superhyper operation Θ where $\Omega \subseteq P^n(S)$. A non-empty subset $\Psi \subseteq \Omega$ is said to be a superhyper submonoid of (Ω, Θ) if Ψ satisfies the following conditions:*

1. ***Non-emptiness:*** *$\Psi \neq \emptyset$.*

2. ***Closure under* Θ:** *For all $x, y \in \Psi$, we have $x \Theta y \in \Psi$.*

3. ***Identity element:*** *There exists an identity element $\theta \in \Psi$ such that, for all $x \in \Psi$, $x \Theta \theta = x \quad \text{and} \quad \theta \Theta x = x$.*

Example 2.5 *Consider the superhyper monoid (Ω, Θ) where*

$$\Omega = \{\{\emptyset\}, \{\{d\}\}, \{\{k\}\}, \{\{d,k\}\}\} \subseteq P^2(\{d,k\}).$$

The superhyper operation Θ is defined as follows: for any $X, Y \in \Omega$,

$$X\Theta Y = \{x \cup y \mid x \in X, y \in Y\}.$$

Let us define the subset $\Psi = \{\{\emptyset\}, \{\{d\}\}, \{\{d,k\}\}\}$.

We need to verify that Ψ is a superhyper submonoid of (Ω, Θ) by checking the three conditions:

1. ***Non-emptiness:*** *Ψ is non-empty as it contains the elements $\{\emptyset\}, \{\{d\}\}, \{\{d,k\}\}$.*

2. ***Closure under* Θ:**

   - *For $x = \{\emptyset\}$:*

$$\begin{aligned} y = \{\emptyset\}, \quad & \{\emptyset\}\Theta\{\emptyset\} = \{\emptyset\} \cup \{\emptyset\} = \{\emptyset\} \in \Psi, \\ y = \{\{d\}\}, \quad & \{\emptyset\}\Theta\{\{d\}\} = \{\emptyset\} \cup \{\{d\}\} = \{\{d\}\} \in \Psi, \\ y = \{\{d,k\}\}, \quad & \{\emptyset\}\Theta\{\{d,k\}\} = \{\emptyset\} \cup \{\{d,k\}\} = \{\{d,k\}\} \in \Psi. \end{aligned}$$

   - *For $x = \{\{d\}\}$:*

$$\begin{aligned} y = \{\emptyset\}, \quad & \{\{d\}\}\Theta\{\emptyset\} = \{\{d\}\} \cup \{\emptyset\} = \{\{d\}\} \in \Psi, \\ y = \{\{d\}\}, \quad & \{\{d\}\}\Theta\{\{d\}\} = \{\{d\}\} \cup \{\{d\}\} = \{\{d\}\} \in \Psi, \\ y = \{\{d,k\}\}, \quad & \{\{d\}\}\Theta\{\{d,k\}\} = \{\{d\}\} \cup \{\{d,k\}\} = \{\{d,k\}\} \in \Psi. \end{aligned}$$

   - *For $x = \{\{d,k\}\}$:*

$$\begin{aligned} y = \{\emptyset\}, \quad & \{\{d,k\}\}\Theta\{\emptyset\} = \{\{d,k\}\} \cup \{\emptyset\} = \{\{d,k\}\} \in \Psi, \\ y = \{\{d\}\}, \quad & \{\{d,k\}\}\Theta\{\{d\}\} = \{\{d,k\}\} \cup \{\{d\}\} = \{\{d,k\}\} \in \Psi, \\ y = \{\{d,k\}\}, \quad & \{\{d,k\}\}\Theta\{\{d,k\}\} = \{\{d,k\}\} \cup \{\{d,k\}\} = \{\{d,k\}\} \in \Psi. \end{aligned}$$

   Therefore, Ψ is closed under Θ.

3. ***Identity element:*** *The element $\{\emptyset\}$ serves as the identity element in Ψ because*

$$\begin{aligned} For\ x = \{\{d\}\} : \quad & \{\{d\}\}\Theta\{\emptyset\} = \{\{d\}\}, \quad \{\emptyset\}\Theta\{\{d\}\} = \{\{d\}\}, \\ For\ x = \{\{d,k\}\} : \quad & \{\{d,k\}\}\Theta\{\emptyset\} = \{\{d,k\}\}, \quad \{\emptyset\}\Theta\{\{d,k\}\} = \{\{d,k\}\}. \end{aligned}$$

   Hence, $\{\emptyset\}$ is the identity element in Ψ.

Thus, (Ψ, Θ) is a superhyper submonoid of (Ω, Θ).

Theorem 2.2 *A superhyper submonoid in a superhyper monoid (Ω, Θ) has the same identity element as in (Ω, Θ).*

Proof: Let (Ω, Θ) be a superhyper monoid with a binary superhyper operation Θ. Consider a superhyper submonoid (Ψ, Θ) of (Ω, Θ).

Let θ_Ω be the identity element of (Ω, Θ) and θ_Ψ be the identity element of Ψ. We aim to show that $\theta_\Omega = \theta_\Psi$.

Since θ_Ω is the identity element of (Ω, Θ), for all $x \in \Omega$, we have $x\Theta\theta_\Omega = x$ and $\theta_\Omega\Theta x = x$

Now, consider any $x \in \Psi$. Since Ψ is a superhyper submonoid, we have $x\Theta\theta_\Psi \in \Psi$ and $\theta_\Psi\Theta x \in \Psi$.

Since θ_Ψ is the identity element of (Ψ, Θ), it follows that

$$x\Theta\theta_\Psi = x \quad \text{and} \quad \theta_\Psi\Theta x = x \quad \text{for all } x \in \Psi.$$

Therefore, $\theta_\Omega = \theta_\Psi$. Hence, a superhyper submonoid in a superhyper monoid (Ω, Θ) has the same identity element as in (Ω, Θ).

□

Theorem 2.3 *The intersection of any two superhyper submonoids in a superhyper monoid is a superhyper submonoid.*

Proof: Let (Ω, Θ) be a superhyper monoid with a binary superhyper operation Θ where $\Omega \subseteq P^n(S)$. Suppose Ψ_1 and Ψ_2 are two superhyper submonoids of (Ω, Θ). We need to show that $\Psi_1 \cap \Psi_2$ is also a superhyper submonoid. By the theorem stated earlier, each superhyper submonoid has the same identity element as in (Ω, Θ). Let θ be the common identity element of Ψ_1 and Ψ_2.

To prove this, we will verify the three conditions for $\Psi_1 \cap \Psi_2$:

1. **Non-emptiness:** Since Ψ_1 and Ψ_2 are non-empty subsets of Ω, as they have the same identity element θ, their intersection $\Psi_1 \cap \Psi_2$ is also non-empty, since it contains at least the identity element θ. i.e., $\theta \in \Psi_1 \cap \Psi_2 \neq \emptyset$.

2. **Closure under Θ:** Let $x, y \in \Psi_1 \cap \Psi_2$. Since $x, y \in \Psi_1$ and $x, y \in \Psi_2$, it follows that $x\Theta y \in \Psi_1$ and $x\Theta y \in \Psi_2$ by the closure property of Ψ_1 and Ψ_2. Therefore, $x\Theta y \in \Psi_1 \cap \Psi_2$.

3. **Identity element:** For any $x \in \Psi_1 \cap \Psi_2$, we have $x\Theta\theta = x$ and $\theta\Theta x = x$ by the properties of θ as the identity element in both Ψ_1 and Ψ_2.

Since $\Psi_1 \cap \Psi_2$ satisfies all the properties of a superhyper submonoid, we conclude that the intersection of any two superhyper submonoids in a superhyper monoid (Ω, Θ) is a superhyper submonoid.

□

Theorem 2.4 *The union of any two superhyper submonoids in a superhyper monoid is not necessarily a superhyper submonoid.*

Proof: To prove this theorem, we provide a counter example.

Let $S = \{\kappa, \eta, \lambda\}$ and let $\Omega = \{\{\emptyset\}, \{\{\kappa\}\}, \{\{\eta\}\}, \{\{\lambda\}\}, \{\{\kappa, \eta\}\}, \{\{\eta, \lambda\}\}, \{\{\kappa, \eta, \lambda\}\}\} \subseteq P^2(S)$ be a superhyper monoid with a superhyper operation Θ, such that $X\Theta Y$ is the union of all pairs $\{x \cup y \mid x \in X, y \in Y\}$ for any $X, Y \in \Omega$.

Suppose we have two superhyper submonoids Ψ_1 and Ψ_2 in Ω.

Assume:

$$\Psi_1 = \{\{\emptyset\}, \{\{\kappa\}\}, \{\{\eta, \lambda\}\}\}$$

$$\Psi_2 = \{\{\emptyset\}, \{\{\eta\}\}, \{\{\kappa, \lambda\}\}\}$$

The union $\Psi_1 \cup \Psi_2$ contains:

$$\Psi_1 \cup \Psi_2 = \{\{\emptyset\}, \{\{\kappa\}\}, \{\{\eta\}\}, \{\{\kappa, \lambda\}\}, \{\{\eta, \lambda\}\}\}$$

Now, let's check if $\Psi_1 \cup \Psi_2$ satisfies the conditions to be a superhyper submonoid:

1. **Non-emptiness:** $\Psi_1 \cup \Psi_2$ is non-empty.
2. **Closure under Θ:** Let's consider $\{\{\kappa\}\}$ and $\{\{\eta\}\}$ from $\Psi_1 \cup \Psi_2$. Their union $\{\{\kappa\}\}\Theta\{\{\eta\}\} = \{\{\kappa, \eta\}\}$ is not in $\Psi_1 \cup \Psi_2$, violating closure.

Since $\Psi_1 \cup \Psi_2$ fails to satisfy the closure under Θ, it is not a superhyper submonoid, confirming the theorem. □

3. SuperHyper Ideal

In this section, we introduce the concept of superhyper Ideals, providing a comprehensive study of their properties and characteristics. We explore the theoretical underpinnings of these ideals within the framework of the superhyper model, highlighting their significance in addressing uncertainties and contradictions in mathematical structures.

Definition 3.1 *In a superhyper monoid (Ω, Θ), a subset I of Ω is called a superhyper ideal if the following conditions hold:*

1. *For all $x, y \in I$, $x\Theta y \in I$.*
2. *For all $x \in I$ and $y \in \Omega$, $x\Theta y \in I$, and $y\Theta x \in I$.*

Example 3.1 *Consider the superhyper monoid (Ω, Θ) where*

$$\Omega = \{\{\emptyset\}, \{\{\kappa\}\}, \{\{\eta\}\}, \{\{\kappa, \eta\}\}\} \subseteq P^2(\{\kappa, \eta\}).$$

The superhyper operation Θ is defined as follows: for any $X, Y \in \Omega$,

$$X\Theta Y = \{x \cup y \mid x \in X, y \in Y\}.$$

Let's define the set $I = \{\{\{\kappa\}\}, \{\{\kappa, \eta\}\}\}$. We need to verify two properties for I to be a superhyper ideal in (Ω, Θ):

1. *For $x, y \in I$, we need to show that $x\Theta y \in I$.*
   - *If $x = \{\{\kappa\}\}$ and $y = \{\{\kappa\}\}$, then $x\Theta y = \{\{\kappa\}\}\Theta\{\{\kappa\}\} = \{\{\kappa\}\}$, which is in I.*
   - *If $x = \{\{\kappa\}\}$ and $y = \{\{\kappa, \eta\}\}$, then $x\Theta y = \{\{\kappa\}\}\Theta\{\{\kappa, \eta\}\} = \{\{\kappa, \eta\}\}$, which is in I.*
   - *If $x = \{\{\kappa, \eta\}\}$ and $y = \{\{\kappa\}\}$, then $x\Theta y = \{\{\kappa, \eta\}\}\Theta\{\{\kappa\}\} = \{\{\kappa, \eta\}\}$, which is in I.*
   - *If $x = \{\{\kappa, \eta\}\}$ and $y = \{\{\kappa, \eta\}\}$, then $x\Theta y = \{\{\kappa, \eta\}\}\Theta\{\{\kappa, \eta\}\} = \{\{\kappa, \eta\}\}$, which is in I. Therefore, for all $x, y \in I$, $x\Theta y \in I$.*
2. *For $x \in I$ and $y \in \Omega$, we need to show that $y\Theta x \in I$.*
   - *If $x = \{\{\kappa\}\}$ and $y \in \Omega$, then $y\Theta x \in I$.*
     - *For $y = \{\emptyset\}$, $y\Theta x = x\Theta y = \{\{\kappa\}\}$.*
     - *For $y = \{\{\kappa\}\}$, $y\Theta x = x\Theta y = \{\{\kappa\}\}$.*
     - *For $y = \{\{\eta\}\}$, $y\Theta x = x\Theta y = \{\{\kappa, \eta\}\}$.*
     - *For $y = \{\{\kappa, \eta\}\}$, $y\Theta x = x\Theta y = \{\{\kappa, \eta\}\}$.*
   - *If $x = \{\{\kappa, \eta\}\}$ and $y \in \Omega$, then $y\Theta x \in I$.*
     - *For $y = \{\emptyset\}$, $y\Theta x = x\Theta y = \{\{\kappa, \eta\}\}$.*
     - *For $y = \{\{\kappa\}\}$, $y\Theta x = x\Theta y = \{\{\kappa, \eta\}\}$.*
     - *For $y = \{\{\eta\}\}$, $y\Theta x = x\Theta y = \{\{\kappa, \eta\}\}$.*
     - *For $y = \{\{\kappa, \eta\}\}$, $y\Theta x = x\Theta y = \{\{\kappa, \eta\}\}$.*

   Therefore, for $x \in I$ and $y \in \Omega$, $y\Theta x, x\Theta y \in I$.

Therefore, I is closed under Θ.

Hence, $I = \{\{\{\kappa\}\}, \{\{\kappa, \eta\}\}\}$ is a superhyper ideal in (Ω, Θ).

Comparable developments have been studied in neutrosophic and fuzzy soft multiset-based systems [6, 8, 13–15, 22, 31], where maximality and closure properties under uncertain environments are central to decision-making and data analysis.

Theorem 3.1 *The intersection of any two superhyper ideals in a superhyper monoid is also an ideal.*

Proof: Assume Γ_1 and Γ_2 be two superhyper ideals in the superhyper monoid (Ω, Θ). We need to show that $\Gamma_1 \cap \Gamma_2$ is also a superhyper ideal.

To do this, we need to verify that $\Gamma_1 \cap \Gamma_2$ satisfies the two conditions for being a superhyper ideal:

1. **Closure under Θ:** For all $x, y \in \Gamma_1 \cap \Gamma_2$, we need to show that $x\Theta y \in \Gamma_1 \cap \Gamma_2$.
   - Since $x \in \Gamma_1 \cap \Gamma_2$ and $y \in \Gamma_1 \cap \Gamma_2$ it follows that $x \in \Gamma_1$ and $x \in \Gamma_2$ and $y \in \Gamma_1$ and $y \in \Gamma_2$.
   - Because Γ_1 is a superhyper ideal, we have $x\Theta y \in \Gamma_1$.
   - Similarly, because Γ_2 is a superhyper ideal, we have $x\Theta y \in \Gamma_2$.
   - Therefore, $x\Theta y \in \Gamma_1 \cap \Gamma_2$.
2. **Closure under Reverse Θ:** For all $x \in \Gamma_1 \cap \Gamma_2$ and $y \in \Omega$, we need to show that $x\Theta y \in \Gamma_1 \cap \Gamma_2$ and $y\Theta x \in \Gamma_1 \cap \Gamma_2$.
   - Since $x \in \Gamma_1 \cap \Gamma_2$, it follows that $x \in \Gamma_1$ and $x \in \Gamma_2$.
   - Because Γ_1 is a superhyper ideal, for any $y \in \Omega$, we have $x\Theta y \in \Gamma_1$ and $y\Theta x \in \Gamma_1$.
   - Similarly, because Γ_2 is a superhyper ideal, for any $y \in \Omega$, we have $x\Theta y \in \Gamma_2$ and $y\Theta x \in \Gamma_2$.
   - Therefore, $x\Theta y \in \Gamma_1 \cap \Gamma_2$ and $y\Theta x \in \Gamma_1 \cap \Gamma_2$.

Since both conditions are satisfied, $\Gamma_1 \cap \Gamma_2$ is a superhyper ideal in (Ω, Θ). □

Theorem 3.2 *The union of any two superhyper ideals in a superhyper monoid is a superhyper ideal.*

Proof: Let Γ_1 and Γ_2 be two superhyper ideals in superhyper monoid (Ω, Θ). We need to show that $\Gamma_1 \cup \Gamma_2$ is also a superhyper ideal.

According to the definition, this requires that:

1. For all $x, y \in \Gamma_1 \cup \Gamma_2$, $x\Theta y \in \Gamma_1 \cup \Gamma_2$
2. For all $x \in \Gamma_1 \cup \Gamma_2$ and $y \in \Omega$, $x\Theta y \in \Gamma_1 \cup \Gamma_2$ and $y\Theta x \in \Gamma_1 \cup \Gamma_2$

1. **Closure under Θ:** Consider any $x, y \in \Gamma_1 \cup \Gamma_2$. There are four cases to consider:
   - $x \in \Gamma_1$ and $y \in \Gamma_1$
   - $x \in \Gamma_1$ and $y \in \Gamma_2$
   - $x \in \Gamma_2$ and $y \in \Gamma_1$
   - $x \in \Gamma_2$ and $y \in \Gamma_2$

   For the first case, since Γ_1 is a superhyper ideal, $x\Theta y \in \Gamma_1 \subseteq \Gamma_1 \cup \Gamma_2$.

   For the fourth case, since Γ_2 is a superhyper ideal, $x\Theta y \in \Gamma_2 \subseteq \Gamma_1 \cup \Gamma_2$.

   For the second and third cases, we need to show that $x\Theta y \in \Gamma_1 \cup \Gamma_2$.

   Assume $x \in \Gamma_1$, and $y \in \Gamma_2 \subseteq \Omega$. Since Γ_1 is ideal, from the definition of ideal, for all $x \in \Gamma_1$ and $y \in \Omega$, $x\Theta y \in \Gamma_1$, and $y\Theta x \in \Gamma_1$ implies for all $x, y \in \Gamma_1 \cup \Gamma_2$, $x\Theta y \in \Gamma_1 \cup \Gamma_2$ and $y\Theta x \in \Gamma_1 \cup \Gamma_2$

2. **Closure under Θ with elements from Ω:** Consider $x \in \Gamma_1 \cup \Gamma_2$ and any $y \in \Omega$. There are two cases to consider:

   - $x \in \Gamma_1$
   - $x \in \Gamma_2$

   For the first case, since Γ_1 is a superhyper ideal, $x\Theta y \in \Gamma_1 \subseteq \Gamma_1 \cup \Gamma_2$ and $y\Theta x \in \Gamma_1 \subseteq \Gamma_1 \cup \Gamma_2$.
   For the second case, since Γ_2 is a superhyper ideal, $x\Theta y \in \Gamma_2 \subseteq \Gamma_1 \cup \Gamma_2$, and $y\Theta x \in \Gamma_2 \subseteq \Gamma_1 \cup \Gamma_2$.

Therefore, under the given assumptions, $\Gamma_1 \cup \Gamma_2$ is closed under Θ and reverse Θ, implying $\Gamma_1 \cup \Gamma_2$ is a superhyper ideal. □

Theorem 3.3 *Let S_1, S_2 be two non-empty sets, and let $\Omega_1 \subseteq P^n(S_1), \Omega_2 \subseteq P^m(S_2)$. If $f : \Omega_1 \to \Omega_2$ is a homomorphism between two superhyper monoids (Ω_1, Θ_1) and (Ω_2, Θ_2), and $I \subseteq \Omega_1$ is a superhyper ideal, then $f(I) \subseteq \Omega_2$ is also a superhyper ideal.*

Proof: Let $f : \Omega_1 \to \Omega_2$ be a homomorphism between two superhyper monoids (Ω_1, Θ_1) and (Ω_2, Θ_2). Let $I \subseteq \Omega_1$ be an ideal. We need to show that $f(I) \subseteq \Omega_2$ is also an ideal.

To do this, we need to verify that $f(I)$ satisfies the two conditions for being a superhyper ideal:

1. **Closure under Θ_2:** For all $x', y' \in f(I)$, we need to show that $x'\Theta_2 y' \in f(I)$.

   - Since $x', y' \in f(I)$, there exist $x, y \in I$ such that $f(x) = x'$ and $f(y) = y'$.
   - Because I is a superhyper ideal, we have $x\Theta_1 y \in I$.
   - Since f is a homomorphism, we have $f(x\Theta_1 y) \subseteq f(x)\Theta_2 f(y) = x'\Theta_2 y'$.
   - Therefore, $x'\Theta_2 y' \in f(I)$.

2. **Closure under an element in Ω_2:** For all $x' \in f(I)$, and $y' \in \Omega_2$, we need to show that $x'\Theta_2 y' \in f(I)$ and $y'\Theta_2 x' \in f(I)$.

   - Since $x' \in f(I)$, there exists $x \in I$ such that $f(x) = x'$.
   - Because I is a superhyper ideal, for any $y \in \Omega_1$, we have $x\Theta_1 y \in I$ and $y\Theta_1 x \in I$.
   - Let $y' \in \Omega_2$. There exists $y \in \Omega_1$ such that $f(y) = y'$.
   - Since f is a homomorphism, we have $f(x\Theta_1 y) \subseteq f(x)\Theta_2 f(y) = x'\Theta_2 y'$, which implies $x'\Theta_2 y' \in f(I)$.
   - Similarly, we have $f(y\Theta_1 x) \subseteq f(y)\Theta_2 f(x) = y'\Theta_2 x'$, which implies $y'\Theta_2 x' \in f(I)$.

Since both conditions are satisfied, $f(I) \subseteq \Omega_2$ is a superhyper ideal. This completes the proof. □

Theorem 3.4 *Let (Ω, Θ) be a superhyper monoid. If T is a superhyper submonoid of (Ω, Θ) and I is a superhyper ideal of (Ω, Θ), then $T \cap I$ is a superhyper ideal of T.*

Proof: Let (Ω, Θ) be a superhyper monoid, $T \subseteq \Omega$ be a superhyper submonoid, and $I \subseteq \Omega$ be an ideal. We need to show that $T \cap I$ is a superhyper ideal of T. Recall that T being a submonoid means:

1. $T \neq \emptyset$.

2. $\forall x, y \in T, x\Theta y \in T$.

3. There exists an identity element $\theta \in T$ such that $\forall x \in T, x\Theta\theta = x$ and $\theta\Theta x = x$.

Also, I being an ideal means:

1. $\forall x, y \in I, x\Theta y \in I$.
2. $\forall x \in I$ and $y \in \Omega, x\Theta y \in I$ and $y\Theta x \in I$.

We need to verify that $T \cap I$ satisfies the conditions for being an ideal in the submonoid T:

1. **Non-emptiness:** Since $\theta \in T$ and $I \neq \emptyset$, $T \cap I \neq \emptyset$.
2. **Closure under Θ:** Let $x, y \in T \cap I$. Since $x, y \in T$ and T is a submonoid, we have $x\Theta y \in T$. Since $x, y \in I$ and I is an ideal, we have $x\Theta y \in I$. Therefore, $x\Theta y \in T \cap I$.
3. **Absorption:** Let $x \in T \cap I$ and $y \in T$. Since $x \in I$ and I is an ideal, for all $y \in \Omega$, $x\Theta y \in I$ and $y\Theta x \in I$. In particular, since $y \in T \subseteq \Omega$, we have $x\Theta y \in I$ and $y\Theta x \in I$. Also, since $x, y \in T$ and T is a submonoid, we have $x\Theta y \in T$ and $y\Theta x \in T$. Therefore, $x\Theta y \in T \cap I$ and $y\Theta x \in T \cap I$.

Hence, $T \cap I$ satisfies the conditions for being an ideal in T. Thus, $T \cap I$ is an ideal of T. □

Definition 3.2 *Let (Ω, Θ) be a superhyper monoid and I an ideal of (Ω, Θ). The superhyper quotient set Ω/I consists of equivalence classes of Ω modulo the ideal I. Each equivalence class $x + I$ contains all elements in Ω that are related to x via the ideal I, with the superhyper operation defined as:*

$$(x + I)\Theta(y + I) = (x\Theta y) + I.$$

Theorem 3.5 *If (Ω, Θ) is a superhyper monoid and I is an ideal of (Ω, Θ), then the quotient set Ω/I with the superhyper operation $(x + I)\Theta(y + I) = (x\Theta y) + I$ forms a superhyper monoid.*

Proof: To prove that Ω/I with the superhyper operation $(x+I)\Theta(y+I) = (x\Theta y)+I$ form a superhyper monoid, we need to show that the quotient set satisfies the axioms of a superhyper monoid.

1. **Closure:** For any $x, y \in \Omega$, $x\Theta y \in \Omega$. Thus, $(x\Theta y)+I \in \Omega/I$. Therefore, the superhyper operation is closed in Ω/I.
2. **Associativity:** For any $x, y, z \in \Omega$,

   $$((x + I)\Theta(y + I))\Theta(z + I) = ((x\Theta y) + I)\Theta(z + I) = ((x\Theta y)\Theta z) + I$$

   and

   $$(x + I)\Theta((y + I)\Theta(z + I)) = (x + I)\Theta((y\Theta z) + I) = (x\Theta(y\Theta z)) + I.$$

   Since Θ is associative in (Ω, Θ), we have

   $$((x\Theta y)\Theta z) + I = (x\Theta(y\Theta z)) + I.$$

   Therefore, the superhyper operation Θ is associative in Ω/I.
3. **Identity element:** Let θ be the identity element in (Ω, Θ). For any $x \in \Omega$,

   $$(x + I)\Theta(\theta + I) = (x\Theta\theta) + I = x + I$$

   and

   $$(\theta + I)\Theta(x + I) = (\theta\Theta x) + I = x + I.$$

   Thus, $\theta + I$ is the identity element in Ω/I.

Therefore, Ω/I with the superhyper operation $(x+I)\Theta(y+I) = (x\Theta y)+I$ forms a superhyper monoid. □

Definition 3.3 *Let (Ω, Θ) be a superhyper monoid. A superhyper ideal I is said to be maximal if there does not exist any proper superhyper ideal J such that $I \subset J$.*

In simpler terms, a maximal superhyper ideal is an ideal within the monoid for which there is no larger ideal contained within it. Any additional element or subset that could be added to the maximal ideal would either break closure under the superhyper operation or violate the properties of an ideal.

Maximal ideals are important because they represent the largest possible ideals within a monoid that maintain the desired algebraic properties, making them a focus of study in the context of monoid theory and algebraic structures.

Example 3.2 *Consider the superhyper monoid (Ω, Θ) where*

$$\Omega = \{\{\emptyset\}, \{\{d\}\}, \{\{h\}\}, \{\{d,h\}\}, \{\{k,h\}\}, \{\{d,k,h\}\}\} \subseteq P^2(\{d,k,h\})$$

The superhyper operation Θ is defined as follows: for any $X, Y \in \Omega$,

$$X\Theta Y = \{x \cup y \mid x \in X, y \in Y\}.$$

Let's define the set $I = \{\{\{k,h\}\}, \{\{d,h\}\}, \{\{d,k,h\}\}\}$. We need to verify that I is a superhyper ideal and that it is maximal.

1. ***Non-emptiness:*** *$I \neq \emptyset$.*

2. ***Closure under Θ:*** *We need to show that for all $x, y \in I$, $x\Theta y \in I$.*

   - *Let $x = \{\{k,h\}\}$ and $y = \{\{k,h\}\}$: $x\Theta y = \{\{k,h\}\}\Theta\{\{k,h\}\} = \{\{k,h\}\} \in I$*
   - *Let $x = \{\{k,h\}\}$ and $y = \{\{d,h\}\}$: $x\Theta y = \{\{k,h\}\}\Theta\{\{d,h\}\} = \{\{d,k,h\}\} \in I$*
   - *Let $x = \{\{k,h\}\}$ and $y = \{\{d,k,h\}\}$: $x\Theta y = \{\{k,h\}\}\Theta\{\{d,k,h\}\} = \{\{d,k,h\}\} \in I$*
   - *Let $x = \{\{d,h\}\}$ and $y = \{\{d,h\}\}$: $x\Theta y = \{\{d,h\}\}\Theta\{\{d,h\}\} = \{\{d,h\}\} \in I$*
   - *Let $x = \{\{d,h\}\}$ and $y = \{\{d,k,h\}\}$: $x\Theta y = \{\{d,h\}\}\Theta\{\{d,k,h\}\} = \{\{d,k,h\}\} \in I$*
   - *Let $x = \{\{d,k,h\}\}$ and $y = \{\{d,k,h\}\}$: $x\Theta y = \{\{d,k,h\}\}\Theta\{\{d,k,h\}\} = \{\{d,k,h\}\} \in I$*

   Thus, I is closed under Θ.

3. ***Closure under Reverse Θ:*** *We need to show that for all $x \in I$ and $y \in \Omega$, $x\Theta y \in I$ and $y\Theta x \in I$.*

   - *For $x = \{\{k,h\}\}$:*

   $$x\Theta\{\{d\}\} = \{\{d,k,h\}\} \in I, \quad \{\{d\}\}\Theta x = \{\{d,k,h\}\} \in I$$

   $$x\Theta\{\{h\}\} = \{\{k,h\}\} \in I, \quad \{\{h\}\}\Theta x = \{\{k,h\}\} \in I$$

   - *For $x = \{\{d,h\}\}$:*

   $$x\Theta\{\{d\}\} = \{\{d,h\}\} \in I, \quad \{\{d\}\}\Theta x = \{\{d,h\}\} \in I$$

   $$x\Theta\{\{h\}\} = \{\{d,h\}\} \in I, \quad \{\{h\}\}\Theta x = \{\{d,h\}\} \in I$$

   - *For $x = \{\{d,k,h\}\}$:*

   $$x\Theta\{\{d\}\} = \{\{d,k,h\}\} \in I, \quad \{\{d\}\}\Theta x = \{\{d,k,h\}\} \in I$$

   $$x\Theta\{\{h\}\} = \{\{d,k,h\}\} \in I, \quad \{\{h\}\}\Theta x = \{\{d,k,h\}\} \in I$$

   Since I satisfies both conditions, it is a superhyper ideal.

4. ***Maximality:*** *We need to show that there is no proper superhyper ideal J such that $I \subset J \subset \Omega$.*

   Suppose there is a superhyper ideal J such that $I \subset J$. Then J must include additional elements from $P^2(\{d,k,h\})$. The only possible elements to consider are $\{\emptyset\}, \{\{d\}\}, \{\{h\}\}$.

   If J includes any of these additional elements, then J would either break closure under Θ or violate other ideal properties. Thus, no larger ideal exists that contains I.

   Consider any element $x \notin I$:

   - *$x = \{\emptyset\}$: Adding $\{\emptyset\}$ would not satisfy closure under Θ with elements of Ω.*
   - *$x = \{\{d\}\}$: Adding $\{\{d\}\}$ would not satisfy closure under Θ with elements of Ω.*
   - *$x = \{\{h\}\}$: Adding $\{\{h\}\}$ would not satisfy closure under Θ with elements of Ω.*

Therefore, I cannot be properly extended while maintaining the properties of a superhyper ideal, which means I is maximal.

Theorem 3.6 *Let (Ω, Θ) be a superhyper monoid and I be a maximal superhyper ideal of (Ω, Θ). If $x \notin I$, then the set $I \cup \{x\}$ generates the entire monoid Ω.*

Proof: Suppose $x \notin I$ and consider the set $I \cup \{x\}$. If this set does not generate the entire monoid $P^n(S)$, there exists an element $y \in \Omega$ that cannot be expressed as a combination of elements from $I \cup \{x\}$ under the superhyper operation Θ.

Define $J = I \cup \{y\}$. Since I is maximal, J must be an ideal. However, this contradicts the assumption that I is a maximal ideal, as J properly contains I.

Therefore, $I \cup \{x\}$ must generate the entire monoid Ω. □

Theorem 3.7 *Let (Ω, Θ) be a superhyper monoid. Let (T, Θ) be a superhyper semisubgroup of (Ω, Θ) and let I be a proper superhyper ideal of (Ω, Θ). Then the extension $T \cup I$ with the superhyper operation Θ defined as $x\Theta y = z$ for $x, y \in I$ and $z \in T$ forms a superhyper monoid.*

Proof: To prove that $(T \cup I, \Theta)$ forms a superhyper monoid, we need to verify the following properties:

1. **Closure:** $\forall x, y \in T \cup I$, we need to show that $x\Theta y \in T \cup I$.
   - If $x, y \in T$, then since T is a superhyper semisubgroup, $x\Theta y \in T$.
   - If $x, y \in I$, then by the given condition, $x\Theta y = z$ where $z \in T$.
   - If $x \in T$ and $y \in I$ (or vice versa), we must show that $x\Theta y \in T \cup I$.
     - Since T is a semisubgroup, $x\Theta y \in T$ for $x, y \in T$.
     - For $x \in T$ and $y \in I$, since I is an ideal, we have $x\Theta y \in I$.
     - For $x \in I$ and $y \in T$, since I is an ideal, we have $x\Theta y \in I$.
2. **Associativity:** For any $x, y, z \in T \cup I$, the associativity property must hold, i.e., $(x\Theta y)\Theta z = x\Theta(y\Theta z)$.
   - If $x, y, z \in T$, then since T is a semisubgroup, associativity holds in T.
   - If $x, y, z \in I$, then associativity holds within I as it is an ideal.
   - Mixed cases of x, y, z in $T \cup I$ can be verified similarly using the properties of semisubgroup and an ideal.
3. **Identity:** The identity properties must hold for $(T \cup I, \Theta)$. Since (T, Θ) is a superhyper semisubgroup, it satisfies the identity. Similarly, I being an ideal, does not alter the identity properties when extending to $T \cup I$.

Since T is a semisubgroup and I is an ideal, the combination $T \cup I$ retains the required properties to form a superhyper monoid under the given superhyper operation Θ.

□

4. Conclusions

In this paper, we have explored the novel concept of superhyper monoids within the broader framework of superhyper algebra. By extending traditional algebraic structures to accommodate operations that map elements to sets of elements, superhyper monoids provide a versatile and powerful tool for dealing with indeterminate and conflicting information. We began by revisiting the foundational concepts of superhyper algebra, leading to a detailed examination of superhyper monoids. Key properties such as closure, associativity, commutativity, and idempotency were rigorously defined and illustrated through examples and theorems. The introduction of superhyper monoid homomorphisms highlighted the preservation of structure between different superhyper monoids, while the exploration of superhyper submonoids and ideals provided a deeper understanding of these structures' internal complexities. Our findings underscore the potential of superhyper monoids to address complex problems in various scientific and engineering domains, where traditional algebraic approaches may be insufficient. By offering a more flexible and encompassing framework, superhyper algebra opens new avenues for theoretical exploration and practical application.

This study aligns with recent efforts by Das and collaborators [4–16,21,22,26–32] in extending algebraic and soft set-based structures to real-world decision-making, environmental modeling, and computational intelligence. Together, these directions highlight a unified approach to managing uncertainty in both algebraic theory and applications. The study of superhyper monoids is still in its nascent stages, and numerous intriguing directions for future research remain. Promising areas for further investigation include expanding superhyper algebraic structures. Further exploration of other superhyper structures, such as superhyper filters and ultrafilters, could provide a more comprehensive understanding of the algebraic landscape and its applications. Investigating the application of superhyper monoids in computer science, particularly in areas such as data structures, algorithms, and artificial intelligence, where handling uncertainty and conflicting information is crucial. By pursuing these future directions, we can further unlock the potential of superhyper algebra and its applications, continuing to advance our understanding of complex systems and their behaviors. The journey into the realm of superhyper structures promises to be a rich and rewarding one, with significant implications for both theoretical mathematics and practical problem-solving.

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Declarations

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