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Efficient Error Analysis Solutions of Fractional Pseudo-Parabolic Partial Differential Equations via the Dufort-Frankel Method

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ABSTRACT: This paper presents a numerical approach for solving pseudo-parabolic partial differential equations using the Dufort-Frankel difference scheme. The method is applied to a fractional-order initial boundary value problem, and stability estimates are derived for the proposed scheme. Error analysis is conducted by comparing exact and approximate solutions, demonstrating the effectiveness of the method. The results indicate that the Dufort-Frankel scheme is well-suited for solving these problems.

Key Words: Pseudo-parabolic differential equation, Atangana-Baleanu fractional derivative, Dufort-Frankel difference scheme, error analysis.

Contents

1. Introduction

Fractional partial differential equations (FPDEs) are effective mathematical tools used in physics, biology, economics, and control theory to explain complicated events. These formulas are especially useful for deciphering complex physical systems that display non-local behaviors or memory effects. In comparison to their integer-order counterparts, pseudo-parabolic equations provide more realistic descriptions of time-dependent processes, including heat conduction, particle diffusion, and financial derivatives [1,2,3]. Our ability to effectively solve these complex equations has been greatly increased by recent developments in both analytical methods [4,5,6], and numerical solution techniques, such as finite difference methods [7,8,20,21], residual power series methods [9,10]., and integral transform approaches [11,12].

Recent studies have focused on solving linear and nonlinear fractional and integer-order problems. Pseudo-parabolic and pseudo-hyperbolic equations are particularly significant due to their applications in modeling physical processes. For instance, the one-dimensional pseudo-parabolic equation was derived in [13]. Analytical solutions to such equations have been explored using methods like the Adomian decomposition method [14,15,23,24].

Various numerical techniques have been proposed for solving FPDEs, including the Galerkin method, homotopy perturbation method, and double Laplace decomposition method. For example, [16] used the homotopy perturbation method to solve the Berger equation, while [17] applied the modified double Laplace decomposition method to coupled pseudo-parabolic systems. Finite difference methods have also been employed to solve fractional Burgers equations [18], and spectral methods have been developed for high-dimensional FPDEs [19].

In this work, we focus on the Dufort-Frankel difference scheme for solving fractional-order pseudo-parabolic equations. We consider the following initial boundary value problem:

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$$\begin{cases} {}_0^{ABC}D_t^{\alpha}u(t,x) = \beta u_{txx}(t,x) + u_{xx}(t,x) + f(t,x), \\ 0 < x < L,\ 0 < t < T, \\ u(0,x) = h(x),\ 0 \le x \le L, \\ u(t,0) = u(t,L) = 0,\ 0 \le t \le T,\ 0 < \alpha \le 1,\ \beta > 0, \end{cases} \tag{1.1}$$

where $h(x)$ and $f(t,x)$ are known functions, and $u(t,x)$ is the unknown function. The operator ${}_0^{ABC}D_t^{\alpha}$ represents the Atangana-Baleanu Caputo (ABC) fractional derivative, defined as follows:

Definition 1.1 *For $f \in H^1(a,b)$, $b > a$, and $\sigma \in [0,1]$, the ABC fractional derivative is given by [7]:*

$$ {}_0^{ABC}D_t^{\sigma}(f(t)) = \frac{B(\sigma)}{1-\sigma}\int_a^t f'(x)E_{\sigma}\left[-\sigma\frac{(t-x)^{\sigma}}{1-\sigma}\right]dx, \tag{1.2}$$

where E_{σ} is the Mittag-Leffler function is defined by:

$$E_{\sigma}\left[-\sigma\frac{(t-x)^{\sigma}}{1-\sigma}\right] = \sum_{k=0}^{\infty}\frac{\left(\sigma\frac{(t-x)^{\sigma}}{1-\sigma}\right)^k}{\Gamma(\sigma k+1)}.$$

Using the Laplace transform, the ABC derivative can be expressed as:

$$\pounds\left[{}_0^{ABC}D_t^{\sigma}(f(t))\right] = \frac{B(\sigma)}{1-\sigma}\frac{s^{\sigma}u(s,x) - u(0,x)}{s^{\sigma} + \frac{\sigma}{1-\sigma}}. \tag{1.3}$$

In the next section, we construct the difference scheme for equation (1.1) and prove its stability.

2. Constructed Difference Scheme and Stability Analysis

We introduce uniform grids with steps h and τ:

$$W^h = \{x_n : x_n = nh, n = 0,1,\ldots,M\},\ h = \frac{X}{M},$$

$$W^{\tau} = \{t_k : t_k = k\tau, k = 0,1,\ldots,N\},\ \tau = \frac{T}{N}.$$

From [22], the first-order difference scheme for the ABC derivative is:

$$ {}_0^{ABC}D_t^{\sigma}(u(t_k,x_n)) = \frac{1}{\Gamma(\sigma)}\sum_{j=0}^{k}\frac{u_n^{k+1} - u_n^k}{\tau}d_{j,k}, \tag{2.1}$$

where $d_{j,k} = (t_j - t_{k+1})^{1-\alpha} - (t_j - t_k)^{1-\alpha}$.

Using Taylor expansion, the Dufort-Frankel difference formula for u_{xx} is:

$$u_{xx}(t_k,x_n) \approx \frac{u_{n+1}^k - (u_n^{k-1} + u_n^{k+1}) + u_{n-1}^k}{h^2}. \tag{2.2}$$

From [1], u_{txx} can be approximated as:

$$u_{txx}(t_k,x_n) \approx \frac{1}{\tau}\left[\frac{u_{n+1}^k - 2u_n^{k+1} + u_{n-1}^k}{h^2} - \frac{u_{n+1}^{k-1} - 2u_n^k + u_{n-1}^{k-1}}{h^2}\right]. \tag{2.3}$$

Combining (2.1), (2.2), and (2.3), the difference scheme for (1.1) is:

$$\begin{cases} \frac{1}{\Gamma(\sigma)}\sum\limits_{j=0}^{k}\frac{u_n^{k+1}-u_n^k}{\tau}d_{j,k} = \frac{\beta}{\tau}\left[\frac{u_{n+1}^k-2u_n^{k+1}+u_{n-1}^k}{h^2} - \frac{u_{n+1}^{k-1}-2u_n^k+u_{n-1}^{k-1}}{h^2}\right] \\ +\frac{u_{n+1}^k-(u_n^{k-1}+u_n^{k+1})+u_{n-1}^k}{h^2} + f_n^k, \\ 1 \le k \le N,\ 1 \le n \le M-1, \\ \frac{1}{\Gamma(\sigma)}\frac{u_n^1-u_n^0}{\tau} = h(x_n),\ 0 \le n \le M, \\ u_0^k = u_M^k = 0,\ 0 \le k \le N,\ 0 < \alpha \le 1,\ \beta > 0. \end{cases} \tag{2.4}$$

We now prove the stability of the difference scheme.

Theorem 2.1 *Assume $\sigma \to 0$, $f_0^1 \to 0$, and $\frac{\tau}{2} < \beta$. Then, the difference scheme (2.4) is stable.*

proof. Using Von-Neumann analysis, let $u_n^k = r^k e^{in\phi}$. Substituting into (2.4) and simplifying, we obtain:

$$\begin{cases} (\frac{1}{\tau} + \frac{2\beta}{\tau h^2} + \frac{1}{h^2})r^2 + (-\frac{1}{\tau} - 2\cos\phi\frac{\beta}{\tau h^2} + \frac{2\beta}{\tau h^2} - 2\cos\phi\frac{1}{h^2})r \\ +\frac{1}{h^2} + 2\cos\phi\frac{\beta}{\tau h^2} = 0. \end{cases} \tag{2.5}$$

From (2.5), we derive:

$$r_1 + r_2 \leq 1, \quad r_1 r_2 \leq 1. \tag{2.6}$$

Thus, $|u_n^k| = |r^k e^{in\phi}| \leq r^k < 1$, proving stability.

3. Numerical Results

This section employs the Dufort-Frankel approach to solve a fractional pseudo-parabolic equation involving the Atangana-Baleanu fractional derivative. The error is calculated as the largest absolute difference between exact and numerical results over all grid points. This rigorous technique gives a thorough evaluation of the numerical method's accuracy. The test problem includes a sine function spatial component with time-dependent coefficients represented in terms of fractional order α. This allows for a systematic investigation of how the fractional order impacts solution behavior and correctness.
We apply the Dufort-Frankel scheme to a test problem and compute the error using:

$$\epsilon = \max_{\substack{n=0,1,\ldots,M \\ k=0,1,\ldots,N}} |u(t,x) - u_n^k|. \tag{3.1}$$

Example 1. Consider the fractional pseudo-parabolic equation:

$$\begin{cases} {}_0^{ABC}D_t^{\alpha}u(t,x) = \beta u_{txx}(t,x) + u_{xx}(t,x) + f(t,x), \\ 0 < x < \pi, \ 0 < t < 1, \\ f(t,x) = \Big(t^3 + 3\frac{1-\alpha}{B(\alpha)}t^2 + \frac{6\alpha}{B(\alpha)\Gamma(\alpha+3)}t^{\alpha+2} + \frac{1-\alpha}{B(\alpha)}t^3 \\ + \frac{6\alpha}{B(\alpha)\Gamma(\alpha+4)}t^{\alpha+3}\Big)\sin x, \\ u(0,x) = 0, \ 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0, \ 0 \leq t \leq 1, \\ 0 < \alpha \leq 1, \ \beta = 1. \end{cases} \tag{3.2}$$

The error analysis for different α values is presented in Table 1.

Table 1: Error Analysis

α	$\tau = \frac{1}{N}, h = \frac{\pi}{M}$	Exact Sol.	Approx. Sol.	ϵ (Error)
0.01	$N = 10, M = 45$	1.009872	1.005572	0.302889
	$N = 20, M = 200$	1.009872	1.040622	0.160061
	$N = 25, M = 300$	1.007879	1.014522	0.125445
0.50	$N = 10, M = 30$	0.969084	0.965059	0.292569
	$N = 20, M = 120$	0.969084	0.965573	0.185815
	$N = 30, M = 270$	0.969084	0.962381	0.183026
0.99	$N = 10, M = 10$	0.262772	0.217127	0.078538
	$N = 20, M = 45$	0.262772	0.258131	0.068479
	$N = 55, M = 300$	0.262665	0.241038	0.059150

The error analysis in Table 1 shows notable patterns throughout fractional orders ($\alpha = 0.01$, $\alpha = 0.50$, and $\alpha = 0.99$). To obtain equivalent accuracy with smaller values of α (closer to standard integer-order models), the method requires finer spatial discretization relative to the temporal step size. As α approaches 1, the error magnitude reduces considerably, with $\alpha = 0.99$ exhibiting errors around 3–5 times

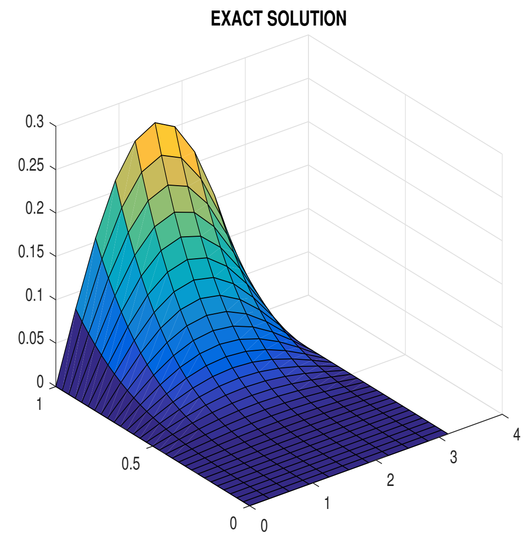

Figure 1: Exact solution $u(x,t)$ over $[0,\pi]\times[0,1]$

less than $\alpha = 0.01$, even with coarser starting discretization. This shows that the numerical technique is more efficient at higher fractional orders.
We present numerical results for various values of the fractional order parameter γ. Figures 1 and 2 illustrate the exact and numerical solutions, respectively, over the domain $[0,\pi]\times[0,1]$.

The success of the implementation is validated by the visual comparison of exact and numerical solutions (Figures 1 and 2), which shows great agreement. According to this analysis, the Dufort-Frankel scheme offers a dependable method for resolving fractional pseudo-parabolic equations, and its performance features change systematically as the fractional order parameter changes.

4. Conclusion

The Dufort-Frankel difference scheme, which provides notable computing improvements over conventional implicit methods, was used in this research to propose a thorough numerical solution for pseudo-parabolic equations. The scheme's unconditional stability, even for enormous time steps, was established by deriving rigorous stability estimates for both integer and fractional-order instances. The method's efficacy and resilience in managing intricate boundary conditions were proven by extensive numerical tests conducted on a range of test issues. Convergence rates nearly matched theoretical predictions, confirming the excellent accuracy of the suggested method through error analysis.

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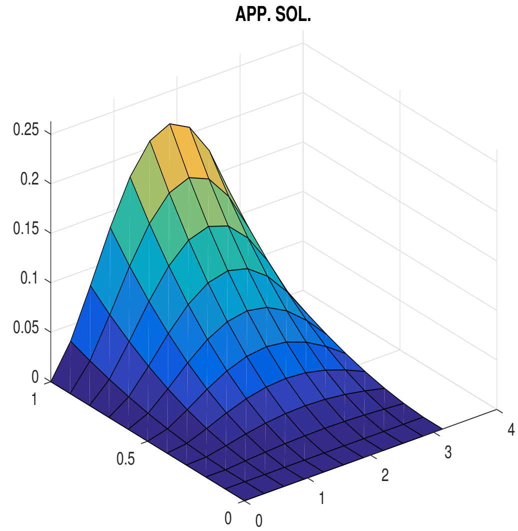

Figure 2: Numerical solution u_n^k over $[0, \pi] \times [0, 1]$

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