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Certain properties of generalized and Higher–order q-Hermite polynomials: monomiality and applications to their zero distributions

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ABSTRACT: In the present paper, we demonstrate 3-variable 2-parameter q-Hermite polynomials via generating functions along with their series definitions, q-derivatives and operational identities, then we deduce some properties for 2-variable 1-parameter q-Hermite polynomials. Also, we present the same mentioned features for multi-index q-Hermite polynomials and their associated formalism. Moreover, we utilize the techniques of quasi-monomial extension to explain and implement q-multiplicative and q-derivative operators for q-Hermite polynomials in three variables and multi-index q-Hermite polynomials. Finally, we present applications that can be derived using these polynomials, where the graphs of the zero functions and the meshes are displayed.

Key Words: Quantum calculus, Extending the monomiality principle, q-Hermite polynomials, 3-variable 2-parameter Hermite polynomials, Higher order Hermite polynomials, q-dilatation operator.

Contents

1. Introduction

Hermite polynomials provide flexible and straightforward solutions to boundary value problems, which have numerous applications in applied sciences, such as probability, numerical analysis and combinatorics, umbral calculus, in the quantum harmonic oscillator and in also to investigate the statistical properties of chaotic light. Moreover, the applications of Hermite polynomials in optic wave transfer and the theory of quantum mechanics problems were identified by Dattoli and his colleagues [10]. The two variables one parameter Hermite polynomials is provided Dattoli and Torre [11] and extension by others. Moreover, the three variables two parameters Hermite polynomials is defined by Subuhi and Rehana Khan [20,21]. The multi-variable and multi-index Hermite polynomials have been applied to the formulation of quantum-phase-space mechanics and applications to wave propagation [22,27] and the study of charged-beam transport issues in classical mechanics [9]. On the topic of polynomial families and their various extensions, a remarkably large amount of research has appeared in the literature (see,

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for example, [1,13,14,23,24]).

Quantum calculus is the modern term for the study of calculus without limits. Jackson's calculus, sometimes recognized as quantum calculus, was invented firstly by Jackson's calculus, sometimes called quantum calculus, was first invented by Jackson [19] and others further developed it as a significant extension of ordinary calculus. The increased requirement for mathematics that simulates quantum computing has recently sparked interest in this area. To begin, we cover some basic notions, notations and outcomes from research in quantum mathematics that will be pertinent to the remainder of the paper's discussion (for $0 < q < 1$).

The q-analogue of every real number can be characterized as [2,17]:

$$[\gamma]_q = \frac{1-q^\gamma}{1-q}.$$

The presented quantity for the q-factorial [2,17]:

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} \ (q \neq 1, n \in \mathbb{N}) \text{ and } [0]_q! := 1,$$

where for $d \in \mathbb{R}$,

$$(d;q)_n = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1}(1-dq^k), & n \in \mathbb{N}. \end{cases}$$

The descriptions for both q-exponential expressions are as follows [2,17]:

$$e_q(x) = \frac{1}{(x(1-q);q)_\infty} = \sum_{l=0}^{\infty} \frac{x^l}{[l]_q!}, \qquad |x| < \frac{1}{1-q}, 0 < q < 1 \tag{1.1}$$

and

$$E_q(x) = (-x(1-q);q)_\infty = \sum_{l=0}^{\infty} q^{\binom{l}{2}} \frac{x^l}{[l]_q!}, \qquad x \in \mathbb{C}, 0 < q < 1.$$

The preceding q-exponential functions are associated with each other as [2,17]:

$$e_q(x)E_q(-x) = 1, \qquad |x| < \frac{1}{1-q}. \tag{1.2}$$

The one that follows is a definition of the q-derivative of a formula f concerning the variable x [18]:

$$D_{q,x}f(x) = \frac{f(qx) - f(x)}{qx - x}, \qquad x \neq 0, \qquad \text{with } (D_{q,x}f)(0) = f'(0).$$

According to [18], we recall some properties of q-derivative by the subsequent formulas [18]:

$$D_{q,x}\, x^n = [n]_q x^{n-1}, \tag{1.3}$$

$$D_{q,x}e_q(\alpha x) = \alpha e_q(\alpha x), \qquad \alpha \in \mathbb{C} \tag{1.4}$$

and the formula of multiplication [18]:

$$D_{q,x}\big(f(x)g(x)\big) = f(x)D_{q,x}g(x) + g(qx)D_{q,x}f(x). \tag{1.5}$$

We were drawn to quantum calculus because of its numerous applications in modeling quantum computing, non-commutative probability, combinatorics, functional analysis, mathematical physics and approximation theory. Very recently, Wani *et al.* provided multi-dimensional q-Hermite polynomials and their monomiality features [28]. Raza and Fadel *et al.* [15,25] defined and built the two- and three-variable q-Hermite polynomials as the specific respective generating functions:

$$e_q(xt)e_q(yt^2) = \sum_{n=0}^{\infty} H_{n,q}(x,y)\frac{t^n}{[n]_q!}$$

and

$$e_q(xt)\ e_q(yt^2)e_q(zt^3) = \sum_{n=0}^{\infty} H_{n,q}(x,y,z)\frac{t^n}{[n]_q!}. \tag{1.6}$$

The q-derivative for $H_{n,q}(x,y,z)$ is given [15]:

$$D_{q,x}H_{n,q}(x,y,z) = [n]_q H_{n-1,q}(x,y,z), \qquad n \geq 1. \tag{1.7}$$

That is how the q-dilatation operator T_z on any function $f(z)$ [16]:

$$T_z^k f(z) = f(q^k z), \qquad k \in \mathbb{R}. \tag{1.8}$$

The monomiality principle allowed for the simple derivation of several properties of traditional and generalized polynomials using the relevant operators. The monomiality concept is offered as a powerful tool for studying the properties of families of special functions and particular polynomials. This sense was initially defined by J.F. Steffensen [26]. Dattoli and C. Cesarado et al. [6,8] developed and extended the concept of quasi-monomiality. In this context, a polynomial $p_n(x)_{n\in\mathbb{N}}$ is considered quasi-monomial if it has two operators acting as derivative operator $\hat{P}\{p_n(x)\} = np_{n-1}(x)$ and multiplicative operator $\hat{M}\ \{p_n(x)\} = p_{n+1,}(x)$.
In [4,12], the respective q-multiplicative and q-derivative operators for $\{p_{n,q}(x)\}_{n=0}^{\infty}$ are realized as:

$$\hat{M}_q\ \{p_{n,q}(x)\} = p_{n+1,q}(x) \tag{1.9}$$

and

$$\hat{P}_q\{p_{n,q}(x)\} = [n]_q p_{n-1,q}(x). \tag{1.10}$$

These q-operators satisfies the subsequent commutation relation [4,12]:

$$[\hat{P}_q, \hat{M}_q] = \hat{P}_q\hat{M}_q - \hat{M}_q\hat{P}_q. \tag{1.11}$$

More specifically, we have [4,12]

$$\hat{M}_q\hat{P}_q\ \{p_{n,q}(x)\} = [n]_q\ p_{n,q}(x). \tag{1.12}$$

Within the framework of (1.11) and (1.12), we note that [4,12]:

$$[\hat{P}_q, \hat{M}_q] = [n+1]_q - [n]_q.$$

In addition, according to (1.9), we determine that [4,12]

$$p_{n,q}(x) = \hat{M}_q^n\{p_{0,q}(x)\} = \hat{M}_q^n\{1\},$$

where $p_{0,q}(x) = 1$ symbolizes the q-sequel within the polynomials $p_{n,q}(x)$. In other words [4,12], the polynomials $p_{n,q}(x)$ is provided by a particular generating formula:

$$e_q(\hat{M}_q t)\{1\} = \sum_{n=0}^{\infty}\ p_{n,q}(x)\frac{t^n}{[n]_q!}.$$

This prompted us to generate 3-variable 2-parameter q-Hermite polynomials and multi-index q-Hermite polynomials, as well as investigate the associated formalism. These polynomials play a crucial role in a variety of mathematical and scientific fields, including mathematical physics, quantum-phase-space mechanics, wave propagation, and classical mechanics. Furthermore, it has commonly been used in classical optics to explore overlapping Hermite-Gauss modes and quantum mechanics to study transition matrix elements with harmonic oscillators. Scientists will pay particular attention to the 3-variable 2-parameter q-Hermite polynomials and multi-index q-Hermite polynomials, which stem from their q-analogue investigation and have identified novel and exciting opportunities, giving the potential of extending the theory of q-special functions, which have garnered more attention. They are useful tools for the formulation of quantum-phase-space mechanics and applications to wave propagation in the study of charged-beam transport concerns in classical mechanics, as well as resolving a number of mathematical issues, due to their adaptability and application. Similarly, the recent revelation of the huge importance of q-Hermite polynomials in a variety of areas have opened up new avenues for research and application. In the next sections, we demonstrate 3-variable 2-parameter q-Hermite polynomials via generating functions along with their series definitions, q-derivatives, operational identities, then we deduce some properties for 2-variable 1-parameter q-Hermite polynomials. Also, we present the same mentioned features for multi-index q-Hermite polynomials and their associated formalism. Moreover, we utilize the techniques of quasi-monomial extension to explain and implement q-multiplicative and q-derivative operators for q-Hermite polynomials in three variables and multi-index q-Hermite polynomials.

2. Generalized and Higher orders q-Hermite polynomials

In this part, we create 3-variable 2-parameter q-Hermite polynomials $H_{n,q}(x,y,z;s_1,s_2)$ through generating formula along with their series definition and q-derivative relations, then we deduce the same properties for 2-variable 1-parameter q-Hermite polynomials $H_{n,q}(x,y;s_1)$. Also, we present certain results for multi-dimensional and multi-index q-Hermite polynomials with their associated formalism.

2.1. Three variables two parameters q-Hermite polynomials

We create 3V2PqHP $H_{n,q}(x,y,z;s_1,s_2)$ via the corresponding generating value:

$$e_q(xt)e_q(ys_1t^2)e_q(zs_2t^3) = \sum_{n=0}^{\infty} H_{n,q}(x,y,z;s_1,s_2)t^n. \tag{2.1}$$

After utilizing equation (1.1) to expand the previously given equation and comparing the corresponding powers of t of each component of the ensuing equation, we receive the subsequent series description of 3V2PqHP $H_{n,q}(x,y,z;s_1,s_2)$:

$$H_{n,q}(x,y,z;s_1,s_2) = \sum_{r=0}^{[n/3]} \sum_{k=0}^{[(n-3r)/2]} \frac{(zs_2)^r (ys_1)^k x^{n-3r-2k}}{[r]_q![k]_q![n-3r-2k]_q!}. \tag{2.2}$$

Theorem 2.1 *For 3V2PqHP $H_{n,q}(x,y,z;s_1,s_2)$, the following facts about q-partial derivatives are valid:*

$$D_{q,x}H_{n,q}(x,y,z;s_1,s_2) = H_{n-1,q}(x,y,z;s_1,s_2), \qquad n \geq 1, \tag{2.3}$$

$$D_{q,y}H_{n,q}(x,y,z;s_1,s_2) = s_1H_{n-2,q}(x,y,z;s_1,s_2), \qquad n \geq 2, \tag{2.4}$$

$$D_{q,z}H_{n,q}(x,y,z;s_1,s_2) = s_2H_{n-3,q}(x,y,z;s_1,s_2) \qquad n \geq 3, \tag{2.5}$$

$$D_{q,s_1}H_{n,q}(x,y,z;s_1,s_2) = yH_{n-2,q}(x,y,z;s_1,s_2), \qquad n \geq 2, \tag{2.6}$$

$$D_{q,s_2}H_{n,q}(x,y,z;s_1,s_2) = zH_{n-3,q}(x,y,z;s_1,s_2), \qquad n \geq 3. \tag{2.7}$$

Proof: Considering the q-partial derivative for each aspect of (2.1) with respect to x via equation (1.4), we receive

$$\sum_{n=0}^{\infty} D_{q,x}H_{n,q}(x,y,z;s_1,s_2)t^n = te_q(xt)e_q(ys_1t^2)e_q(zs_2t^3). \tag{2.8}$$

Implementing the equation (2.1) upon the right portion of the equation (2.8), we yield

$$\sum_{n=0}^{\infty} D_{q,x} H_{n,q}(x,y,z;s_1,s_2)t^n = \sum_{n=0}^{\infty} H_{n,q}(x,y,z;s_1,s_2)t^{n+1}.$$

Thereby, when the matching quantities of t of each part are compared, we determine assertion (2.3).

Afterward, considering the q-partial derivatives for each part of the equation (2.1) concerning $y, z.s_1$ and s_2. Then again, carry out the steps in the equation's proof (2.3). This will yield the assertions (2.4), (2.5), (2.6) and (2.7), in that order.

Theorem 2.1 has been fully proved. □

Remark 2.1 *We obtain the generating function for 2V1PqHP $H_{n,q}(x,y;s_1)$ by taking $z=0$ in equation (2.1) as follows:*

$$e_q(xt)e_q(ys_1t^2) = \sum_{n=0}^{\infty} H_{n,q}(x,y;s_1)t^n. \tag{2.9}$$

Expanding a previous equation via (1.1) then juxtaposing identical powers of t on the each part, provides the series description for 2V1PqHP $H_{n,q}(x,y;s_1)$:

$$H_{n,q}(x,y;s_1) = \sum_{r=0}^{[n/2]} \frac{x^{n-2r}(ys_1)^r}{[r]_q![n-2r]_q!}. \tag{2.10}$$

By differentiating equation (2.9), with respect to x, y, s_1, we obtain the respective subsequent q-derivatives for 2V1PqHP $H_{n,q}(x,y;s_1)$:

$$D_{q,x}H_{n,q}(x,y;s_1) = H_{n-1,q}(x,y;s_1), \qquad n \geq 1, \tag{2.11}$$

$$D_{q,y}H_{n,q}(x,y;s_1) = s_1 H_{n-2,q}(x,y;s_1), \qquad n \geq 2$$

and

$$D_{q,s_1}H_{n,q}(x,y;s_1) = yH_{n-2,q}(x,y;s_1), \qquad n \geq 2.$$

From equation (2.10), equation (2.2) gives the equivalent series relation for $H_{n,q}(x,y,z;s_1;s_2)$:

$$H_{n,q}(x,y,z;s_1,s_2) = \sum_{r=0}^{[n/3]} \frac{(s_2z)^r H_{n-3r,q}(x,y;s_1)}{[r]_q!}. \tag{2.12}$$

2.2. Multi-index q-Hermite polynomials

The multi-index Hermite polynomials $h_{m,n}(x,y;\tau)$ (also known incomplete form of two-variable two-index Hermite polynomials) is provided via a subsequent generating formula [3,5,7]:

$$\sum_{m,n=0}^{\infty} h_{m,n}(x,y;\tau)\frac{u^m v^n}{m!n!} = exp(\tau uv + xu + yv), \qquad m,n \geq 0, \tau \in \mathbb{R}.$$

In view of previous equation, we generate incomplete 2-dimensional q-Hermite polynomials $h_{m,n,q}(x,y;\tau)$ via a subsequent generating formula:

$$\sum_{m,n=0}^{\infty} h_{m,n,q}(x,y;\tau)\frac{u^m v^n}{[m]_q![n]_q!} = e_q(\tau uv)e_q(xu)e_q(yv), \qquad m,n \geq 0, \tau \in \mathbb{R}. \tag{2.13}$$

By extending the right part of the earlier formula and then matching the corresponding values of t on each part of the resultant formula, we obtain the subsequent series formula:

$$h_{m,n,q}(x,y;\tau) = [m]_q![n]_q! \sum_{s=0}^{min[m,n]} \frac{\tau^s x^{m-s} y^{n-s}}{[m-s]_q![n-s]_q![s]_q!}. \tag{2.14}$$

The boundaries condition for $h_{m,n,q}(x,y;\tau)$ is derived by inserting $\tau = 0$ into the formula (2.13):

$$h_{m,n,q}(x,y;0) = x^m y^n.$$

The theorem that is employed for demonstrating the q-partial derivatives for $h_{m,n,q}(x,y;\tau)$:

Theorem 2.2 *For $h_{m,n,q}(x,y;\tau)$, the subsequent q-partial derivatives are valid:*

$$D_{q,\tau} h_{m,n,q}(x,y,\tau) = [m]_q[n]_q h_{m-1,n-1,q}(x,y;\tau), \qquad m,n \geq 1, \tag{2.15}$$

$$D_{q,x} h_{m,n,q}(x,y;\tau) = [m]_q h_{m-1,n,q}(x,y;\tau), \qquad m \geq 1, \tag{2.16}$$

$$D_{q,y} h_{m,n,q}(x,y;\tau) = [n]_q h_{m,n-1,q}(x,y;\tau), \qquad n \geq 1. \tag{2.17}$$

Proof: Considering the q-partial derivative of every part on formula (2.13) with the parameter τ using formula (1.4), we receive

$$\sum_{m,n=0}^{\infty} D_{q,\tau} h_{m,n,q}(x,y;\tau) \frac{u^m v^n}{[m]_q![n]_q!} = uve_q(\tau uv)e_q(xu)e_q(yv). \tag{2.18}$$

So using the formula (2.13) onto the right half within the formula (2.18), that we gain

$$\sum_{m,n=0}^{\infty} D_{q,\tau} h_{m,n,q}(x,y;\tau) \frac{u^m v^n}{[m]q![n]_q!} = \sum_{m,n=0}^{\infty} h_{m,n,q}(x,y;\tau) \frac{u^{m+1} v^{n+1}}{[m]_q![n]_q!}.$$

As a result, comparing the connected values of t across every part produces statement (2.15).

Afterward, we calculate the q-partial derivatives for every value of the formula (2.13) in terms of x and y. This offers us

$$\sum_{m,n=0}^{\infty} D_{q,x} h_{m,n,q}(x,y;\tau) \frac{u^m v^n}{[m]_q![n]_q!} = ue_q(\tau uv)e_q(xu)e_q(yv),$$

$$\sum_{m,n=0}^{\infty} D_{q,y} h_{m,n,q}(x,y;\tau) \frac{u^m v^n}{[m]_q![n]_q!} = ve_q(\tau uv)e_q(xu)e_q(yv). \tag{2.19}$$

Then, to produce assertions (2.16) as well as (2.17), respectively, we replicate the procedure described in the equation's proof (2.15).

Theorem 2.2 has been fully proved. □

3. Operational forms

In this part, we establish the operational definitions for 2-variable 1-parameter q-Hermite polynomials $H_{n,q}(x,y;s_1)$ and 3-variable 2-parameter q-Hermite polynomials $H_{n,q}(x,y,z;s_1,s_2)$ with provide some of their analogous. Also, we provide the operational definition for multi-index q-Hermite polynomials $h_{m,n,q}(x,y;\tau)$.

It has become clear that the use of operational identities has facilitated the study of q-special polynomials, as we obtain the operational definitions for 2V1PqHP $H_{n,q}(x,y;s_1)$ and 3V2PqHP $H_{n,q}(x,y,z;s_1,s_2)$.

Presently, we arrive at the subsequent outcome:

Theorem 3.1 *The 2-variable 1-parameter q-Hermite polynomials $H_{n,q}(x,y;s_1)$ and 3-variable 2-parameter q-Hermite polynomials $H_{n,q}(x,y,z;s_1,s_2)$ satisfy the following respective operational identities:*

$$H_{n,q}(x,y;s_1) = e_q(s_1 y D^2_{q,x}) \ \frac{x^n}{[n]_q!} \tag{3.1}$$

and

$$H_{n,q}(x,y,z;s_1,s_2) = e_q(s_1 y D^2_{q,x}) e_q(s_2 z D^3_{q,x}) \frac{x^n}{[n]_q!}, \tag{3.2}$$

where $D^2_{q,x}$ and $D^3_{q,x}$ are the 2^{nd} and 3^{rd} q-derivative operators.

Proof: In view of equation (1.3), we have

$$D^{2r}_{q,x} \ x^n = \frac{[n]_q!}{[n-2r]_q!} x^{n-2r}.$$

Utilizing the preceding equation to the right aspect of the formula (2.10), we acquire

$$H_{n,q}(x,y;s) = \sum_{r=0}^{\infty} \frac{\left(s_1 y D^2_{q,x}\right)^r x^n}{[r]_q! [n]_q!}.$$

Utilizing the expression (1.1) on the right part of the preceding equation, we arrive at the statement (3.1).

In view of equation (2.11), we have

$$D^{3r}_{q,x} \ H_{n,q}(x,y;s_1) = H_{n-3r,q}(x,y;s_1).$$

Utilizing the preceding formula on the right aspect of the formula (2.12), we gain

$$H_{n,q}(x,y,z;s_1,s_2) = \sum_{r=0}^{\infty} \frac{(s_2 z)^r D^{3r}_{q,x} H_{n,q}(x,y;s_1)}{[r]_q!},$$

or, equivalently

$$H_{n,q}(x,y,z;s_1,s_2) = \sum_{r=0}^{\infty} \frac{\left(s_2 z D^3_{q,x}\right)^r H_{n,q}(x,y;s_1)}{[r]_q!}.$$

Employing (1.1) to the right aspect of the preceding formula produces

$$H_{n,q}(x,y,z;s_1,s_2) = e_q(s_2 z D^3_{q,x}) H_{n,q}(x,y;s_1).$$

Using equation (3.1), we get the statement (3.2).

Theorem 3.1 has been fully proved. □

The operational definitions (3.1) and (3.2) greatly simplifies the study of the properties of 2V1PqHP $H_{n,q}(x,y;s_1)$ and 3V2PqHP $H_{n,q}(x,y,z;s_1,s_2)$ and of their generalizations as well as from these definitions. We can now establish the analogous operational identities for 2V1PqHP $H_{n,q}(x,y;s_1)$ and 3V2PqHP $H_{n,q}(x,y,z;s_1,s_2)$.

Corollary 3.1 *The 2V1PqHP $H_{n,q}(x,y;s_1)$ and 3V2PqHP $H_{n,q}(x,y,z;s_1,s_2)$ satisfy the following equivalent operational identities:*

$$E_q(-s_1 y D^2_{q,x}) H_{n,q}(x,y;s_1) = \frac{x^n}{[n]_q!}, \tag{3.3}$$

$$e_q(s_1 z D^2_{q,x}) H_{n,q}(x,y;s_1) = H_{n,q}(x,y+z;s_1), \tag{3.4}$$

$$E_q(-s_2 z D^3_{q,x}) H_{n,q}(x,y,z;s_1,s_2) = H_{n,q}(x,y;s_1), \tag{3.5}$$

$$E_q(-s_1 y D^2_{q,x}) H_{n,q}(x,y,z;s_1,s_2) = H_{n,q}(x,z;s_2). \tag{3.6}$$

Proof: In view of operational definition (3.1) and formula (1.2), we get assertion (3.3).

Replacing y by $y+z$ in equation (2.9), we have

$$e_q(xt)e_q(ys_1t^2)e_q(zs_1t^2) = \sum_{n=0}^{\infty} H_{n,q}(x, y+z; s_1)t^n.$$

Using equation (2.9) in the aforementioned equation, we have

$$e_q(s_1zt^2)\sum_{n=0}^{\infty} H_{n,q}(x, y; s_1)t^n = \sum_{n=0}^{\infty} H_{n,q}(x, y+z; s_1)t^n. \tag{3.7}$$

Using equation (2.8) for $z = 0$ on the left-part for formula (3.7), then, comparing the values of coefficients powers of t that are equal to the two aspects of the consequent formula, we arrive at statement (3.4).

We obtain assertion (3.5) by operating $E_q(-s_2zD^3_{q,x})$ on each aspect of formula (3.2) and then using formula (3.1) in the equation that results.

Similarly, we obtain affirmation (3.6) by operating $E_q(-s_1yD^2_{q,x})$ over each aspect of formula (3.2), then employing formula (3.1) within the outcome of the formula.

Corollary (3.1) has a thorough proof. □

Theorem 3.2 *Regarding incomplete 2-dimensional q-Hermite polynomials $h_{m,n,q}(x, y; \tau)$, the operational definition afterward is true:*

$$h_{m,n,q}(x, y; \tau) = e_q(\tau D_{q,x}D_{q,y})\ x^m y^n. \tag{3.8}$$

where $D_{q,x}$ and $D_{q,y}$ are the q-derivative operators.

Proof: In view of equation (1.3), we have

$$D^s_{q,x}D^s_{q,y}\ x^m y^n = \frac{[m]_q!}{[m-s]_q!}\frac{[n]_q!}{[n-s]_q!}x^{m-s}y^{n-s}.$$

With the use of the previous equation on the right part of the formula (2.13), we acquire

$$h_{m,n,q}(x, y; \tau) = \sum_{s=0}^{min[m,n]} \frac{\tau^s D^s_{q,x}D^s_{q,y}\ x^m y^n}{[s]_q!},$$

or, equivalently

$$h_{m,n,q}(x, y; \tau) = \sum_{s=0}^{min[m,n]} \frac{\left(\tau D_{q,x}D_{q,y}\right)^s x^m y^n}{[s]_q!}.$$

We gain the statement (3.8) through the use of the expression (1.1) for the right part of the previous formula.

We have finished proving Theorem 3.2. □

4. Quasi-monomials characteristics

Extending the concept of quasi-monomials to q-special functions has enabled research into their features, particularly several q-operators and other key identities. In this part, we cultivate the $H_{n,q}(x,y,z)$ and $h_{m,n,q}(x,y,\tau)$ are quasi-monomials.

Theorem 4.1 *The 3VqHP $H_{n,q}(x,y,z)$ is subjected to the following q-multiplicative and q-derivative operator, which produces quasi-monomials:*

$$\hat{M}_{3VqH} = xT_yT_z + yD_{q,x}T_z\Big(1+qT_y\Big) + zD^2_{q,x}\Big(1+qT_z+q^2T_z^2\Big) \tag{4.1}$$

and

$$\hat{P}_{3VqH} = D_{q,x}, \tag{4.2}$$

respectively, where T_x, T_y and T_z denote the q-dilatation operators given by equation (1.8).

Proof:

First, we recall the q-derivative of the two aspects of equation (1.6), (see page 10, equation 53 [15]):

$$\begin{aligned}
&\sum_{n=1}^{\infty} H_{n,q}(x,y,z)\frac{t^{n-1}}{[n-1]_q\,!}\\
&= xe_q(xt)e_q(qyt^2)e_q(qzt^3) + yte_q(xt)\ e_q(yt^2)e_q(qzt^3) + qyte_q(xt)e_q(qyt^2)e_q(qzt^3)\\
&+ zt^2e_q(xt)e_q(yt^2)e_q(zt^3) + qzt^2\ e_q(xt)e_q(yt^2)e_q(qzt^3) + q^2zt^2\ e_q(xt)e_q(yt^2)e_q(q^2zt^3).
\end{aligned}$$

We get at the claim (4.1) by attaching the formula (1.6) onto the right part of our previous formula, employing formula (1.8), and then comparing the two corresponding powers of t across both parts of our result formula.

In view of formula (1.10), formula (1.7), gives assertion (4.2).

Theorem 4.1 has been fully proved. □

Theorem 4.2 *The following q-differential equation for 3VqHP $H_{n,q}(x,y,z)$ hold true:*

$$\Big(xT_yT_zD_{q,x} + yD_{q,x}T_z\Big(D_{q,x}+qT_yD_{q,x}\Big) + zD^2_{q,x}\Big(D_{q,x}+qT_zD_{q,x}+q^2T_z^2D_{q,x}\Big) - [n]_q\Big)H_{n,q}(x,y,z) = 0. \tag{4.3}$$

Proof: In view of equations (1.12), (4.1) and (4.2), we have

$$\Big(xT_yT_z + yD_{q,x}T_z\Big(1+qT_y\Big) + zD^2_{q,x}\Big(1+qT_z+q^2T_z^2\Big)\Big)D_{q,x}H_{n,q}(x,y,z) = [n]_qH_{n,q}(x,y,z).$$

According to the equation previously, (4.3) is true.

Theorem 4.2 has been fully proved. □

Theorem 4.3 *The quasi-monomials for incomplete 2-dimensional q-Hermite polynomials $h_{m,n,q}(x,y;\tau)$ are affected via the subsequent q-multiplicative and q-derivative operators:*

$$\hat{M}_{xqh} = xT_\tau + \tau D_{q,y}, \tag{4.4}$$

or, alternatively

$$\hat{M}_{xqh} = x + \tau T_\tau D_{q,y}, \tag{4.5}$$

and

$$\hat{M}_{yqh} = yT_\tau + \tau D_{q,x}, \tag{4.6}$$

or, alternatively

$$\hat{M}_{yqh} = y + \tau T_\tau D_{q,x} \tag{4.7}$$

and

$$\hat{P}_{qh} = D_{q,\tau}. \tag{4.8}$$

Here T_x and T_y indicate for q-dilatation operators specified in formula (1.8).

Proof: Considering equation (1.5) to evaluate the q-partial derivative of both sides of equation (2.13) with respect to u, we acquire

$$\sum_{m,n=0}^{\infty} h_{m,n,q}(x,y;\tau) D_{q,u} \frac{u^m v^n}{[m]_q![n]_q!} = e_q(q\tau uv) D_{q,u} e_q(xu) e_q(yv) + D_{q,u} e_q(\tau uv) e_q(xu) e_q(yv),$$

which on using equation (1.4), we get

$$\sum_{m=1,n=0}^{\infty} h_{m,n,q}(x,y;\tau) \frac{u^{m-1} v^n}{[m-1]_q![n]_q!} = x e_q(q\tau uv) e_q(xu) e_q(yv) + \tau v e_q(\tau uv) e_q(xu) e_q(yv). \tag{4.9}$$

Using equations (2.13) and (1.8) in the right part of equation (4.9), we get

$$\sum_{m=1,n=0}^{\infty} h_{m,n,q}(x,y;\tau) \frac{u^{m-1} v^n}{[m-1]_q![n]_q!} = \sum_{n=0}^{\infty} \big(xT_\tau + \tau v\big) h_{m,n,q}(x,y;\tau) \frac{u^m v^n}{[m]_q![n]_q!}. \tag{4.10}$$

From formula (2.19) then we compare the coefficients that have same powers of t on each part of formula (4.10), we gain

$$h_{m+1,n,q}(x,y;\tau) = \big(xT_\tau + \tau D_{q,y}\big) h_{m,n,q}(x,y;\tau),$$

which from formula (1.9), we get assertion (4.4).

Similarly, differentiating both sides of equation (2.13) with respect to u by using equation (1.5) for $f_q(t) = e_q(\tau uv)$ and $g_q(t) = e_q(xu)e_q(yv)$, then using equation (2.13), then comparing the coefficients of equal powers of t from both sides of the resultant equation, we obtain

$$h_{m+1,n,q}(x,y;\tau) = \big(x + \tau T_\tau v\big) h_{m,n,q}(x,y;\tau).$$

In considering formula (1.9), we have assertion (4.5).

Moreover, to obtain the claims (4.6) and (4.7), we employ the q-derivative of each side of formula (2.13) for v and apply the procedure for claims (4.4) and (4.5).

From equations (1.10) and (2.15), gives assertion (4.8).

Theorem 4.3 has been fully proved. □

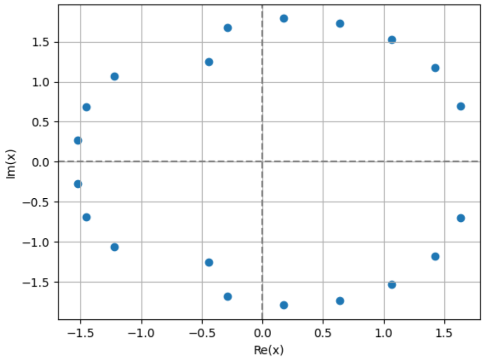
5. Some values with graphical representations and zeros of the two parameters three variables q-Hermite polynomials

We present graphical representations of selected zeros for the two parameters three variables q-Hermite polynomials, denoted by $H_{n.q}(x,y;s_1.s_2)$. Additionally, we employ a Python script to provide illustrative examples that further substantiate the existence and structure of these polynomial families.

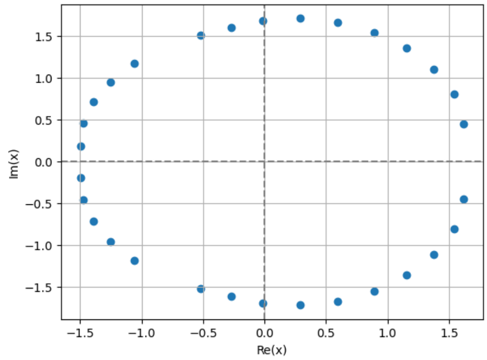
For any $n \in \mathbb{N}_0$, $s_1 = 1$, $s_2 = 2$, the first few two parameters three variables q-Hermite polynomials are given as:

$$
\begin{aligned}
&H_{0,q}(x,y,z;1,1) = 1,\\
&H_{1,q}(x,y,z;1,1) = x,\\
&H_{2,q}(x,y,z;1,1) = \frac{qy + x^2 + y}{q+1},\\
&H_{3,q}(x,y,z;1,1) = \frac{q^2x^3 - 2qx^3 + x^3 + (xy+z)(q^5 - q^3 - q^2 + 1)}{q^5 - q^3 - q^2 + 1},\\
&H_{4,q}(x,y,z;1,1) = \frac{\begin{aligned}&x^4(q^2-1)(q^3 - 3q^2 + 3q - 1)\\ &+ xz(q^2-1)(q^9 - q^7 - q^6 - q^5 + q^4 + q^3 + q^2 - 1)\\ &+ y(qx^2 + qy - x^2 - y)(q^9 - q^7 - q^6 - q^5 + q^4 + q^3 + q^2 - 1)\end{aligned}}{(q^2-1)(q^9 - q^7 - q^6 - q^5 + q^4 + q^3 + q^2 - 1)}.
\end{aligned}
$$

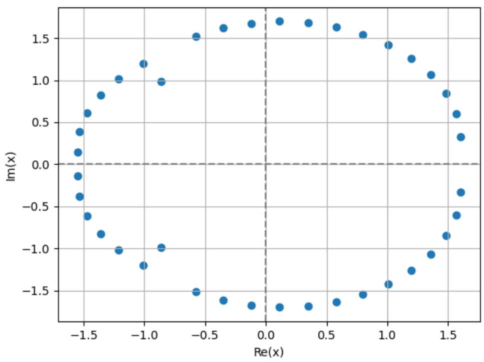
Subsequently, we present numerical values of the zeros of these polynomial families, obtained by assigning specific values to the parameters. These computations allow us to illustrate their behavior and distribution. The corresponding graphical representations of $H_{n,q}(x,y,z;s_1,s_2) = 0$ for $s_1 = 1$, $s_2 = 1$ and $q = \frac{1}{2}$ are provided in Figure1.



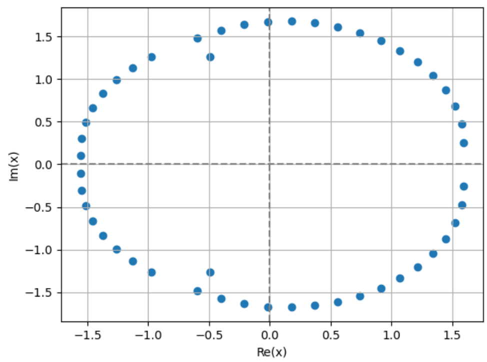
(a) Zeros of $H_{20,\frac{1}{2}}(x, y, z; 1, 1) = 0$



(b) Zeros of $H_{30,\frac{1}{2}}(x, y, z; 1, 1) = 0$



(c) Zeros of $H_{40,\frac{1}{2}}(x, y, z; 1, 1) = 0$



(d) Zeros of $H_{50,\frac{1}{2}}(x, y, z; 1, 1) = 0$

Figure 1: Zeros of $H_{n,\frac{1}{2}}(x, y, z; 1, 1) = 0$, for $n = 20, 30, 40, 50$

In Figure 1, we set $y = 1$,$z = 1$, $s_1 = 1$, $s_2 = 1$ and $q = \frac{1}{2}$ while varying the order of the polynomial to examine the behavior of its zeros. Specifically, we considered different values of n: in the top-left panel, we set $n = 20$; in the top-right panel, $n = 30$; in the bottom-left panel, $n = 40$; and in the bottom-right panel, $n = 50$. This allows us to observe how the distribution of zeros evolves as the polynomial order increases.

Figure 2 illustrates the evolution of the zeros' behavior as the polynomial order increases, specifically for $H_{n,q}(x, y, z; s_1, s_2) = 0$ with $0 \le n \le 50$, considering different values of the variables y, z and parameter q, fixing the parameters $s_1 = 1$ and $s_2 = 1$.

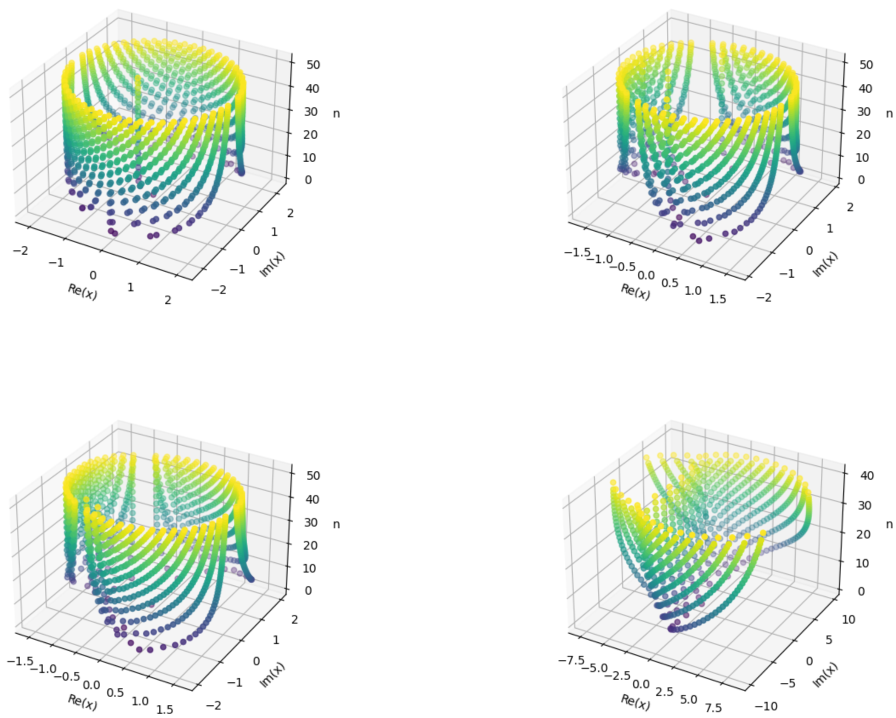


Figure 2: Zeros of $H_{n,q}(x, y, z; s_1, s_2) = 0$, for $n \in [0, 50]$

In Figure 2, we vary n from 0 to 50 while adjusting the variables y,z and the parameter q, keeping $s_1 = s_2 = 1$, to analyze the behavior of the zeros. Specifically, in the top-left panel, we set $y = 4$, $z = 1.5$ and $q = \frac{1}{10}$; in the top-right panel, $y = 1$, $z = 2$ and $q = \frac{1}{3}$; in the bottom-left panel, $y = 1$, $z = 1$ and $q = \frac{1}{2}$; and in the bottom-right panel, $y = 2\pi$, $z = 2$ and $q = 0.9$. This setup allows us to examine how the distribution of zeros evolves as the polynomial order increases.

We then calculated an approximate solution of the two parameters three variables q-Hermite polynomials $H_n(x, y, z; s_1, s_2) = 0$ for $s_1 = 1$, $s_2 = 1$, $y = 1$ and $z = 1$. The results are given in Table 1

Order n	$\mathbf{x}$
0	
1	0
2	$-1.225\mathbf{i},\ 1.225\mathbf{i}$
3	$-0.8029,\ 0.4014 - 1.763\mathbf{i},\ 0.4014 + 1.763\mathbf{i}$
4	$-0.6789 - 0.5377\mathbf{i},\ -0.6789 + 0.5377\mathbf{i},\ 0.6789 - 1.978\mathbf{i},\ 0.6789 + 1.978\mathbf{i}$
5	$-1.314,\ -0.2301 - 1.235\mathbf{i},\ -0.2301 + 1.235\mathbf{i},\ 0.8871 - 1.952\mathbf{i},\ 0.8871 + 1.952\mathbf{i}$
6	$-1.159 - 0.4467\mathbf{i},\ -1.159 + 0.4467\mathbf{i},\ 0.08184 - 1.704\mathbf{i},\ 0.08184 + 1.704\mathbf{i},\ 1.077 - 1.794\mathbf{i},\ 1.077 + 1.794\mathbf{i}$
7	$-1.193,\ -0.8812 - 0.6816\mathbf{i},\ -0.8812 + 0.6816\mathbf{i},\ 0.2153 - 1.94\mathbf{i},\ 0.2153 + 1.94\mathbf{i},\ 1.263 - 1.676\mathbf{i},\ 1.263 + 1.676\mathbf{i}$
8	$-1.428 - 0.5627\mathbf{i},\ -1.428 + 0.5627\mathbf{i},\ -0.2979 - 1.235\mathbf{i},\ -0.2979 + 1.235\mathbf{i},\ 0.3543 - 1.952\mathbf{i},\ 0.3543 + 1.952\mathbf{i},\ 1.371 - 1.561\mathbf{i},\ 1.371 + 1.561\mathbf{i}$
9	$-1.131,\ -1.335 - 0.6983\mathbf{i},\ -1.335 + 0.6983\mathbf{i},\ -0.1011 - 1.671\mathbf{i},\ -0.1011 + 1.671\mathbf{i},\ 0.556 - 1.862\mathbf{i},\ 0.556 + 1.862\mathbf{i},\ 1.446 - 1.436\mathbf{i},\ 1.446 + 1.436\mathbf{i}$
10	$-1.418 - 0.572\mathbf{i},\ -1.418 + 0.572\mathbf{i},\ -0.8137 - 0.777\mathbf{i},\ -0.8137 + 0.777\mathbf{i},\ -0.03268 - 1.857\mathbf{i},\ -0.03268 + 1.857\mathbf{i},\ 0.7546 - 1.836\mathbf{i},\ 0.7546 + 1.836\mathbf{i},\ 1.51 - 1.326\mathbf{i},\ 1.51 + 1.326\mathbf{i}$

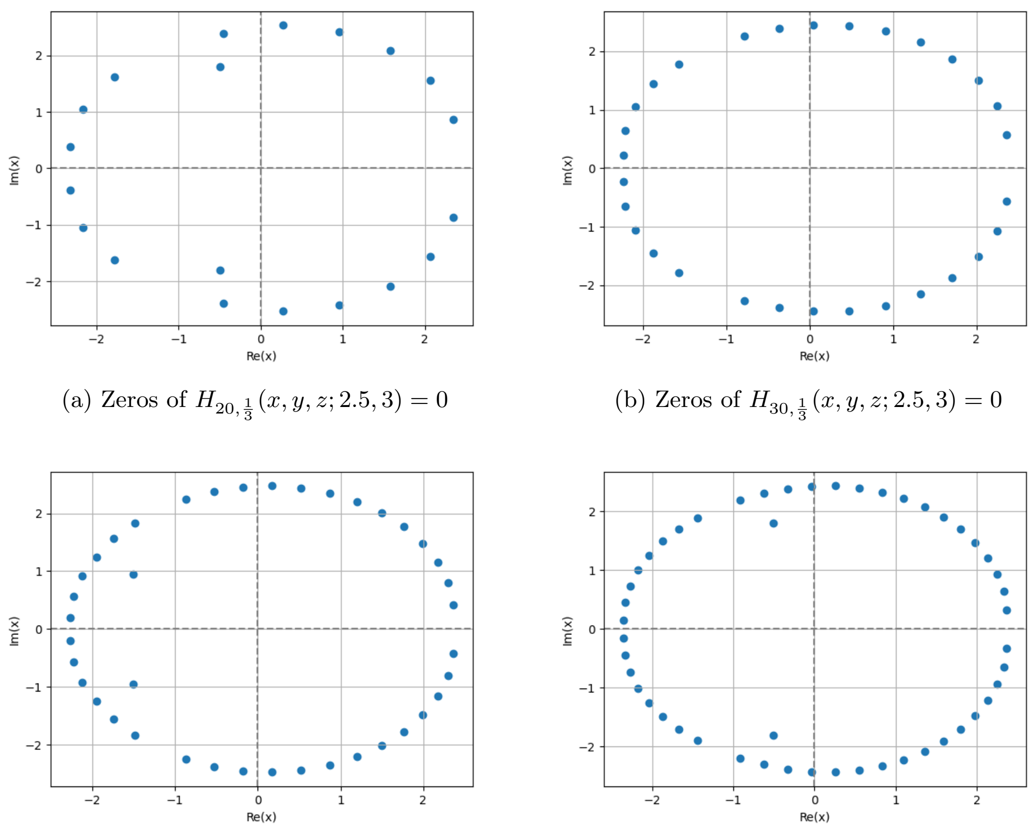
Table 1: Values of x for different two parameters three variables q-Hermite polynomials orders n.

We now present a different set of polynomials and their numerical values of the zeros of the two parameters three variables q-Hermite polynomials, obtained by assigning different values to the parameters.

For any $n \in \mathbb{N}_0$, $s_1 = 2.5$, $s_2 = 3$, the first few two parameters three variables q-Hermite polynomials are given as:

$$
\begin{aligned}
H_{0,q}(x, y, z; 2.5, 3) &= 1,\\
H_{1,q}(x, y, z; 2.5, 3) &= x,\\
H_{2,q}(x, y, z; 2.5, 3) &= \frac{1.0(2.5qy + 1.0x^2 + 2.5y)}{q+1},\\
H_{3,q}(x, y, z; 2.5, 3) &= \frac{x^3(q-1)^2 + (q^2-1)(q^3-1)(2.5xy + 3z)}{(q^2-1)(q^3-1)},\\
H_{4,q}(x, y, z; 2.5, 3) &= \frac{\begin{array}{l} x^4(q-1)^3 \\ + 3xz(q^2-1)(q^3-1)(q^4-1) \\ + y(q-1)(q^3-1)(q^4-1)(2.5x^2 + 6.25y) \end{array}}{(q^2-1)(q^3-1)(q^4-1)}.
\end{aligned}
$$

Subsequently, we present numerical values of the zeros of these polynomial families, obtained by assigning specific values to the parameters. These computations allow us to illustrate their behavior and distribution. The corresponding graphical representations of $H_{n,q}(x, y, z; s_1, s_2) = 0$ for $s_1 = 2.5$, $s_2 = 3$ and $q = \frac{1}{3}$ are provided in Figure3.



(a) Zeros of $H_{20,\frac{1}{3}}(x, y, z; 2.5, 3) = 0$

(b) Zeros of $H_{30,\frac{1}{3}}(x, y, z; 2.5, 3) = 0$

(c) Zeros of $H_{40,\frac{1}{3}}(x, y, z; 2.5, 3) = 0$

(d) Zeros of $H_{50,q=\frac{1}{3}}(x, y, z; 2.5, 3) = 0$

Figure 3: Zeros of $H_{n,\frac{1}{3}}(x, y, z; 2.5, 3) = 0$, for $n = 20, 30, 40, 50$

In Figure 3, we set $y = 1$,$z = 2$, $s_1 = 2.5$, $s_2 = 3$ and $q = \frac{1}{3}$ while varying the order of the polynomial to examine the behavior of its zeros. Specifically, we considered different values of n: in the top-left panel, we set $n = 20$; in the top-right panel, $n = 30$; in the bottom-left panel, $n = 40$; and in the bottom-right panel, $n = 50$. This allows us to observe how the distribution of zeros evolves as the polynomial order increases.

Figure 4 illustrates the evolution of the zeros' behavior as the polynomial order increases, specifically for $H_{n,q}(x, y, z; s_1, s_2) = 0$ with $0 \leq n \leq 50$, considering different values of the variables y, z and parameter q, fixing the parameters $s_1 = 2.5$ and $s_2 = 3$.

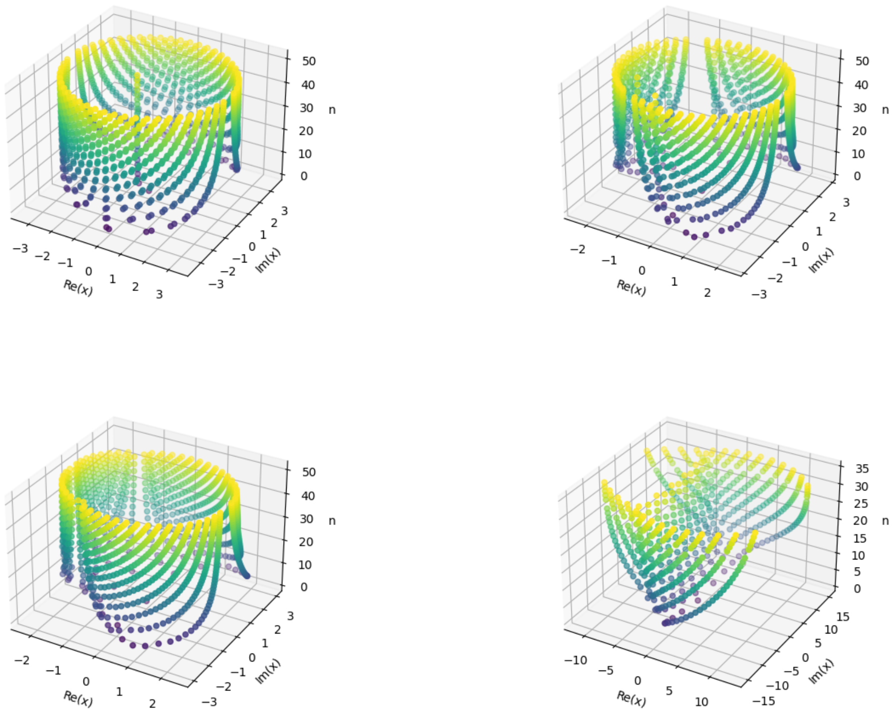


Figure 4: Data visualization of zeros of $H_{n,q}(x, y, z; s_1, s_2) = 0$, for $n \in [0, 50]$

In Figure 4, we vary n from 0 to 50 while adjusting the variables y,z and the parameter q, keeping $s_1 = 2.5$ and $s_2 = 3$, to analyze the behavior of the zeros. Specifically, in the top-left panel, we set $y = 4$, $z = 1.5$ and $q = \frac{1}{10}$; in the top-right panel, $y = 1$, $z = 2$ and $q = \frac{1}{3}$; in the bottom-left panel, $y = 1$, $z = 1$ and $q = \frac{1}{2}$; and in the bottom-right panel, $y = 2\pi$, $z = 2$ and $q = 0.9$. This setup allows us to examine how the distribution of zeros evolves as the polynomial order increases.

We then calculated an approximate solution of the two parameters three variables q-Hermite polynomials $H_n(x, y, z; s_1, s_2) = 0$ for $s_1 = 2.5$, $s_2 = 3$, $y = 1$ and $z = 2$. The results are given in Table 2

Order n	$\mathbf{x}$
0	
1	0
2	$-1.826\mathbf{i},\ 1.826\mathbf{i}$
3	$-1.580,\ 0.7901-2.585\mathbf{i},\ 0.7901+2.585\mathbf{i}$
4	$-1.135-0.4405\mathbf{i},\ -1.135+0.4405\mathbf{i},\ 1.135-2.78\mathbf{i},\ 1.135+2.78\mathbf{i}$
5	$-2.217,\ -0.2929-1.783\mathbf{i},\ -0.2929+1.783\mathbf{i},\ 1.401-2.618\mathbf{i},\ 1.401+2.618\mathbf{i}$
6	$-1.918-0.7316\mathbf{i},\ -1.918+0.7316\mathbf{i},\ 0.2073-2.545\mathbf{i},\ 0.2073+2.545\mathbf{i},\ 1.71-2.286\mathbf{i},\ 1.71+2.286\mathbf{i}$
7	$-1.241,\ -1.651-0.7731\mathbf{i},\ -1.651+0.7731\mathbf{i},\ 0.3485-2.773\mathbf{i},\ 0.3485+2.773\mathbf{i},\ 1.923-2.154\mathbf{i},\ 1.923+2.154\mathbf{i}$
8	$-2.227-0.8689\mathbf{i},\ -2.227+0.8689\mathbf{i},\ -0.3686-1.789\mathbf{i},\ -0.3686+1.789\mathbf{i},\ 0.5678-2.694\mathbf{i},\ 0.5678+2.694\mathbf{i},\ 2.028-1.983\mathbf{i},\ 2.028+1.983\mathbf{i}$
9	$-1.932,\ -1.998-1.097\mathbf{i},\ -1.998+1.097\mathbf{i},\ -0.1122-2.486\mathbf{i},\ -0.1122+2.486\mathbf{i},\ 0.9607-2.519\mathbf{i},\ 0.9607+2.519\mathbf{i},\ 2.116-1.779\mathbf{i},\ 2.116+1.779\mathbf{i}$
10	$-1.908-0.8889\mathbf{i},\ -1.908+0.8889\mathbf{i},\ -1.429-0.7607\mathbf{i},\ -1.429+0.7607\mathbf{i},\ -0.04331-2.661\mathbf{i},\ -0.04331+2.661\mathbf{i},\ 1.188-2.518\mathbf{i},\ 1.188+2.518\mathbf{i},\ 2.193-1.654\mathbf{i},\ 2.193+1.654\mathbf{i}$

Table 2: Values of x for different two parameters three variables q-Hermite polynomials orders n.

6. Conclusions

Traditional Hermite polynomials and their generalizations are well-known for providing flexible and straightforward solutions to boundary value problems. They have numerous applications in applied sciences, including probability, numerical analysis, combinatorics, umbral calculus, quantum harmonic oscillators, optic wave transfer and quantum mechanics theory. Moreover, the multi-variable, multi-index Hermite polynomials pose challenges in the formulation of quantum-phase-space mechanics and its applications to wave propagation, charged-beam transport issues in classical mechanics, classical optics to investigate interspersed Hermite-Gauss modes and study transition matrix elements with harmonic oscillators. In this clarification, we display a weaving of new facets relevant to the 3-variable 2-parameter q-Hermite polynomials, 2-variable 1-parameter q-Hermite polynomials, and multi-index q-Hermite polynomials and their associated formalism. Through rigorous examination, we expose their generating functions, together with their series definitions, q-derivatives, operational identities, and probing into the realm of quasi-monomial extension to and execute q-multiplicative and q-derivative operators for three variables and multi-index q-Hermite polynomials. The analysis of these polynomials has revealed new possibilities for extending the theory of q-special functions, including significant statements and techniques. It has also led to the emergence of new multi-variable and multi-index q-polynomial families and hybrid forms from various classes, paving the way for further research.

7. Data availability.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

8. Conflict of interest.

The authors have no relevant financial or non-financial interests to disclose.

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