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Function spaces under various operators

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ABSTRACT: Various topologies on the function space Y^X will be determined through this paper. To do this, application of generalized open sets will be discussed. Topological ideal is also an applicable part to determine the topologies on Y^X. Topological group and the continuous functions will be helpful to determine the topologies on Y^X (or $C(X,Y)$). This paper also discusses the huge changes of the topologies on Y^X by the small displacement of the generalized open sets from the space Y.

Key Words: Topological group, topological ideal, function space Y^X.

Contents

1. Introduction

The study of function spaces is a unique study for all over fields of mathematics and other pure science subjects. The study of this field is growing so far through [4], [25], [28], etc. The paper further considering the function space and discuss the situation of the space under various limit points as well the $*$ operator's points in the field of ideal topological space. To do this, we consider the generalized open sets and their properties from literature. After obtaining the situation of the function space via local function [11,12] we may supposed to be characterized the Hayashi-Samuel space rigorously.

2. Preliminaries

General definition of point-open topology on Y^X is,

Definition 2.1 *[25] Given a point x of the set X and an open set U of the topological space Y, let*

$$S(x,U) = \{f \in Y^X \mid f(x) \in U\}.$$

The collection of all such sets $S(x,U)$ forms a subbasis for a topology on Y^X. This topology is called the **point-open topology on Y^X**.

In this paper, we shall discuss various new topologies on Y^X. For this, the following generalized open sets are important tools:

Definition 2.2 *A subset A of a topological space Y is said to be*

1. *semiopen [18] if $A \subseteq Cl(In(A))$;*
2. *preopen [19] if $A \subseteq In(Cl(A))$;*

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3. *β-open [9] or semi-preopen[2] if $A \subseteq Cl(In(Cl(A)))$;*

4. *b-open[3] if $A \subseteq In(Cl(A)) \cup Cl(In(A))$;*

where 'In' and 'Cl' denote the usual notion for interior and closure operators respectively.

We denote the collection of all semiopen sets, preopen sets, β-open sets and b-open sets in a topological space Y as $SO(Y)$, $PO(Y)$, $\beta O(Y)$ and $BO(Y)$ respectively. These collections obey the relations, collection of open sets $\subseteq PO(Y) \subseteq BO(Y) \subseteq \beta O(Y)$ and collection of open sets $\subseteq SO(Y) \subseteq BO(Y) \subseteq \beta O(Y)$. The complement of a semiopen (resp. preopen, β-open, b-open) set is addressed as a semi-closed (resp. pre-closed, β-closed, b-closed) set. The intersection of all semi-closed (resp. pre-closed, β-closed, b-closed) sets containing A is called semi-closure (resp. pre-closure, β-closure, b-closure) of A and is denoted as $sCl(A)$ (resp. $pCl(A)$, $\beta Cl(A)$, $bCl(A)$). Also, the union of all semiopen (resp. preopen, β-open, b-open) sets contained in A is called semi-interior (resp. pre-interior, β-interior, b-interior) of A and is denoted as $sIn(A)$ (resp. $pIn(A)$, $\beta In(A)$, $bIn(A)$).

The following is one way to obtain weaker and stronger topologies on Y^X, and it is an introductory result of the paper.

Lemma 2.1 *Suppose σ and σ' are two topologies on the set Y and $\sigma \subseteq \sigma'$. Then, the point-open topology induced by σ' is finer than the point-open topology induced by σ.*

Note that, if σ' is strictly finer than σ, then the point-open topology induced by σ' is strictly finer than the point-open topology induced by σ.

We concentrate on the topologies of the space Y^X through the various limit points and we know that limit points of a set can be derived via ideal [1,17]. Following is a few words about the ideal.

An ideal $\mathbb{I}$ on a topological space (Y, σ) is a collection of subsets of Y and satisfying the hereditary property as well as the finite additivity property. The same was first introduced by Kuratowski [17] in 1933. The study of the local function (or generalization of limit points) is a remarkable part for the study of various topological notions (recently it was determined, the * compactification [15]). It is formally defined as:

$A^\star = \{y \in Y \mid U_y \cap A \notin \mathbb{I},\ U_y \in \sigma(y)\}$, where $\sigma(y)$ is the collection of all open sets of (Y, σ) containing y. Associated set-valued set function [23] of the operator $()^*$ is the operator ψ [22,26] and it has been introduced in the literature by the relation, $\psi(A) = Y \setminus (Y \setminus A)^*$.

Throughout this paper, we denote that $(Y, \sigma, \mathbb{I})$ (or simply $\mathbb{I}_Y$) is an ideal topological space. Furthermore, an ideal $\mathbb{I}$ on the topological space (Y, σ) is called codense ideal [7] (or the ideal topological space $\mathbb{I}_Y$ is called Hayashi-Samuel space [6]) if $\mathbb{I} \cap \sigma = \{\emptyset\}$. Sometimes this type of ideal is called σ boundary ideal [10,27].

3. Function spaces via generalized open sets related operators

Before starting this section, we considered X as a set and Y as a topological space (or simply space).

Lemma 3.1 *[16] Let Y be a topological space. Given a point $x \in X$ and a subset $A \in SO(Y)$ (resp. $PO(Y)$, $\beta O(Y)$, $BO(Y)$), let*

$$S(x, A) = \{f \in Y^X \mid f(x) \in A\}.$$

The sets $S(x, A)$ form a subbasis for a topology on Y^X.

The topology generated by the above subbasis is called **point-semiopen** (resp. **point-preopen, point-β-open, point-b-open**) topology on Y^X.

Theorem 3.1 *[16] Suppose Y is a topological space. Then, the point-semiopen (resp. point-β-open, point-b-open) topology on Y^X is finer than the point-open topology on Y^X.*

Lemma 3.2 *Given a point $x \in X$ and a subset A of the topological space Y, let*

$$S(x, sIn(A)) = \{f \in Y^X \mid f(x) \in sIn(A)\} \text{ (resp. } S(x, pIn(A)) = \{f \in Y^X \mid f(x) \in pIn(A)\},$$
$$S(x, bIn(A)) = \{f \in Y^X \mid f(x) \in bIn(A)\},\ S(x, \beta In(A)) = \{f \in Y^X \mid f(x) \in \beta In(A)\}).$$

The sets $S(x, sIn(A))$ (resp. $S(x, pIn(A))$, $S(x, bIn(A))$, $S(x, \beta In(A))$) form a subbasis for a topology on Y^X.

Proof: Proof is obvious from the fact that $sIn(Y)$ (or $pIn(Y)$, $bIn(Y)$, $\beta In(Y)$) $= Y$. □

The topology generated by the above subbasis is called **point-sIn topology (resp. point-pIn topology, point-bIn topology, point-βIn topology)** on Y^X.

As we know sCl (resp. pCl, bCl, βCl) is associated set-valued set function [23] with sIn (resp. pIn, bIn, βIn), thus if we treat the sets $S(x, sIn(A))$ (resp. $S(x, pIn(A))$, $S(x, bIn(A))$, $S(x, \beta In(A))$) by $\{f \in Y^X | \ f(x) \in X \setminus sCl(X \setminus A)\}$ (resp. $\{f \in Y^X | \ f(x) \in X \setminus pCl(X \setminus A)\}$, $\{f \in Y^X | \ f(x) \in X \setminus bCl(X \setminus A)\}$, $\{f \in Y^X | \ f(x) \in X \setminus \beta Cl(X \setminus A)\}$) or $\{f \in Y^X | \ f(x) \notin sCl(X \setminus A)\}$ (resp. $\{f \in Y^X | \ f(x) \notin pCl(X \setminus A)\}$, $\{f \in Y^X | \ f(x) \notin bCl(X \setminus A)\}$), $\{f \in Y^X | \ f(x) \notin \beta Cl(X \setminus A)\}$, then we also reach the same topology.

Now, we state that 'sCl' (resp. 'pCl', 'bCl', 'βCl') operator makes independently a topology on Y^X by the following:

Lemma 3.3 *Given a point $x \in X$ and a subset A of the topological space Y, let*

$$S(x, sCl(A)) = \{f \in Y^X \mid f(x) \in sCl(A)\} \text{ (resp. } S(x, pCl(A)) = \{f \in Y^X \mid f(x) \in pCl(A)\},$$
$$S(x, bCl(A)) = \{f \in Y^X \mid f(x) \in bCl(A)\},\ S(x, \beta Cl(A)) = \{f \in Y^X \mid f(x) \in \beta Cl(A)\}).$$

The sets $S(x, sCl(A))$ (resp. $S(x, pCl(A))$, $S(x, bCl(A)$, $S(x, \beta Cl(A))$)) form a subbasis for a topology on Y^X.

The topology generated by the above subbasis is called **point-sCl topology (resp. point-pCl topology, point-bCl topology, point-βCl topology)** on Y^X.

As $sIn \sim^Y sCl$ (resp. $pIn \sim^Y pCl$, $bIn \sim^Y bCl$, $\beta In \sim^Y \beta Cl$) [23], thus one can rewrite the above Lemma using 'sIn' (resp. 'pIn', 'bIn', 'βIn') operator.

The point-semiopen (resp. point-preopen, point-b-open, point- β open) topology and the point-sIn topology (resp. point-pIn topology, point-bIn topology, point-βIn topology) on Y^X coincide. Again, it is not always possible that for an basis element $S(y_1, kIn(A_1)) \cap S(y_2, kIn(A_2)) \cdots \cap S(y_j, kIn(A_j))$ there is no basis element $S(y_1, kCl(A_1)) \cap S(y_2, kCl(A_2)) \cdots \cap S(y_j, kCl(A_j))$ in other topology on Y^X such that $S(y_1, kCl(A_1)) \cap S(y_2, kCl(A_2)) \cdots \cap S(y_j, kCl(A_j)) \subseteq S(y_1, kIn(A_1)) \cap S(y_2, kIn(A_2)) \cdots \cap S(y_j, kIn(A_j))$ and vice versa, where $k \in \{s, p, b, \beta\}$. In this point of view, point-sIn topology (resp. point-pIn topology, point-bIn topology, point-βIn topology) and point-sCl topology (resp. point-pCl topology, point-bCl topology, point-βCl topology) are not comparable.

Lemma 3.4 *Given a point $x \in X$ and a subset A of the topological space Y. Then, the sets*

$$S(x, kIn(kCl(A))) = \{f \in Y^X \mid f(x) \in kIn(kCl(A))\}$$

form a subbasis for a topology on Y^X, where $k \in \{s, p, b, \beta\}$.

The topology generated by the above subbasis is called **point-kIn-kCl topology** on Y^X, where $k \in \{s, p, b, \beta\}$.

Due to $kInkCl \sim^Y kClkIn$ [23], we may rewrite the subbasis of the point-interior-closure topology on Y^X using 'kCl' and 'kIn' operators.

Proposition 3.1 *Suppose Y is a topological space. Then, the point-semiopen (resp. point-preopen, point-b-open, point-β-open) topology on Y^X is finer than the point-sIn-sCl (resp. point-pIn-pCl, point-bIn-bCl, point-βIn-βCl) topology on Y^X.*

Proof: Let β_τ and $\beta_{\tau'}$ be bases for point-sIn-sCl topology and point-semiopen topology on Y^X respectively. Let $B = S(x_1, sIn(sCl(A_1))) \cap S(x_2, sIn(sCl(A_2))) \cap \cdots \cap S(x_n, sIn(sCl(A_n)))$ be a member of β_τ and $f \in B$. Then, $f \in S(x_i, sIn(sCl(A_i)))$, $\forall\ i = 1, 2, \cdots, n$. This implies that $f \in S(x_i, U_i)$ where $U_i = sIn(sCl(A_i))$, $\forall\ i = 1, 2, \cdots, n$. So, $f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n) = B' \in \beta_{\tau'}$, as $U_1,\ U_2,\ \cdots,\ U_n$ are semi-open subsets of Y. Thus, for every $f \in B$, $\exists\ B' \in \beta_{\tau'}$ such that $B' \subseteq B$. □

For converse of this proposition, we have:

Let $B_1' = S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n)$ be a member of $\beta_{\tau'}$ and $g \in B_1'$. Then, $g \in S(x_i, U_i) \implies g \in S(x_i, sIn(sCl(U_i)))$ (as $U_i \subseteq sCl(U_i) \implies U_i \subseteq sIn(sCl(U_i)))$, $\forall\ \ i = 1,\ 2,\ \cdots,\ n$. So, $g \in S(x_1, sIn(sCl(U_1))) \cap S(x_2, sIn(sCl(U_2))) \cap \cdots \cap S(x_n, sIn(sCl(U_n))) = B_1 \in \beta_\tau$. Thus, for $B_1' \in \beta_{\tau'}$, we have a B_1 belongs to β_τ. But, $B_1 \subseteq B_1'$ is not true in general. For justification of this statement we give following example:

Example 3.1 *Let (Y, σ) be a topological space, where $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{c\}\}$. Then, $SO(Y) = \{A \subseteq Cl(In(A)) \mid\ \ A \subseteq Y\} = \{\emptyset, Y, \{c\}, \{a, c\}, \{b, c\}\}$. Therefore, $\{sIn(sCl(A)) \mid\ \ A \subseteq Y\} = \{\emptyset, Y\}$ Thus, $\{sIn(sCl(A)) \mid\ A \subseteq Y\}$ is not equal to $SO(Y)$.*

Lemma 3.5 *Given a point $x \in X$ and a subset A of the topological space Y, let*

$$S(x, kCl(kIn(A))) = \{f \in Y^X \mid\ f(x) \in kCl(kIn(A))\}.$$

The sets $S(x, kCl(kIn(A)))$ form a subbasis for a topology on Y^X, where $k \in \{s, p, b, \beta\}$.

The topology generated by the above subbasis is called **point-kCl-kIn topology** on Y^X, where $k \in \{s, p, b, \beta\}$.

For the fact that, for a subset A of X, $kInsCl(A) \subseteq$ (resp. $\supseteq$) $kClsIn(A)$ is not true in general. Thus, point-kIn-kCl topology and point-kCl-kIn topology on Y^X are not comparable.

We learnt from [20], the collection of dense sets and the collection of semi-dense sets are equal. Further, the collection of dense sets and collections of predense sets are not equal except for every dense set D, $In(D) \neq \emptyset$. In this connection we have mentioned that the collection of all rationals $\mathbb{Q}$ in the lower limit topology $\mathbb{R}_l$ is a dense set but $In(\mathbb{Q}) = \emptyset$. Furthermore, for $X = \{t_1, t_2, t_3, t_4\}$, $\tau = \{\emptyset, X, \{t_3, t_4\}, \{t_2, t_3, t_4\}, \{t_1, t_3, t_4\}\}$, $\{t_1, t_4\}$ is a β as well as b-open set but does not contain any non-empty open set. Therefore, the collection of β dense sets, b dense sets and dense sets are equal in general.

Therefore, we have the following lemma:

Lemma 3.6 *Let Y be a topological space. Given a point $x \in X$ and a subset $A \in \mathcal{D}^k(Y)$ (set of all k dense sets in Y), let*

$$S(x, A) = \{f \in Y^X \mid\ f(x) \in A\}.$$

The sets $S(x, A)$ form a subbasis for a topology on Y^X, where $k \in \{p, b, \beta\}$.

The topology generated by the above subbasis is called **point-k dense topology** on Y^X.

4. Role of ideals to make topologies on Y^X

It is well known from [5,10,21,22] that ψ is not an interior operator. The following Lemma shows that this non-interior operator may also be an essential tool for obtaining topologies on Y^X.

Lemma 4.1 *Let $\mathbb{I}$ be a codense ideal on the topological space Y. Given a point $x \in X$ and a subset A of the topological space Y, let*

$$S_{\mathbb{I}}(x, \psi(A^*)) = \{f \in Y^X \mid\ f(x) \in \psi(A^*)\}.$$

The sets $S_{\mathbb{I}}(x, \psi(A^*))$ form a subbasis for a topology on Y^X.

Proof: Let $f \in Y^X$. Then, $f \in S_{\mathbb{I}}(x, Y) = S_{\mathbb{I}}(x, \psi(Y^*)) \subseteq \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i^*))$, where $x_i \in X$ and A_i are subsets of Y. So, $f \in \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i^*))$. Thus, $Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, \psi(A_i^*))$. Hence, the sets $S_{\mathbb{I}}(x_i, \psi(A_i^*))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called **point-$\psi*$ topology** on Y^X.

Due to $\psi* \sim^Y *\psi$ [23], one can rewrite the subbasis of point-$\psi*$ topology on Y^X using $*\psi$-operator. Relations between several topologies on Y^X with point-$\psi*$ topology are followed by following:

Proposition 4.1 *Suppose $\mathbb{I}$ is a codense ideal on the topological space Y. The point-open topology on Y^X is finer than the point-$\psi*$ topology on Y^X.*

Proof: Let β_τ and $\beta_{\tau'}$ be bases for point-$\psi*$ topology and point-open topology on Y^X respectively. Let $B = S_{\mathbb{I}}(x_1, \psi(A_1^*)) \cap S_{\mathbb{I}}(x_2, \psi(A_2^*)) \cap \cdots \cap S_{\mathbb{I}}(x_n, \psi(A_n^*))$ be a member of β_τ and $f \in B$. Then, $f \in S_{\mathbb{I}}(x_i, \psi(A_i^*))$, $\forall\ i = 1, 2, \cdots, n$. This implies that $f \in S(x_i, U_i)$ where $U_i = \psi(A_i^*)$ (as for each i, $\psi(A_i^*)$ is open [10,22]), $\forall\ i = 1, 2, \cdots, n$. So, $f \in S(x_1, U_1) \cap S(x_2, U_2) \cap \cdots \cap S(x_n, U_n) = B' \in \beta_{\tau'}$ as $U_1,\ U_2, \cdots,\ U_n$ are open subsets of Y. Thus, for each $f \in B$, $\exists\ B' \in \beta_{\tau'}$ such that $B' \subseteq B$. □

For the converse of this proposition we give the following example:

Example 4.1 *Let $X = \{a, b\}$ and (Y, σ) be a topological space, where $Y = \{1, 2, 3\}$, $\sigma = \{\emptyset, Y, \{3\}, \{1, 3\}, \{2, 3\}\}$ and $\mathbb{I} = \{\emptyset, \{1\}\}$. All possible functions from X to Y are defined by*
$f_1(a) = 1,\ f_1(b) = 2;$ $f_2(a) = 1,\ f_2(b) = 3;$ $f_3(a) = 2,\ f_3(b) = 3;$ $f_4(a) = 2,\ f_4(b) = 1;$ $f_5(a) = 3,\ f_5(b) = 1;$ $f_6(a) = 3,\ f_6(b) = 2;$ $f_7(a) = 1,\ f_7(b) = 1;$ $f_8(a) = 2,\ f_8(b) = 2;$ $f_9(a) = 3,\ f_9(b) = 3.$
Then, a basis of the point-$\psi*$ topology τ on Y^X is $\beta_\tau = \{\emptyset, Y^X\}$.
A basis of the point-open topology τ' on Y^X is $\beta_{\tau'} = \{\emptyset, Y^X, \{f_9\}, \{f_2, f_9\}, \{f_3, f_9\}, \{f_5, f_9\}, \{f_6, f_9\}, \{f_2, f_3, f_9\}, \{f_5, f_6, f_9\}, \{f_1, f_2, f_6, f_9\}, \{f_2, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_9\}, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\}$.
Here, $f_9 \in \{f_9\} \in \beta_{\tau'}$ but there exists no $B_1 \in \beta_\tau$ such that $f_9 \in B_1 \subseteq \{f_9\}$. Thus, τ is not finer than τ'.

Example 4.2 *Let $(Y, \sigma, \mathbb{I})$ be an ideal topological space, where $Y = \{1, 2, 3\}$, $\sigma = \{\emptyset, Y, \{3\}, \{1, 3\}, \{2, 3\}\}$ and $\mathbb{I} = \{\emptyset, \{1\}\}$. Then, $\{\psi(A^*) \mid\ A \subseteq Y\} = \{\emptyset, Y\}$. In this example, we see that $\{\psi(A^*) \mid\ A \subseteq Y\} \neq \sigma$ on Y.*

Proposition 4.2 *Suppose $\mathbb{I}$ is a codense ideal on the topological space Y. Given a point $x \in X$ and a subset A of the topological space Y, let*

$$S_{\mathbb{I}}(x, (\psi(A))^*) = \{f \in Y^X \mid\ f(x) \in (\psi(A))^*\}.$$

The sets $S_{\mathbb{I}}(x, (\psi(A))^*)$ form a subbasis for a topology on Y^X.

Proof: Let $f \in Y^X$. Then, $f \in S_{\mathbb{I}}(x, Y) = S_{\mathbb{I}}(x, (\psi(Y))^*)$ (as $\mathbb{I}$ is a codense ideal)$\subseteq \bigcup_i S_{\mathbb{I}}(x_i, (\psi(A_i))^*)$, where $x_i \in X$ and A_i are subsets of Y. So, $f \in \bigcup_i S_{\mathbb{I}}(x_i, (\psi(A_i))^*)$. Thus, $Y^X \subseteq \bigcup_i S_{\mathbb{I}}(x_i, (\psi(A_i))^*)$. Hence, the sets $S_{\mathbb{I}}(x_i, (\psi(A_i))^*)$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called **point-$*\psi$ topology** on Y^X.

Theorem 4.1 *Let $\mathbb{I}$ be an ideal on the topological space Y. Given a point $x \in X$, let*

$$S_{\mathbb{I}}(x, A) = \{f \in Y^X \mid\ f(x) \in Cl(\psi(A))\}.$$

The sets $S_{\mathbb{I}}(x, A)$ form a subbasis for a topology on Y^X.

For our next discussion, we will call the topology obtained in the Theorem 4.1 as **point-$L\psi$ topology** on Y^X.

Following is the comparison of point-$L\psi$ topology on Y^X with earlier topologies on Y^X.

Remark 4.1 *Suppose $\mathbb{I}$ is an ideal on the topological space Y. Then, the point-$L\psi$ topology on Y^X and the point-open topology on Y^X are not comparable.*

Example 4.3 *Let $X = \{a, b\}$ and (Y, σ) be a topological space, where $Y = \{1, 2, 3\}$, $\sigma = \{\emptyset, Y, \{2\}, \{3\}, \{2, 3\}\}$ and $\mathbb{I} = \{\emptyset, \{2\}\}$. All possible functions from X to Y are defined by*
$f_1(a) = 1,\ f_1(b) = 2;\ f_2(a) = 1,\ f_2(b) = 3;\ f_3(a) = 2,\ f_3(b) = 3;\ f_4(a) = 2,\ f_4(b) = 1;\ f_5(a) = 3,\ f_5(b) = 1;\ f_6(a) = 3,\ f_6(b) = 2;\ f_7(a) = 1,\ f_7(b) = 1;\ f_8(a) = 2,\ f_8(b) = 2;\ f_9(a) = 3,\ f_9(b) = 3.$
Then, a basis of the point-open topology τ on Y^X is $\beta_\tau = \{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_3, f_8\}, \{f_6, f_9\}, \{f_6, f_8\}, \{f_3, f_9\}, \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}, \{f_3, f_6, f_8, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}\}$.
A basis of the point-$L\psi$ topology τ' on Y^X is $\beta_{\tau'} = \{Y^X, \{f_1, f_4, f_7, f_8\}, \{f_1, f_2, f_3, f_4, f_7, f_8\}, \{f_1, f_4, f_5, f_6, f_7, f_8\}\}$.
Here, $f_3 \in \{f_3\} \in \beta_\tau$ but there exists no $B' \in \beta_{\tau'}$ such that $f_3 \in B' \subseteq \{f_3\}$. Thus, τ' is not finer than τ. Again, $f_1 \in \{f_1, f_4, f_7, f_8\} \in \beta_{\tau'}$ but there exists no $B \in \beta_\tau$ such that $f_1 \in B \subseteq \{f_1, f_4, f_7, f_8\}$. Thus, τ is not finer than τ'.
Hence, point-open topology and point-$L\psi$ topology of Y^X are not comparable.

Remark 4.2 *Suppose $\mathbb{I}$ is an ideal on the topological space Y. Then, the point-$L\psi$ topology on Y^X and the point-ψ topology [16] on Y^X are not comparable.*

Example 4.4 *We consider Example 4.3. Then, $\{\psi(A) : A \subseteq Y\} = \{Y, \{2\}, \{2, 3\}\}$ and $\{Cl(\psi(A)) : A \subseteq Y\} = \{Y, \{1, 2\}\}$.*
A basis of the point-ψ topology τ on Y^X is $\beta_\tau = \{Y^X, \{f_8\}, \{f_3, f_8\}, \{f_6, f_8\}, \{f_1, f_6, f_8\}, \{f_3, f_4, f_8\}, \{f_3, f_6, f_8, f_9\}, \{f_1, f_2, f_3, f_6, f_8, f_9\}, \{f_3, f_4, f_5, f_6, f_8, f_9\}\}$.
A basis of the point-$L\psi$ topology τ' on Y^X is $\beta_{\tau'} = \{Y^X, \{f_1, f_4, f_7, f_8\}, \{f_1, f_2, f_3, f_4, f_7, f_8\}, \{f_1, f_4, f_5, f_6, f_7, f_8\}\}$.
Here, $f_8 \in \{f_8\} \in \beta_\tau$ but there exists no $B' \in \beta_{\tau'}$ such that $f_8 \in B' \subseteq \{f_8\}$. Thus, τ' is not finer than τ. Again, $f_1 \in \{f_1, f_4, f_7, f_8\} \in \beta_{\tau'}$ but there exists no $B \in \beta_\tau$ such that $f_1 \in B \subseteq \{f_1, f_4, f_7, f_8\}$. Thus, τ is not finer than τ'.
Hence, the point-ψ topology and the point-$L\psi$ topology of Y^X are not comparable.

Remark 4.3 *Suppose $\mathbb{I}$ is a codense ideal on the topological space Y. Then, the point-$L\psi$ topology on Y^X and the point-$\psi*$ topology on Y^X are not comparable.*

Example 4.5 *We consider Example 4.3 with $\mathbb{I} = \{\emptyset, \{1\}\}$. Then, $\{\psi(A^*) : A \subseteq Y\} = \{\emptyset, Y, \{2\}, \{3\}\}$ and $\{Cl(\psi(A)) : A \subseteq Y\} = \{\emptyset, Y, \{1, 2\}, \{1, 3\}\}$.*
A basis of the point-$L\psi$ topology τ on Y^X is $\beta_{\tau'} = \{\emptyset, Y^X, \{f_7\}, \{f_1, f_7\}, \{f_2, f_7\}, \{f_4, f_7\}, \{f_5, f_7\}, \{f_1, f_2, f_7\}, \{f_4, f_5, f_7\}, \{f_1, f_4, f_7, f_8\}, \{f_2, f_3, f_4, f_7\}, \{f_1, f_5, f_6, f_7\}, \{f_2, f_5, f_7, f_9\}, \{f_1, f_2, f_3, f_4, f_7, f_8\}, \{f_1, f_2, f_5, f_6, f_7, f_9\}, \{f_1, f_4, f_5, f_6, f_7, f_8\}, \{f_2, f_3, f_4, f_5, f_7, f_9\}\}$.
A basis of the point-$\psi*$ topology τ' on Y^X is $\beta_{\tau'} = \{\emptyset, Y^X, \{f_3\}, \{f_6\}, \{f_8\}, \{f_9\}, \{f_1, f_6, f_8\}, \{f_2, f_3, f_9\}, \{f_3, f_4, f_8\}, \{f_5, f_6, f_9\}\}$.
Here, $f_7 \in \{f_7\} \in \beta_\tau$ but there exists no $B' \in \beta_{\tau'}$ such that $f_7 \in B' \subseteq \{f_7\}$. Thus, τ' is not finer than τ. Again, $f_3 \in \{f_3\} \in \beta_{\tau'}$ but there exists no $B \in \beta_\tau$ such that $f_3 \in B \subseteq \{f_3\}$. Thus, τ is not finer than τ'.
Hence, point-$L\psi$ topology and point-$\psi*$ topology on Y^X are not comparable.

Theorem 4.2 *Let $\mathbb{I}$ is a codense ideal on the topological space Y. Given a point $x \in X$ and a subset A of the topological space Y, let*

$$S_{\mathbb{I}}(x, In(A^*)) = \{f \in Y^X \mid f(x) \in In(A^*)\}.$$

The sets $S_{\mathbb{I}}(x, A)$ form a subbasis for a topology on Y^X.

Proof: Let $f \in Y^X$. Then, $f \in S_{\mathbb{I}}(x, Y) = S_{\mathbb{I}}(x, In(Y^*) \subseteq \bigcup\limits_i S_{\mathbb{I}}(x_i, In(A_i^*))$, where $x_i \in X$ and $A_i \subseteq Y$. So, $f \in \bigcup\limits_i S(x_i, In(A_i^*))$. Thus, $Y^X \subseteq \bigcup\limits_i S_{\mathbb{I}}(x_i, In(A_i^*))$. Hence, the sets $S_{\mathbb{I}}(x_i, In(A_i^*))$ form a subbasis for a topology on Y^X. □

The topology generated by the above subbasis is called **point-$In*$ topology** on Y^X.

For a codense ideal space Y, the collection of dense sets and the collection of dense sets in the $*$ topology of Y are equal. Thus, the topology defined by the dense sets is equal to the topology defined by the dense sets of the $*$ topology of Y.

5. $*$ function spaces

We have seen that function space can also be derived from the operators, closure, interior and their mixed operators. If we replace the closure operator with $*$ operator and interior with ψ operator, then we can get the similar topologies of the section 3. To do this, we go through the generalized open sets defined by the $*$ operator. These are $f_{\mathbb{I}}$-set [14], $\mathbb{I}$-open set [13], b^*-set [24] and almost $\mathbb{I}$-open set [8].

For the ideal topological space $\mathbb{I}_X$, a subset A of X is said to be $f_{\mathbb{I}}$-set (resp. $\mathbb{I}$-open set, b^*-set, almost $\mathbb{I}$-open set) if $A \subseteq (InA)^*$ (resp. $A \subseteq (In(A^*))$, $A \subseteq (InA)^* \cup In(A^*)$, $A \subseteq Cl(InA^*)$).

The collection of these sets are respectively denoted as $f_{\mathbb{I}}O(\mathbb{I}_X)$, $\mathbb{I}O(\mathbb{I}_X)$, $B^*O(\mathbb{I}_X)$ and $A\mathbb{I}O(\mathbb{I}_X)$.

For the function space Y^X, sets $S(x, A) = \{f \in Y^X |\ f(x) \in kO(\mathbb{I}_Y)\}$, where $k \in \{f_{\mathbb{I}}, \mathbb{I}, B^*, A\mathbb{I}\}$ do not form a topology on Y^X in general. Because Y is not a member of $kO(\mathbb{I}_Y)$. Due to the above question, we shall try to find way to determine the topologies on Y^X through $kO(\mathbb{I}_Y)$.

The condition $Y^* = Y$ gives some topologies on Y^X. This condition is a characterization of the condition $\mathbb{I} \cap Y = \{\emptyset\}$. That is, $Y^* = Y \Longleftrightarrow \mathbb{I} \cap Y = \{\emptyset\}$.

Lemma 5.1 *For the ideal topological space $\mathbb{I}_Y$, the sets $\xi(x, A) = \{f \in Y^X |\ f(x) \in A \in kO(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X, where $\mathbb{I} \cap \tau = \{\emptyset\}$ and $k \in \{f_{\mathbb{I}}, \mathbb{I}, B^*, A\mathbb{I}\}$.*

One can called these topologies along with the set Y^X by $*$ function spaces. That is, for $k = f_{\mathbb{I}}$, we may supposed that Y^X is a $*$-$f_{\mathbb{I}}$ space.

Next, we shall try to find out the converse part of the Lemma 5.1.

Suppose the sets $\xi(x, A) = \{f \in Y^X |\ f(x) \in A \in \mathbb{I}O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, for $f \in Y^X$, there exists at least one $\xi(x, A)$ such that $f \in \xi(x, A)$. That is, $f(x) \in A \in \mathbb{I}O(\mathbb{I}_Y) \Longrightarrow f(x) \in In(A^*)$. Thus, there exists an open set $U_{f(x)} \in \sigma(f(x), Y)$ such that $U_{f(x)} \subseteq A^*$. Then, for $t \in U_{f(x)}$, for all $V_t \in \sigma(t, Y)$, $V_t \cap A \notin \mathbb{I}$. Hence, $U_{f(x)} \cap V_t \cap A \notin \mathbb{I} \Longrightarrow U_{f(x)} \notin \mathbb{I}$. This does not mean that $\mathbb{I} \cap \sigma = \{\emptyset\}$. Thus, we have:

Theorem 5.1 *Let $\mathbb{I}_Y$ be an ideal topological space and the sets $\xi(x, A) = \{f \in Y^X |\ f(x) \in A \in \mathbb{I}O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, for each $f \in Y^X$, there exists an open set $U_{f(x)} \in \sigma(f(x), Y)$ such that $U_{f(x)} \notin \mathbb{I}$.*

Next, we suppose that the sets $\xi(x, A) = \{f \in Y^X |\ f(x) \in A \in f_{\mathbb{I}}O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, for $f \in Y^X$, there exists at least one $\xi(x, A)$ such that $f \in \xi(x, A)$. That is, $f(x) \in A \in f_{\mathbb{I}}O(\mathbb{I}_Y) \Longrightarrow f(x) \in (InA)^*$. Thus, for all open sets $U_{f(x)} \in \sigma(f(x), Y)$ such that $U_{f(x)} \cap InA \notin \mathbb{I}$.

Thus, for all open sets $U_{f(x)} \in \sigma(f(x), Y)$, $U_{f(x)} \notin \mathbb{I}$. This does not mean that $\mathbb{I} \cap \sigma = \{\emptyset\}$. Thus, we have:

Theorem 5.2 *Let $\mathbb{I}_Y$ be an ideal topological space and the sets $\xi(x, A) = \{f \in Y^X |\ f(x) \in A \in f_{\mathbb{I}}O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, $\mathbb{I} \cap \sigma = \{\emptyset\}$.*

Next, we suppose that the sets $\xi(x, A) = \{f \in Y^X |\ f(x) \in A \in B^*O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, for $f \in Y^X$, there exists at least one $\xi(x, A)$ such that $f \in \xi(x, A)$. That is, $f(x) \in A \in B^*O(\mathbb{I}_Y) \Longrightarrow f(x) \in In(A^*) \cup (In(A))^*$.

1. If $f(x) \in In(A^*)$, then we get the Theorem 5.1.

2. If $f(x) \in (InA)^*$, then we get the Theorem 5.2.

3. If $f(x) \in In(A^*) \cap (InA)^*$. Thus, there exists an open set $U_{f(x)} \in \sigma(f(x), Y)$ such that $U_{f(x)} \subseteq A^*$ and for all open sets $V_{f(x)} \in \sigma(f(x), Y)$ such that $V_{f(x)} \cap InA \notin \mathbb{I}$. This gives that for all $V_{f(x)} \in \sigma(f(x), Y)$, $V_{f(x)} \notin \mathbb{I}$. Therefore, $\mathbb{I} \cap \sigma = \{\emptyset\}$.

Thus, we get the following result.

Theorem 5.3 *Let $\mathbb{I}_Y$ be an ideal topological space and the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in A \in B^*O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, $\mathbb{I} \cap \sigma = \{\emptyset\}$.*

Now, we suppose that the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in A \in A\mathbb{I}O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, for $f \in Y^X$, there exists at least one $\xi(x, A)$ such that $f \in \xi(x, A)$. That is, $f(x) \in A \in A\mathbb{I}O(\mathbb{I}_Y) \implies f(x) \in Cl(In(A^*))$. Thus, for all open sets $U_{f(x)} \in \sigma(f(x), Y)$ such that $U_{f(x)} \cap In(A^*) \neq \emptyset$. Let $t \in U_{f(x)} \cap In(A^*)$, then $t \in U_{f(x)}$ and there exists an open set $V_t \in \sigma(t, Y)$ such that $V_t \subseteq A^*$. Therefore, for all open set W_t containing t such that $W_t \cap A \notin \mathbb{I}$. Therefore, for all open sets $U_{f(x)} \in \sigma(f(x), Y)$, $U_{f(x)} \notin \mathbb{I}$ and hence $\mathbb{I} \cap \sigma = \{\emptyset\}$.

Thus, we get the following result.

Theorem 5.4 *Let $\mathbb{I}_Y$ be an ideal topological space and the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in A \in A\mathbb{I}O(\mathbb{I}_Y)\}$ form a subbasis for a topology on Y^X. Then, $\mathbb{I} \cap \sigma = \{\emptyset\}$.*

We learnt from [16], the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in Cl(A)\}$ form a subbasis for a topology on Y^X. If we replace the closure operator with the local function then we get following:

Lemma 5.2 *For the ideal topological space $\mathbb{I}_Y$, the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in A^*\}$ form a subbasis for a topology on Y^X, where $\mathbb{I} \cap \sigma = \{\emptyset\}$.*

For the converse of the Lemma, we suppose that the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in A^*\}$ form a subbasis for a topology on Y^X. Then, for $f \in Y^X$, there exists at least one $\xi(x, A)$ such that $f \in \xi(x, A)$. That is, $f(x) \in A^*$. Thus, for all $U_{f(x)} \in \sigma(f(x), Y)$, $U_{f(x)} \cap A \notin \mathbb{I}$. Thus, $\mathbb{I} \cap \sigma = \{\emptyset\}$. Thus, we have the following Theorem:

Theorem 5.5 *Let $\mathbb{I}_Y$ be an ideal topological space. Then, the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in A^*\}$ form a subbasis for a topology on Y^X if and only if $\mathbb{I} \cap \sigma = \{\emptyset\}$.*

If we replace the operators 'Interior' and 'Closure' with the operators 'ψ' and $*$ and their mixed operators in the above Theorems, then we have also reached various topologies on Y^X. Actually, for obtaining the topologies on Y^X, our aim will be, can functional value cover the set Y? The answer will be the following operators:

$(\psi(A))^* \subseteq (\psi(Y))^* \subseteq Y$; $(\psi(A^*)) \subseteq (\psi(Y^*)) \subseteq Y$; $(\psi(A^*))^* \subseteq Y$; $Cl(\psi(Y)) \subseteq Y$; $Int((\psi(Y))^*) \subseteq Y$. Therefore, following sets form topologies on Y^X.

Lemma 5.3 *Let $\mathbb{I}_Y$ be an ideal topological space and $\mathbb{I}$ does not contain non-empty open sets. Then,*

1. *the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in (\psi(A))^*\}$ form a subbasis for a topology (called point-$*\psi$ topology) on Y^X [Proposition 4.2].*
2. *the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in (\psi(A^*))\}$ form a subbasis for a topology (called point-$\psi*$ topology) on Y^X [Lemma 4.1].*
3. *the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in (\psi(A^*)) \cup (\psi(A))^*\}$ form a subbasis for a topology (called point-$\psi^* *^\psi$ topology) on Y^X.*
4. *the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in (\psi(A^*))^*\}$ form a subbasis for a topology (called point-$*^{\psi^*}$ topology) on Y^X.*
5. *the sets $\xi(x, A) = \{f \in Y^X | \ f(x) \in Cl(\psi(A))\}$ form a subbasis for a topology (called point-$L\psi$ topology) on Y^X [Theorem 4.1].*

6. *the sets* $\xi(x,A)=\{f\in Y^X|\ f(x)\in In((\psi(A))^*)\}$ *form a subbasis for a topology (called point-*$In\psi^*$ *topology) on* Y^X.

- Since $(\psi(A))^*\neq(\psi(A^*))$, $(\psi(A))^*$ not subset of $(\psi(A^*))$ and $(\psi(A^*))$ not subset of $(\psi(A))$, point-$*\psi$ topology and point-$\psi*$ topology are not comparable.
- Due to the relations, $(\psi(A))^*\subseteq(\psi(A))^*\cup(\psi(A^*))$ and $(\psi(A^*))\subseteq(\psi(A))^*\cup(\psi(A^*))$, point-$\psi^**^\psi$ topology is finer than both the point-$*\psi$ topology and point-$\psi*$ topology.
- Due to $In(\psi(A))^*\subseteq(\psi(A))^*\subseteq Cl(\psi(A))$, point-$*\psi$ topology is finer than point-$In\psi^*$ topology and point-$L\psi$ topology is finer than point-$*\psi$ topology. However, $\mathbb{I}\cap\tau=\{\emptyset\}\iff$ point-$L\psi$ topology and point-$*\psi$ topology coincide.

Now, the question is, Does the sets $\xi(x,A)=\{f\in Y^X|\ f(x)\in A^*\}$ form a subbasis for a topology on Y^X without meeting the condition $\mathbb{I}\cap\sigma=\{\emptyset\}$? In this purpose, we will find a subset A of Y such that $A^*=Y$. If we consider an $\mathbb{I}$-dense set A in $\mathbb{I}_Y$, then $A^*=Y$ that is for all f and each $x\in X$, $f(x)\in A^*$ implies for all $U_{f(x)}\in\sigma(f(x),Y)$, $U_{f(x)}\cap A\notin\mathbb{I}$. If the sets $\xi(x,A)=\{f\in Y^X|\ f(x)\in A^*\}$ form a subbasis for a topology on Y^X, then from above that $\mathbb{I}\cap A=\{\emptyset\}$. Again, if we consider that $*$-dense in itself set in $\mathbb{I}_Y$, and the sets $\xi(x,A)=\{f\in Y^X|\ f(x)\in A^*\}$ constitute a subbasis for a topology on Y^X, then again we reach the condition $\mathbb{I}\cap\sigma=\{\emptyset\}$.

6. Conclusion

In this paper, it is emphasized that how the non-closure and non-interior operators can made several topologies on Y^X. Their mixed operators are also took a role in this regard. But, local function and ψ-operator are not the interior and closure operator, but they participated to discuss topologies on Y^X. It is mentionable that these operators followed the nature of associated set-valued set function [23].

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