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Analysis of class of 2D nonlinear Rosenau-Regularized Long Wave equation with Neumann boundary conditions

Basheer A. Sadiq and Ghassan A. Al-Juaifri*

ABSTRACT: We analyze nonlinear Rosenau-Regularized Long Wave equation on open bounded convex domains with Neumann boundary conditions. The classical Faedo-Galerkin method, combined with compactness arguments, is employed to establish the existence, continuous dependence and uniqueness of analytic solutions on the initial data. Furthermore, a comprehensive case study is presented to illustrate the application of this approach to the Rosenau equation.

Key Words: Neumann boundary condition, Faedo-Galerkin, Uniqueness, Existence, Rosenau equation, Weak solution, Regularity.

Contents

1. Introduction

The study of wave behavior of nonlinear term in nature remains a compelling subject in scientific research, with numerous researchers historically exploring mathematical models to describe wave dynamics. A broad class of phenomena wave can often be represented by nonlinear partial differential equations [36]. However, due to the complexity introduced by nonlinear terms, obtaining analytical solutions for these equations is rarely feasible. Consequently, the numerical solution of such nonlinear partial differential equations becomes essential, as only a limited subset of these equations can be solved through analytical methods [2,3].
The nonlinear Rosenau-RLW equation class in 2D will be introduce as follow [36]:
Find $\{v\}$ such that

$$\frac{\partial v}{\partial t}+\nabla(\theta\nabla\frac{\partial v}{\partial t})+\Delta^2\frac{\partial v}{\partial t}=f(v),\qquad \text{in}\quad \Gamma\times(0,T), \tag{1.1}$$

with Neumann boundary conditions

$$\frac{\partial v}{\partial \nu}=0,\qquad \text{on}\quad \partial\Gamma\times(0,T), \tag{1.2}$$

* Corresponding author

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and an initial condition

$$v(x,y,0) = v(x,y)_0, \qquad \text{in} \quad \Omega, \tag{1.3}$$

here Γ in $\mathbb{R}^d (d = 1,2)$ open bounded, and $\partial\Gamma$ smooth boundary, ν indicates the exterior unit normal to $\partial\Gamma$, where $f = (f_1, \dots, f_k)$ is a nonlinear term and each $f_k(v), k = 1, \dots, d$, is typically like $-(v + \frac{1}{3}v^3)$, $f(x,y,t)$is a scalar function on $\Gamma \times (0,T)$, we note that ∇ is gradient operator and Δ is Laplace operator. Here, we assume θ is a nonnegative function satisfying:

$$\textbf{A-} \text{ there exists a positive constant } \Pi_1 \text{ and } \Pi_2 \text{ such that } \Pi_1 \leq \theta \leq \Pi_2, \tag{1.4}$$
$$\text{for all } x, y \in \Gamma, \ t \in (0,T),$$

$$\textbf{B-} \text{ there exists a positive constant } \Pi_3 \text{ and } \Pi_4 \text{ such that } \Pi_3 \leq \frac{\partial \theta}{\partial t} \leq \Pi_4, \tag{1.5}$$
$$\text{for all } x, y \in \Gamma, \ t \in (0,T).$$

This condition is essential to preserve the ellipticity of the problem and to guarantee the coercivity of associated bilinear forms, which play a central role in deriving energy estimates. The mathematical modeling of water waves has been a subject of significant interest for a long time, aiming to describe a wide range of wave phenomena, from small-scale ripples on the water surface to large-scale waves such as tsunamis. Several mathematical models have been developed to capture wave dynamics, including the Korteweg?de Vries (KdV) equation [22], the Regularized Long-Wave (RLW) equation [29], and the Rosenau equation [27, 28]. The KdV equation, introduced by Boussinesq and Korteweg & de Vries, has been widely used to model shallow water waves, ion acoustic waves, and longitudinal astigmatic waves. Although the KdV equation possesses an analytical solution, it is known to exhibit numerical instability. As a result, various numerical methods have been employed to solve it, including the finite difference method [19, 21], the collocation method (CM) [20], the finite element method (FEM) [14], the Galerkin method (GM) [7, 35], and the spectral method (SM) [18]. Peregrine [29] initially introduced the Regularized Long-Wave (RLW) equation as an alternative to the classical Korteweg?de Vries (KdV) equation, providing a more accurate description of certain nonlinear dispersive wave phenomena. The RLW equation is widely utilized in various scientific fields, including shallow water waves, magnetohydrodynamic plasma waves, and ion-acoustic plasma waves, which are commonly observed in oceanographic and atmospheric studies. Additionally, the RLW equation plays a significant role in modeling soliton motion within optical fibers in telecommunication systems. While the equation is particularly effective for describing small-amplitude, long-wavelength waves in channels, it does not account for interactions between waves or between waves and solid boundaries, such as walls. Rosenau [32, 33] introduced an equation, now widely recognized as the Rosenau equation, to describe the dynamics of dense discrete systems. Unlike the KdV and RLW equations, which fail to adequately capture wave-wave and wave-wall interactions, the Rosenau equation provides a more comprehensive framework for studying these complex phenomena. Extensive research has been conducted on the Rosenau equation, employing both theoretical analysis and numerical methods to explore its properties and applications.Existence and uniqueness for the Rosenau equation's solution was proved by Park [28]. Also, numerically the odd-ave behavior of the equation has also been well studied in recent years. [6, 10, 16, 30]. Significant research efforts have been dedicated to developing finite difference methods (FDM) for the generalized Rosenau-RLW equation [25, 26, 36, 39]. Although these methods are conceptually straightforward and easy to implement, they face challenges in extending to higher-order approximations and adapting to complex geometries. Zuo et al. [41] introduced a Crank-Nicolson scheme to address these limitations for the equation. However, due to its nonlinear implicit nature, this scheme necessitates extensive iterative computations. Pan and Zhang [25, 26] proposed three-level, conservative implicit linearized difference schemes for the generalized Rosenau-RLW equation, enhancing computational efficiency. Additionally, Wongsaijai et al. [40] developed a fourth-order compact FDM based on a three-level average linear implicit technique, further improving the accuracy and stability of numerical solutions. The numerical solution of the Rosenau-RLW equation has predominantly been explored using FDM, while other numerical techniques have received comparatively less attention. Among alternative numerical approaches, the FEM has been widely employed to address this problem.

For instance, Atouani and Omrani [5] proposed a Galerkin finite element method (GFEM) combined with the Crank-Nicolson scheme to incorporate information from previous time steps. Similarly, Mittal and Jain [24] utilized a FEM with quintic B-splines as basis functions, improving the accuracy of the numerical approximation. Furthermore, by applying a meshless kernel-based method of lines Ari and Dereli [4] investigated the general Rosenau-RLW equation. Unlike conventional finite element or finite difference techniques, this approach eliminates the need for linearization of the nonlinear term. Instead, the solution is obtained as a linear combination of basis functions, such as multiquadratic and Gaussian radial basis functions, improving flexibility and computational efficiency. Extensive mathematical and numerical investigations have been conducted on one-dimensional shallow water waves. However, research on the multi-dimensional case remains limited due to the complexities associated with numerically handling the nonlinear terms. Slow convergence to an exact solution is one of the major drawbacks associated with current numerical algorithms, often requiring a significant increase in grid points to achieve the desired accuracy through grid refinement. This issue is particularly critical in multi-dimensional problems, where the theoretical and computational costs become substantial. A significant challenge in developing efficient numerical techniques lies in the vast spatial domains, which often exceed the capacity of standard computational memory. Addressing this issue requires partitioning the large integration domain into smaller subdomains that fit within memory constraints, a process that is both computationally expensive and technically demanding. Furthermore, in general, there exist no universal theoretical methods for handling variable-coefficient partial differential equations, further complicating the development of accurate and efficient numerical approaches. Historically, the approximate solution of multi-dimensional problems has been primarily explored using the finite difference method, with other numerical approaches receiving comparatively less attention. Each method inherently possesses its own advantages and limitations. For instance, Li [23] developed a fourth-order compact FDM for solving the three-dimensional Rosenau-RLW equation. While the method ensures unique solvability, convergence, and stability within a two-level scheme, it is computationally intensive due to its nonlinear implicit nature. Additionally, it achieves second-order accuracy in space and time while preserving energy. In 2017, Ghiloufi and Kadri [15] introduced conservative difference schemes for the two-dimensional Rosenau-RLW equation. Their scheme effectively preserves both discrete mass and energy, though it requires considerable computational time due to its nonlinear implicit structure. Later, Rouatbi et al. [34] proposed a nonlinear difference scheme with second-order accuracy in space and time for the two-dimensional Rosenau-Burger equation. Although this scheme provided reliable numerical simulations, its performance in preserving mass and energy was found to be suboptimal. Further advancements in numerical methods were made by Wang et al. [38], who using a linear difference scheme to investigated solitary wave solutions of the two-dimensional RLW equation. They analyzed the existence, uniqueness, and conservation properties of mass and energy through the discrete energy method. Additionally, in 2017, Gao and Mei [13] applied GFEM, incorporating the linearized backward Euler formula and the extrapolated Crank-Nicolson technique, to solve the two-dimensional RLW and SRLW equations. The existence and uniqueness of approximate solutions were established using the Brouwer fixed-point theorem, and the accuracy of the methods was validated through simulations of solitary wave propagation and wave interactions. The structure of this paper is organized as follows: fundamental notation and preliminaries have been introduce in Section 2. In Section 3, we establish the existence and uniqueness of weak solutions. Section 5 addresses the presence, regularity, and continuing reliance of strong solutions on the initial conditions.

1.1. Novelty and Contributions

This study contributes to the mathematical analysis of nonlinear wave equations by extending the Rosenau-Regularized Long Wave (RLW) equation to a two-dimensional framework with Neumann boundary conditions an area that remains underexplored in the existing literature. While the classical Rosenau and RLW equations have been extensively analyzed in one dimension, especially with Dirichlet or periodic boundaries, the inclusion of Neumann conditions in higher dimensions introduces additional mathematical complexity, particularly in ensuring compatibility and deriving higher regularity results.

The novelty of the present work lies in the rigorous establishment of existence, uniqueness, and continuous dependence of weak and strong solutions within Sobolev spaces H^2 and H^3, using a well-structured combination of the Faedo-Galerkin method, compactness arguments, and refined energy estimates. This framework not only generalizes earlier one-dimensional results (e.g., Park [28]) but also lays the theoretical foundation for future numerical and physical modeling in higher-dimensional domains, which are essential in applications such as shallow water dynamics and nonlinear optics.

By addressing the well-posedness under Neumann boundary conditions, this study offers a significant step toward modeling more realistic physical systems where reflective or insulated boundaries naturally arise. The approach and methodology presented here also contribute to the broader class of nonlinear dispersive PDEs with potential extensions to coupled systems and variable coefficients.

2. Auxiliary results and Notation

In this study, let bounded domain in $\mathbb{R}^d$ denoted by Γ , where $d = 1, 2$, with $\partial\Gamma$ a Lipschitz boundary . We employ standard $W^{z,o}(\Gamma)$ Sobolev spaces, where $z \in \mathbb{N}$ and $o \in [1, \infty]$, through corresponding norms and semi-norms, symbolize by $\| \cdot \|_{z,o}$ and $| \cdot |_{z,o}$. Specifically, for $o = 2$, we denote $W^{z,2}(\Gamma)$ by $H^z(\Gamma)$, with the norm $\| \cdot \|_z$ and semi-norm $| \cdot |_z$. Additionally, when $z = 0$, we identify $W^{0,2}(\Gamma)$ with $L^2(\Gamma)$. The $L^2(\Gamma)$ inner product over Γ with norm $\| \cdot \|_0 = | \cdot |_0$ is denoted by $(\cdot, \cdot)$. increment, $\langle \cdot, \cdot \rangle$ represents duality pairing between $(H^1(\Gamma))'$ and $H^1(\Gamma)$ where $(H^1(\Gamma))'$ is the dual space of $H^1(\Gamma)$. A norm on $(H^1(\Gamma))'$ is given by:

$$\|\phi\|_{(H^1(\Gamma))'} := \sup_{\eta \neq 0} \frac{|\langle \phi, \eta \rangle|}{\|\eta\|_1} \equiv \sup_{\|\eta\|_1 = 1} \|\langle \phi, \eta \rangle\|. \tag{2.1}$$

Furthermore, the function spaces that depend on time and space are introduced as $L^o(0, T; \Upsilon)$ $(1 \leq o \leq \infty)$, where Υ is a Banach space. This space consists of all functions ϕ such that for almost every $t \in (0, T)$, $\phi \in \Upsilon$, and the following norm is finite:

$$\begin{aligned} \|\phi(t)\|_{L^o(0,T;\Upsilon)} &= \left(\int_0^T |\phi(t)|_\Upsilon^p dt \right)^{\frac{1}{p}}, \\ \|\phi(t)\|_{L^\infty(0,T;\Upsilon)} &= ess \sup_{t \in (0,T)} \|\phi(t)\|_\Upsilon. \end{aligned} \tag{2.2}$$

We realize $L^o(\Gamma_T)$ as $L^o(0, T; L^o(\Gamma))$ for $o \in [1, \infty]$. Additionally, the continuous space functions mapping $[0, T]$ into X was introduce and denoted by $C([0, T]; X)$, consisting of functions $\phi(t) : [0, T] \to X$ s.t $\phi(t) \to \phi(t_0)$ in X as $t \to t_0$. The space $C([0, T]; X)$ is widely recognized as a Banach space equipped with its corresponding norm (refer to [37], p. 43).

From Sobolev's theory, we can derive the following well-known results:

$$H^1(\Gamma) \overset{c}{\hookrightarrow} L^o(\Gamma) \hookrightarrow (H^1(\Gamma))' \text{ holds for } o \in \begin{cases} [1, \infty] & \text{if } d = 1, \\ [1, \infty) & \text{if } d = 2, \\ [1, 6] & \text{if } d = 3, \end{cases} \tag{2.3}$$

Here, $\hookrightarrow$ represent a continuous embedding. Furthermore, according to the Rellich-Kondrachov theorem (see, for example, [9], p. 114, and [11], p. 8), the embedding in (2.3) is compact when the range $o \in [1, 6]$ is changed by $o \in [1, 6)$ when $d = 3$. The notation $\overset{c}{\hookrightarrow}$ used to denote a compact embedding.

Because of their importance, the following inequalities are required: for $1 \leq r_1, r_2 \leq \infty$ such that $\frac{1}{s_1} + \frac{1}{s_2} = 1$ if $\phi \in L^{s_1}(\Gamma)$ and $\psi \in L^{s_2}(\Gamma)$ then $\phi\psi \in L^1(\Gamma)$ and

$$\|\phi\psi\|_{0,1} = \int_\Gamma |\phi\psi| dx \leq \left(\int_\Gamma |\phi|^{s_1} dx \right)^{\frac{1}{r_1}} \left(\int_\Gamma |\psi|^{s_2} dx \right)^{\frac{1}{s_2}} = \|\phi\|_{0,s_1} \|\psi\|_{0,s_2}. \tag{2.4}$$

We generalized the above inequality by using it twice to have

$$\|\phi\psi\theta\|_{0,1} = \int_\Gamma |\phi\psi\theta| dx$$

$$\leq \left(\int_\Gamma |\phi|^{s_1} dx \right)^{\frac{1}{s_1}} \left(\int_\Gamma |\psi|^{s_2} dx \right)^{\frac{1}{s_2}} \left(\int_\Gamma |\theta|^{s_3} dx \right)^{\frac{1}{s_3}} = \|\phi\|_{0,s_1} \|\psi\|_{0,s_2} \|\theta\|_{0,s_3}, \tag{2.5}$$

for $1 \leq s_1, s_2, s_3 \leq \infty$ such that $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1$.

Below inequality called Young's will be used more than once.:

$$\kappa_1\kappa_2 \leq \alpha^{\frac{\alpha_1}{\alpha_2}} \frac{\kappa_1^{\alpha_1}}{\alpha_1} + \alpha^{-1}\frac{\kappa_2^{\alpha_2}}{\alpha_2}, \qquad \frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1, \tag{2.6}$$

valid for any $\kappa_1, \kappa_2 \geq 0$, $\alpha > 0$ and $\alpha_1, \alpha_2 > 1$. Here is another essential result of inequality that Jung arrived at:

$$\kappa_1\kappa_2 \geqslant -\alpha\frac{\kappa_1^2}{2} - \alpha^{-1}\frac{\kappa_2^2}{2}, \qquad \forall \kappa_1, \kappa_2 \in \mathbb{R}, \forall \alpha > 0. \tag{2.7}$$

Lemma 2.1 *The Grönwall lemma differential form is also needed: let $\delta(t) \in W^{1,1}(0,T)$ and $\Phi_1(t), \Psi(t)$, $\Pi(t) \in L^1(0,T)$, where functions are non-negative. It follows from*

$$\frac{d\delta(t)}{dt} + \Phi_1(t) \leq \Phi_2(t)\delta(t) + \Phi_3(t) \text{ a.e. } t \in [0,T],$$

that

$$\delta(T) + \int_0^T \Phi_1(t)dt \leq e^{\int_0^T \Phi_2(\tau)d\tau}\delta(0) + e^{\int_0^T \Phi_2(\tau)d\tau}\int_0^T \Phi_3(\tau)d\tau. \tag{2.8}$$

Finally,

Lemma 2.2 *let $\kappa(t)$ and $\nu(t)$ be nonnegative continuous functions on $(0,T)$, and M is a positive constant, where the inequality*

$$\kappa(t) \leq M + \int_0^t \nu(\tau)\kappa(\tau)d\tau, \ \tau \in (0,T). \tag{2.9}$$

Then

$$\kappa(t) \leq M \exp\Big(\int_0^t \nu(\tau)d\tau\Big), \ \tau \in (0,T). \tag{2.10}$$

3. Weak solutions

The problem (1.1)-(1.3) weak formulation will be introduce.

($\mathcal{B}$) Find $v(.,t) \in H^2(\Gamma)$ such that $v(.,0) = v_0(.)$, and for almost every $t \in (0,T)$

$$(\frac{\partial v}{\partial t}, \lambda) + (\theta\nabla\frac{\partial v}{\partial t}, \nabla\lambda) + (\Delta\frac{\partial v}{\partial t}, \Delta\lambda) = (f(v), \lambda), \qquad \forall\, \lambda \in H^2(\Gamma). \tag{3.1}$$

Lemma 3.1 *Let v be a solution of (3.1), assume that $v_0 \in H_0^2(\Gamma)$, if*

$$E_\theta(t) = \|v(t)\|_0^2 + \|\sqrt{\theta(.,t)}\nabla v(t)\|_0^2 + \|\Delta v(t)\|_0^2,$$

then the following conservation of energy holds:

$$E_\theta(t) \leq CE_\theta(0), \text{for a.e}\, t \in (0,T). \tag{3.2}$$

Proof: *Taking $\lambda = v$ in (3.1), we have*

$$(\frac{\partial v}{\partial t}, v) + (\theta\nabla\frac{\partial v}{\partial t}, \nabla v) + (\Delta\frac{\partial v}{\partial t}, \Delta v) = (f(v), v), \tag{3.3}$$

where $f(v) = -v - \frac{1}{3}v^3$ then, Equation (3.3),implies

$$\frac{1}{2}\frac{d}{dt}(\|v(t)\|_0^2 + \|\triangle v(t)\|_0^2) + (\theta(.,t)\nabla\frac{\partial v}{\partial t}, \nabla v) + \|v\|_0^2 + \frac{1}{3}\|v\|_{0,4}^4 = 0. \tag{3.4}$$

Noting that,

$$(\theta(.,t)\nabla\frac{\partial v}{\partial t},\nabla v)=\int_\Gamma\theta(.,t)\nabla\frac{\partial v}{\partial t},\nabla v d\Gamma=\frac{1}{2}\int_\Gamma\theta(.,t)\frac{\partial}{\partial t}(\nabla v)^2 d\Gamma$$
$$=\frac{1}{2}\frac{d}{dt}\int_\Gamma\theta(.,t)(\nabla v)^2 d\Gamma-\frac{1}{2}\int_\Gamma\frac{\partial\theta}{\partial t}(.,t)(\nabla v)^2 d\Gamma. \tag{3.5}$$

Then Equation (3.4) can be estimated as,

$$\frac{d}{dt}\big(\|v(t)\|_0^2+\|\Delta v(t)\|_0^2+\|\sqrt{\theta}(.,t)\nabla v(t)\|_0^2\big)\leq\int_\Gamma\frac{\partial\theta}{\partial t}(.,t)(\nabla v)^2 d\Gamma. \tag{3.6}$$

Using the hypothesis (1.5), then we have

$$\frac{d}{dt}\big(\|v(t)\|_0^2+\|\sqrt{\theta}(.,t)\nabla v(t)\|_0^2+\|\Delta v(t)\|_0^2\big)\leq\Pi_2\|\nabla v(t)\|_0^2, \tag{3.7}$$

one can easily deduce that,

$$\frac{d}{dt}E_\theta(t)\leq\Pi_2 E_\theta(t). \tag{3.8}$$

By using Lemma 2.1,we obtain

$$E_\theta(T)\leq\exp(\Pi_2 T)E_\theta(0).\ \text{for all t} \tag{3.9}$$

As required.

Corollary 3.1 *Let v be a solution of (3.1), assume that $v_0\in H_0^2(\Gamma)$, then, there exists a positive constant C such that*

$$\|v(t)\|_{L^\infty(0,T;H^2(\Gamma))}\leq C,\text{for a.e}\,t\in(0,T). \tag{3.10}$$

Theorem 3.1 *Suppose $\Gamma\subset\mathbb{R}^d$ $(d=1,2)$ is convex, bounded, open domain, and assume that $v_0(.)\in L^2(\Gamma)$, this problem (1.1)-(1.3) have a unique weak solution $\{v\}$ satisfying*

$$v(x,t)\in L^2(\Gamma_T)\cap L^4(\Gamma_T)\cap L^\infty(0,T;L^2(\Gamma))\cap L^\infty(0,T;H^2(\Gamma))\cap C([0,T];L^2(\Gamma)), \tag{3.11}$$

$$\frac{\partial v(x,t)}{\partial t}\in L^\infty(0,T;H^2(\Gamma)). \tag{3.12}$$

Proof: First,we Rewrite the week form (3.1) as the corresponding Galerkin week form:
$(\mathcal{B}^k)$ Find $v^k(.,t)\in H^2(\Gamma)$ such that $v^k(.,0)=v_0^k(.)$, and for almost every $t\in(0,T)$

$$(\frac{\partial v^k}{dt},\lambda)+(\theta(x,y,t)\nabla\frac{dv^k}{dt},\lambda)+(\Delta\frac{dv^k}{dt},\Delta\frac{d\lambda}{dt})=(f(v^k),\lambda),\quad\forall\,\lambda\in H^2(\Gamma). \tag{3.13}$$

Now, the proof will consist of four parts, as follows:

3.1. Approximations of Local existence

In this section will be use the Faedo-Galerkin approach, similarly to the approach in [1,2,8], it's easy to rewrite (3.13) as the ordinary differential equations is written as equivalent form:

$$\frac{dv^k}{dt}+\nabla(\theta\nabla\frac{dv^k}{dt})+\Delta^2\frac{dv^k}{dt}=P^k f(v^k),\qquad v^k(\cdot,0):=P^k v_0^k. \tag{3.14}$$

We have to prove that the non-linearity of ODEs system is locally Lipschitz. And dealing with function $f(v^k)=-(v^k+\frac{1}{3}(v^k)^3)$, as follows

$$|f(v_1)-f(v_2)|\leq|v_1-v_2|+\frac{1}{3}|v_1^3-v_2^3|$$
$$\leq|v_1-v_2|(1+\frac{1}{3}|v_1^2+v_1v_2+v_2^2|)$$
$$\leq|v_1-v_2|(1+\frac{1}{2}|v_1^2+v_2^2|)\leq L|v_1-v_2|, \tag{3.15}$$

here $L = 1 + \frac{1}{2}|v_1^2 + v_2^2|$. Thus, the function f is locally Lipschitz. By applying the local existence theorem, specifically Picard's Theorem (see, for example, [17], p. 9), and consequently, the system ordinary differential equations admits a unique solution $\{v^k\}$ on finite interval $(0, t_k)$, where $t_k > 0$.

3.2. Approximations of Global existence

To establish global existence of Galerkin estimations, first a priori estimate for the bounds on v^k will be derive which are independent of k, within certain function spaces. By utilizing these estimates, then, global existence of the Galerkin estimations was concluded for any $t_k = T$, where T is independent of k.
Case 3.1 On choosing $\lambda = v^k$ in (3.13), and aggregating the resultant equations reveals:

$$(\frac{\partial v^k}{dt}, v^k) + \big(\theta(x,y,t)\nabla \frac{dv^k}{dt}, v^k(x,t)\big) + \big(\Delta \frac{dv^k}{dt}, \Delta \frac{dv^k}{dt}\big) = (f(v^k), v^k), \tag{3.16}$$

this leads to,

$$\frac{1}{2}\frac{d}{dt}(\|v^k\|_0^2 + \|\sqrt{\theta}\nabla v^k\|_0^2 + \|\Delta v^k\|_0^2) + \|v^k\|_0^2 + \frac{1}{3}\|v^k\|_{0,4}^4 = \frac{1}{2}\int_\Gamma \frac{\partial \theta}{\partial t}|\nabla v^k|^2 d\Gamma. \tag{3.17}$$

Using (1.5) on the right hand side of(3.17) , we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|v^k\|_0^2 + \|\sqrt{\theta}\nabla v^k\|_0^2 + \|\Delta v^k\|_0^2) + \|v^k\|_0^2 + \frac{1}{3}\|v^k\|_{0,4}^4 \\ &\leq \Pi_2(\|v^k\|_0^2 + \|\sqrt{\theta}\nabla v^k\|_0^2 + \|\Delta v^k\|_0^2 + \|v^k\|_0^2),\end{aligned} \tag{3.18}$$

multiplying the result by 2,using Lemma 2.1,we deduce that,

$$\begin{aligned}&\|v^k(T)\|_0^2 + \|\sqrt{\theta}\nabla v^k(T)\|_0^2 + \|\Delta v^k(T)\|_0^2 + 2\int_0^T \|v^k\|_0^2 dt + \frac{2}{3}\int_0^T \|v^k\|_{0,4}^4 dt \\ &\leq 2\Pi_2(\|v^k(0)\|_0^2 + \|\sqrt{\theta}\nabla v^k(0)\|_0^2 + \|\Delta v^k(0)\|_0^2 + \|v^k(0)\|_0^2).\end{aligned} \tag{3.19}$$

Recalling $v_0^k \in H^2(\Gamma)$, and noting (1.4), we have v^k is uniformly bounded in

$$L^\infty(0,T,L^2(\Gamma)) \cap L^\infty(0,T,H^1(\Gamma)) \cap L^\infty(0,T,H^2(\Gamma)) \cap L^2(\Gamma_T) \cap L^4(\Gamma_T).$$

Case 3.2 on choosing $\lambda = \frac{\partial v^k}{\partial t}$ in (3.13), it follows that:

$$\frac{1}{2dt}\Big\|\frac{\partial v^k}{\partial t}\Big\|_0^2 + \Big\|\frac{\partial v^k}{\partial t}\Big\|_0^2 + \Big\|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\Big\|_0^2 + \Big\|\Delta\frac{\partial v^k}{\partial t}\Big\|_0^2 = -\frac{1}{3}\Big((v^k)^3, \frac{\partial v^k}{\partial t}\Big). \tag{3.20}$$

Using Young's inequality (2.6) on first term in (3.20) on the right hand side, and $H^1 \hookrightarrow L^6$ see(2.3), we have that

$$-\frac{1}{3}\Big((v^k)^3, \frac{\partial v^k}{\partial t}\Big) \leq \frac{1}{18}\|v^k\|_{0,6}^6 + \frac{1}{2}\Big\|\frac{\partial v^k}{\partial t}\Big\|_0^2 \leq \frac{1}{18}\|v^k\|_1^6 + \frac{1}{2}\Big\|\frac{\partial v^k}{\partial t}\Big\|_0^2. \tag{3.21}$$

Substitute (3.21) into (3.20), leads to

$$\frac{d}{dt}\Big\|\frac{\partial v^k}{\partial t}\Big\|_0^2 + \Big\|\frac{\partial v^k}{\partial t}\Big\|_0^2 + 2\Big\|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\Big\|_0^2 + 2\Big\|\Delta\frac{\partial v^k}{\partial t}\Big\|_0^2 \leq \frac{1}{9}\|v^k\|_1^6. \tag{3.22}$$

Now, integral over $(0,T)$, gives that

$$\begin{aligned}&\|v^k(T)\|_0^2 + \int_0^T \|\frac{\partial v^k}{\partial t}\|_0^2 dt + 2\int_0^T \|\Delta\frac{\partial v^k}{\partial t}\|_0^2 dt + 2\int_0^T \|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\|_0^2 dt \\ &\leq \frac{1}{9}\int_0^T \|v^k\|_1^6 + \|v^k(0)\|_0^2.\end{aligned} \tag{3.23}$$

Recalling $v^k(0) \in H^2(\Gamma)$, using case 3.2, and $L^\infty(0,T,H^2(\Gamma)) \hookrightarrow L^6(0,T,H^1(\Gamma))$, imply that uniformly bounded in $v^k \in L^\infty(0,T,L^2(\Omega))$ and $\frac{\partial v^k}{\partial t} \in L^2(0,T,H^2(\Omega))$.

3.3. Passage to the limit

Employing traditional compactness arguments (see to [12] Theorems 4, 5), we extract convergent subsequences from the uniformly bounded sequences of functions $\{v^k\}_{k=1}^{\infty}$, designated as $\{v^k\}$, such that

$$\{v^k\} \rightharpoonup \{v\} \quad \text{in} \quad L^2(\Gamma_T) \cap L^4(\Gamma_T) \quad \text{as} \quad k \to \infty, \tag{3.24}$$

$$\{\frac{\partial v^k}{\partial t}\} \rightharpoonup \{\frac{\partial v}{\partial t}\} \quad \text{in} \quad L^2(0,T;H^2(\Gamma)) \quad \text{as} \quad k \to \infty, \tag{3.25}$$

and

$$\{v^k\} \rightharpoonup^* \{v\} \quad \text{in} \quad L^\infty(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H^2(\Gamma)) \quad \text{as} \quad k \to \infty. \tag{3.26}$$

Here, both '$\rightharpoonup$', '$\rightharpoonup^*$' denote weak and weak-star convergence, respectively. We now establish the passage to the limit for the first terms of composite Galerkin approximation (3.14). Consider term $P^k f(v^k)$; it is straightforward to show that

$$|f(v^k)| \leq \Big(|v^k| + \frac{1}{3}|v^k|^3\Big), \tag{3.27}$$

then we have

$$\int_0^T \int_\Gamma \big|f(v^k)\big|^2 dxdt \leq \int_0^T \int_\Gamma \Big(|v^k|^2 + \frac{1}{3}|v^k|^6\Big) dxdt. \tag{3.28}$$

On noting the bounds (3.19), and the injections $L^\infty(0,T;H^2(\Gamma)) \hookrightarrow L^6(\Gamma_T)$, we have that bounded uniformly $f(v^k)$ in $L^2(\Gamma_T)$, then from arguments of weak compactness there exists some $\varepsilon \in L^2(\Gamma_T)$ such that

$$f(v^k) \rightharpoonup \varepsilon \text{ in } L^2(\Gamma_T) \quad \text{as } k \to \infty. \tag{3.29}$$

We demonstrate $P^k f(v^k)$ also has a modest tendency to ε in $L^2(\Gamma_T)$. The projection orthogonal to P^k, describe by $Q^k := I - P^k$. Now recall $(P^k v, \varepsilon^k)_V = (v, \varepsilon^k)_V$ for all $\varepsilon^k \in V^k, v \in H^1(\Gamma)$, which leads $\|P^k v - v\|_1 \leq \|v - \varepsilon^k\|_1$ for all $\varepsilon^k \in V^k, v \in H^1(\Gamma)$. Thus as V^k is dense in $H^1(\Gamma)$ we have $P^k u \longrightarrow u$ in $H^1(\Gamma)$ for all $u \in H^1(\Gamma)$, i.e. $Q^k u \longrightarrow 0$ in $H^1(\Gamma)$ as $k \longrightarrow \infty$. We have also $H^1 \hookrightarrow L^2$ and $Q^k u \longrightarrow 0$ in $L^2(\Gamma)$ for all $u \in L^2(\Gamma)$. Let $\varphi \in L^2(\Gamma_T)$ an arbitrary, then utilizing orthogonality and Hölder's inequality of Q^k

$$\begin{aligned}\Big|\int_0^T (P^k f(v^k) - \varepsilon, \varphi) dt\Big| &= \Big|\int_0^T [(f(v^k) - \varepsilon, \varphi) - (f(v^k), Q^k\varphi)] dt\Big| \\ &\leq \Big|\int_0^T (f(v^k) - \varepsilon, \varphi) dt\Big| + \int_0^T \|f(v^k)\|_0 \|Q^k \varphi\|_0 dt \longrightarrow 0 \\ &\qquad \text{as } k \to \infty,\end{aligned}$$

showing convergence is strong in $Q^k\varphi$ to 0 in $L^2(\Gamma)$ and (3.29). Thus, we have

$$P^k f(v^k) \rightharpoonup \varepsilon \text{ in } L^2(\Gamma_T) \quad \text{as } k \to \infty. \tag{3.30}$$

Noting that $\frac{\Delta \partial v^k}{\partial t} \in L^1(0,T;(H^2(\Gamma))')$ and $P^k f(v^k) \in L^2(\Gamma_T)$ it follows from (1.1) that $\frac{\Delta^2 \partial v^k}{\partial t}$ is uniformly bounded in $L^2(\Gamma_T)$

Finally, we apply a modified version of another classical result for [31] to obtain $v \in C([0,T];L^2(\Gamma))$. We display $v \in L^2(\Gamma_T)$ and $\frac{dv}{dt} \in L^2(0,T;H^2(\Gamma))$. Since $L^2(0,T;(H^2(\Gamma))')$ and $L^2(\Gamma_T)$ are the dual spaces of $L^2(0,T;H^2(\Gamma))$ and $L^2(\Gamma_T)$, then we get $v \in C([0,T];L^2(\Gamma))$.

4. Uniqueness

By assuming the existence of two solutions, u and v, of (3.1) weak form, with conditions $u(.,0) = u_0$, we prove uniqueness, and $v(.,0) = v_0$. Setting $\Sigma = u - v$ in(3.1), subtracting weak forms leads to,

$$(\frac{\partial \Sigma}{\partial t}, \Sigma) + (\theta \nabla \frac{\partial \Sigma}{\partial t}, \nabla \Sigma) + (\Delta \frac{\partial \Sigma}{\partial t} - \Delta \Sigma) + (\Sigma, \Sigma) = -\frac{1}{3}(u^3 - v^3, \Sigma). \tag{4.1}$$

Applying Young's inequality (2.7),and Hölder's inequality (2.4), yields that

$$\begin{aligned} -\frac{1}{3}(u^3 - v^3, \Sigma) &= -\frac{1}{3}(\Sigma(u^2 + uv + v^2), \Sigma) \\ &\leq \frac{1}{2}\int_\Gamma \Sigma^2(u^2 + v^2) d\Gamma \leq \frac{1}{2}\|\Sigma\|_0^2\big(\|u\|_0^2 + \|v\|_0^2\big). \end{aligned} \tag{4.2}$$

Substitute (4.2) into (4.1), and then multiplying the results by 2, leads to

$$\frac{d}{dt}\|\Sigma\|_0^2 + \frac{d}{dt}\|\sqrt{\theta}\nabla\Sigma\|_0^2 + \frac{d}{dt}\|\Delta\Sigma\|_0^2 + \|\Sigma\|_0^2 \leq \|\Sigma\|_0^2\big(\|u\|_0^2 + \|v\|_0^2\big). \tag{4.3}$$

By canceling last term inside inequality's (4.3) left-hand side, and in right hand side, additional non-negative terms $\|\sqrt{\theta}\nabla\Sigma\|_0^2$, $\|\Delta\Sigma\|_0^2$, and application of Grönwall lemma (2.8) gives

$$\begin{aligned} &\|w(T)\|_0^2 + \|\sqrt{\theta}\nabla w(T)\|_0^2 + \|\Delta w(T)\|_0^2 \\ &\leq exp\Big(\int_0^T \|u\|_0^2 dt + \int_0^T \|v\|_0^2 dt\Big)\Big(\|w(0)\|_0^2 + \|\sqrt{\theta}\nabla w(0)\|_0^2 + \|\Delta w(0)\|_0^2\Big). \end{aligned} \tag{4.4}$$

By the uniform bounds in case 3.2, we get to

$$\|w(T)\|_0^2 + \|\sqrt{\theta}\nabla w(T)\|_0^2 + \|\Delta w(T)\|_0^2 \leq C(\|w(T)\|_0^2 + \|\sqrt{\theta}\nabla w(T)\|_0^2 + \|\Delta w(T)\|_0^2). \tag{4.5}$$

Thus, if $u(0) = v(0))$, we deduce uniqueness $u(t) = v(t)$ for all t. However, if $u(0) \neq v(0)$, then we have continuous dependence in $H^2(\Gamma)$. □

5. Higher Regularity

Theorem 5.1 *Let $\Gamma \subset \mathbb{R}^d$ $(d = 1, 2)$ is convex, bounded, open domain with a boundary $\partial\Gamma$ of class C^2. Assume that $v_0 \in H^3(\Gamma)$, then the system (1.1)-(1.3) possesses a unique, strong solution $\{v\}$ satisfying*

$$v(x,t) \in L^\infty(0,T; H^3(\Gamma)) \cap L^2(0,T; H^1(\Gamma)) \cap C([0,T], H^1(\Gamma)), \tag{5.1}$$

$$\frac{\partial v(x,t)}{\partial t} \in L^\infty(0,T; L^2(\Gamma)) \cap L^2(0,T; H^2(\Gamma)), \tag{5.2}$$

Equations (1.1)-(1.3) are valid when applied in $L^2(\Gamma_T)$. Furthermore, the

$$v_0(.) \longmapsto v(.,t; v_0),$$

is continuous in $H^3(\Gamma)$.

Proof: So as to establish strong solutions for existence and uniqueness, it is essential to obtain additional regularity results, which can be accomplished through the application of further a priori estimates.

5.1. Existence

We are going to make the following predictions, which are very important to this part.
Estimate I:
Taking $\lambda = -\Delta v^k$, in the weak forms (3.13) , integrating by parts leads to

$$\frac{1}{2}\frac{d}{dt}\|\nabla v^k\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\Delta v^k\|_0^2 - (\theta\nabla\frac{\partial v^k}{\partial t}, \nabla\Delta v^k) = (f(v^k), -\Delta v^k), \tag{5.3}$$

integrating by parts on the right hand side of (5.3), we get

$$\begin{aligned}&(-v^k, -\Delta v^k) - \frac{1}{3}((v^k)^3 - \Delta v^k) = -(\nabla v^k, \nabla v^k) - (\nabla (v^k)^3, \nabla v^k)\\ &= -\|\nabla v^k\|_0^2 - 3\|v^k\nabla v^k\|_0^2.\end{aligned} \tag{5.4}$$

By using Young's inequality (2.6), with noting (1.4), we have that

$$(\theta\nabla\frac{\partial v^k}{\partial t}, \nabla\Delta v^k) \leq \frac{\Pi_1}{2}\|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\|_0^2 + \frac{1}{2}\|\nabla\Delta v^k\|_0^2. \tag{5.5}$$

Substituting (5.5) into (5.3),noting the definition of function f, with multiplying result by 2, leads to

$$\begin{aligned}&\frac{d}{dt}\|\nabla v^k\|_0^2 + \frac{d}{dt}\|\nabla\Delta v^k\|_0^2 + 2\|\nabla v^k\|_0^2 + 6\|v^k\nabla v^k\|_0^2\\ &\leq \Pi_1\|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\|_0^2 + \|\nabla\Delta v^k\|_0^2 + \|\nabla v^k\|_0^2.\end{aligned} \tag{5.6}$$

Next, Grönwall lemma (2.8) utilization, using bounded (3.23),and recalling $v^k(0) \in H^3(\Gamma)$, we arrive at the following inequality:

$$\begin{aligned}&\|\nabla v^k(T)\|_0^2 + \|\nabla\Delta v^k(T)\|_0^2 + 2\int_0^T \|\nabla v^k\|_0^2 dt + 6\int_0^T \|v^k\nabla v^k\|_0^2 dt\\ &\leq \Pi_1 \exp(T)\int_0^T \|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\|_0^2 dt + \exp(T)\Big(\|\nabla\Delta v^k(0)\|_0^2 + \|\nabla v^k(0)\|_0^2\Big) \leq C.\end{aligned} \tag{5.7}$$

Then, we infer that v^k, is uniformly bounded in $L^\infty(0,T;H^1(\Gamma)) \cap L^\infty(0,T;H^3(\Gamma)) \cap L^2(0,T;H^1(\Gamma))$, and $v^k\nabla v^k$ is uniformly bounded in $L^2(\Gamma_T)$.

We now recall that $L^1(0,T;H^{-1}(\Gamma)) + L^1(0,T;H^{-3}(\Gamma))$, that is a pre-dual of $L^\infty(0,T;H^1(\Gamma)) \cap L^\infty(0,T;H^3(\Gamma))$, is a separable Banach space but not reflexive. Consequently, we deduce from the initial and secondary limits in (5.6) that

$$v^k \rightharpoonup^* v \quad \text{in} \quad L^\infty(0,T;H^1(\Gamma)) \cap L^\infty(0,T;H^3(\Gamma)), \tag{5.8}$$

Then, we have $v \in L^\infty(0,T;H^1(\Gamma)) \cap L^\infty(0,T;H^3(\Gamma))$. Since $L^2(0,T;H^1(\Gamma))$ is a reflexive Banach space, Subsequently, by employing compactness arguments (see to [12], page 289), we infer subsequences of existence $\{v^k\} \in L^2(0,T;H^1(\Gamma))$ such that

$$v^k \rightharpoonup v \quad \text{in} \quad L^2(0,T;H^1(\Gamma)). \tag{5.9}$$

Thus, we arrive at $v \in L^2(0,T;H^1(\Gamma))$.

Estimate II:
Choosing $\lambda = \frac{\partial v^k}{\partial t}$, in the weak forms (3.13) , integrating by parts leads to

$$(\frac{\partial v^k}{\partial t}, \frac{\partial v^k}{\partial t}) + (\theta\nabla\frac{\partial v^k}{\partial t}, \nabla\frac{\partial v^k}{\partial t}) + (\Delta\frac{\partial v^k}{\partial t}, \Delta\frac{\partial v^k}{\partial t}) = (-v^k - \frac{1}{3}(v^k)^3, \frac{\partial v^k}{\partial t}), \tag{5.10}$$

by applying Hölder's inequality(2.4) and Young's inequality (2.6),and noting $H^1 \hookrightarrow L^6$, we have that

$$\frac{-1}{3}((v^k)^3, \frac{dv^k}{dt}) \leq \frac{1}{3}\|v^k\|_{0,6}^3\|\frac{dv^k}{dt}\|_0 \leq \frac{1}{18}\|v^k\|_{0,6}^6 + \frac{1}{2}\|\frac{dv^k}{dt}\|_0^2 \leq \frac{1}{18}\|v^k\|_1^6 + \frac{1}{2}\|\frac{dv^k}{dt}\|_0^2. \tag{5.11}$$

Substituting (5.11) into (5.7), and reorder results, leads to

$$\|\frac{\partial v^k}{\partial t}\|_0^2 + \|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\|_0^2 + \|\Delta\frac{\partial v^k}{\partial t}\|_0^2 + \frac{1}{2}\frac{d}{dt}\|v^k\|_0^2 \leq \frac{1}{18}\|v^k\|_1^6 + \frac{1}{2}\|\frac{\partial v^k}{\partial t}\|_0^2. \tag{5.12}$$

Next, integral over $(0,T)$, gives that

$$\begin{aligned}&\int_0^T \|\frac{\partial v^k}{\partial t}\|dt + \int_0^T \|\sqrt{\theta}\nabla\frac{\partial v^k}{\partial t}\|_0^2 dt + \int_0^T \|\Delta\frac{\partial v^k}{\partial t}dt\|_0^2 + \frac{1}{2}\|v^k(T)\|_0^2\\ &\leq \frac{1}{2}\|v^k(0)\|_0^2 + \frac{1}{18}\int_0^T \|v^k\|_1^6 + \frac{1}{2}\int_0^T \|\frac{\partial v^k}{\partial t}\|_0^2.\end{aligned} \tag{5.13}$$

Now,recalling $v^k(0) \in H^3(\Gamma)$,noting bounds in **Estimate I**: , and $L^\infty(0,T;H^1(\Gamma)) \hookrightarrow L^2(0,T;H^1(\Gamma))$ and $\in L^2(0,T;H^2) \hookrightarrow L^2(\Gamma_T)$, this let to the right is bounded

$$v^k \in L^2(\Gamma_T) \cap L^2(0,T;H^2(\Gamma) \cap L^\infty(0,T;L^2(\Gamma)).$$

Which are a reflexive Banach spaces, thus, by compactness arguments, we infer the subsequences of existence $v^k \in L^2(\Gamma_T) \cap L^2(0,T;H^2(\Gamma)) \cap L^\infty(0,T;L^2(\Gamma))$ such that

$$\frac{\partial v^k}{\partial t} \rightharpoonup^* \frac{\partial v}{\partial t} \quad \text{in} \quad L^\infty(0,T;L^2(\Gamma)), \tag{5.14}$$

$$\frac{\partial v^k}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{in} \quad L^2(\Gamma_T) \cap L^2(0,T;H^2(\Gamma)). \tag{5.15}$$

Thus, we have that $\frac{\partial v}{\partial t} \in L^2(\Gamma_T) \cap L^2(0,T;H^2(\Gamma)) \cap L^\infty(0,T;L^2(\Gamma))$.

Lemma 5.1 *For some $s \geq 0$, suppose that*

$$v \in L^2(0,T;H^{s+1}(\Gamma)), \qquad \frac{\partial v}{\partial t} \in L^2(0,T;H^{s-1}(\Gamma)).$$

It follows that $v \in C([0,T];H^2(\Gamma))$.

5.2. Continuous dependence

Assume v_1 and v_2 satisfy the weak form (3.13) , with initial conditions and $v_1(.,0) = v_{1,0}(.)$, and $v_2(.,0) = v_{2,0}(.)$, such that $v_{1,0}(.) \neq v_{2,0}(.)$. Setting $\Omega = v_1 - v_2$, and setting $\lambda = -\Delta\Omega + \Omega$ in (3.13), after integrating by parts, we subtracting weak forms which leads to

$$\begin{aligned}&(\frac{\partial \Omega}{\partial t}, -\Delta\Omega + \Omega) + (\theta\nabla\frac{\partial \Omega}{\partial t}, \nabla(-\Delta\Omega + \Omega) + (\Delta\frac{\partial \Omega}{\partial t}, \Delta(-\Delta\Omega + \Omega))\\ &= (-(\Omega + \frac{1}{3}(v_1^3 - v_2^3), -\Delta\Omega + \Omega).\end{aligned} \tag{5.16}$$

Integrating by parts and noting (1.4), to give

$$
\begin{aligned}
&(\theta\nabla\frac{\partial\Omega}{\partial t}, \nabla(-\Delta\Omega+\Omega\Omega) = (\nabla\theta\nabla\frac{\partial\Omega}{\partial t}\Delta\Omega) + (\theta\nabla\frac{\partial\Omega}{\partial t}, \nabla\Omega)\\
&\geq \Pi_1\int_\Gamma \Delta\Omega\Delta\frac{\partial\Omega}{\partial t}dx + \Pi_1\int_\Gamma \Delta\frac{\partial\Omega}{\partial t}\nabla\Omega dx = \Pi_1\frac{1}{2}\frac{d}{dt}\|\Delta\Omega\|_0^2 + \Pi_1\frac{1}{2}\frac{d}{dt}\|\nabla\Omega\|_0^2,
\end{aligned}
\tag{5.17}
$$

and

$$
\begin{aligned}
&(\Delta\frac{\partial\Omega}{\partial t}, \Delta(-\Delta\Omega+\Omega)) = -(\Delta\frac{\partial\Omega}{\partial t}, \Delta(\Delta\Omega)) + (\Delta\frac{\partial\Omega}{\partial t}, \Delta\Omega)\\
&= (\nabla\Delta\frac{\partial\Omega}{\partial t}, \nabla\Delta\Omega) + (\Omega_t, \Delta\Omega) = \frac{1}{2}\frac{d}{dt}\|\nabla\Delta\Omega\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\Omega\|_0^2.
\end{aligned}
\tag{5.18}
$$

Applying Young's inequality (2.7),and Hölder's inequality (2.4), yields that

$$
\begin{aligned}
&\frac{1}{3}(v_1^3 - v_2^3, \Delta\Omega) = \frac{1}{3}\int_\Gamma (v_1^3 - v_2^3)\\
&\leq \frac{1}{3}(\|v_1\|_{0,6} + \|v_2\|_{0,6})\|\Delta\Omega\|_0^2 \leq c(\|v_1\|_1 + \|v_2\|_1)\|\Delta\Omega\|_0^2,
\end{aligned}
\tag{5.19}
$$

integrating by parts and Using(4.2), (5.5), and noting (5.19), to give

$$
\begin{aligned}
&\Big(-\Omega - \frac{1}{3}(v_1^3 - v_2^3), -\Delta\Omega + \Omega\Big)\\
&= (\Omega, \Delta\Omega) - (\Omega, \Omega) + \frac{1}{3}(v_1^3 - v_2^3, \Delta\Omega) - \frac{1}{3}(v_1^3 - v_2^3, \Omega)\\
&\leq -\|\nabla\Omega\|_0^2 - \|\Omega\|_0^2 + c(\|v_1\|_1 + \|v_2\|_1)(\|\Omega\|_0^2 + \|\Delta\Omega\|_0^2).
\end{aligned}
\tag{5.20}
$$

Substituting (5.17), (5.18) and (5.20) into (5.16), leads to,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Omega\|_0^2 + (1+\Pi_1)\|\nabla\Omega\|_0^2 + (1+\Pi_1)\|\Delta\Omega\|_0^2 + \|\nabla\Delta\Omega\|_0^2) + \|\nabla\Omega\|_0^2 + \|\Omega\|_0^2\\
&\leq C(\|v_1\|_1 + \|v_2\|_1)(\|\Omega\|_0^2 + \|\Delta\Omega\|_0^2).
\end{aligned}
\tag{5.21}
$$

By deleting positive terms part from the left and adding to the right side, then both multiplying by two, we get that

$$
\begin{aligned}
&\frac{d}{dt}(\|\Omega\|_0^2 + (1+\Pi_1)\|\nabla\Omega\|_0^2 + (1+\Pi_1)\|\Delta\Omega\|_0^2 + \|\nabla\Delta\Omega\|_0^2)\\
&\leq C(\|v_1\|_1 + \|v_2\|_1)\big(\|\Omega\|_0^2 + (1+\Pi_1)\|\nabla\Omega\|_0^2 + (1+\Pi_1)\|\Delta\Omega\|_0^2 + \|\nabla\Delta\Omega\|_0^2\big).
\end{aligned}
\tag{5.22}
$$

Applied the Grönwall lemma (2.8) yields that

$$
\begin{aligned}
&\|\Omega\|_0^2 + (1+\Pi_1)\|\nabla\Omega\|_0^2 + (1+\Pi_1)\|\Delta\Omega\|_0^2 + \|\nabla\Delta\Omega\|_0^2 \leq \exp\big(C\int_0^T (\|v_1\|_1 + \|v_2\|_1)dt\big)\\
&\times\big[\|\Omega(0)\|_0^2 + (1+\Pi_1)\|\nabla\Omega(0)\|_0^2 + (1+\Pi_1)\|\Delta\Omega(0)\|_0^2 + \|\nabla\Delta\Omega(0)\|_0^2\big].
\end{aligned}
\tag{5.23}
$$

From the uniform bounds in (3.19), and the continuous injections $L^\infty(0,T;H^2(\Gamma)) \hookrightarrow L^2(0,T;H^2(\Gamma))$ for $d = 1, 2, 3$, thus we have

$$
\|\Omega\|_3^2 \leq C\|\Omega(0)\|_3^2. \tag{5.24}
$$

Thus if $v_1(0) = v_2(0)$ then $\Omega(0) = 0$ hence $v_1(t) = v_2(t)$ for all t. However, if $(v_1(0) \neq v_2(0)$, then we have continuous dependence in $H^3(\Gamma)$. Now, the proof of Theorem 5.1 completes .

6. Conclusions

In this study, we conducted a rigorous theoretical analysis of the two-dimensional nonlinear Rosenau-Regularized Long Wave (RLW) equation under homogeneous Neumann boundary conditions within a convex bounded domain. By employing the Faedo-Galerkin approximation method in combination with compactness arguments and precise energy estimates, we established the existence, uniqueness, and continuous dependence of weak and strong solutions. Additionally, higher regularity of solutions was obtained in appropriate Sobolev spaces under natural compatibility conditions on the initial data. A notable aspect of this work is the handling of Neumann boundary conditions in a multidimensional setting, which introduces analytical challenges not typically present in one-dimensional formulations. Our results extend the theoretical foundation of Rosenau-type models and contribute to the broader understanding of nonlinear dispersive partial differential equations with reflective boundary behavior. While the current manuscript focuses on the theoretical aspects, it lays the groundwork for future numerical investigations, including benchmark simulations, error analysis, and applications to physical models such as shallow water dynamics and nonlinear wave propagation in bounded media. □

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Basheer A. Sadiq,
Department of Mathematics,
College of Sciences,
University of Basrah,
Iraq.
E-mail address: basheer.abdulridha@uobasrah.edu.iq

and

Ghassan A. Al-Juaifri,

Department of Mathematics,
Faculty of Computer Science and Mathematics,
University of Kufa,
Iraq.
E-mail address: ghassana.ali@uokufa.edu.iq