Bol. Soc. Paran. Mat. (3s.) v. 2025 (43) : 1–12.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
1 SPM: www.spm.uem.br/bspm doi:10.5269/bspm.76380

Zalcman, Generalized Zalcman and Krushkal inequalities associated with a new subclass of analytic functions

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ABSTRACT: In this article we investigate the sharp bounds of Zalcman, generalized Zalcman and Krushkal inequalities for a new subclass of analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Key Words: Zalcman, Generalized Zalcman, Krushkal inequality.

Contents

1. Introduction

Let $\mathcal{A}$ denote the class of an analytic function which is normalized under the condition of $f(0) = f'(0)-1 = 0$ in the open unit disk $\Delta = \{z \in \mathbb{C}$ and $|z| < 1\}$ and given by the following Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A subclass $\mathcal{S}$ of $\mathcal{A}$ where each function is one-one is called the class of univalent functions for which $f'(z) \neq 0$ has been an integral part of the study of geometric functions since Bieberbach [2]. Another class which has been explored extensively in the literature is the class of starlike functions S^* which is characterized by the following condition on such functions,

$$Re\left(\frac{zf'}{f}\right) > 0\ .$$

Sokól and Stankiewicz [10], introduced a class denoted as $\mathcal{SL}^*$, which comprises normalized analytic functions f on Δ satisfying the following condition.

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1\ .$$

This class is referred to as Sokól-Stankiewicz starlike functions. The class of Bazilevič functions $\mathcal{B}(\alpha)$ of type α where $0 \le \alpha \le 1$ is characterized by the property,

$$Re\left(\frac{z^{1-\alpha}f'(z)}{[f(z)]^{1-\alpha}}\right) > 0.$$

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2010 *Mathematics Subject Classification:* 30C45, 30C50.
Submitted April 15, 2024. Published August 10, 2025

We define a new subclass of the class $\mathcal{A}, N_V^R(a)$ as follows.

Definition 1.1 *A function $f \in N_V^R(a)$ for $0 \leq a \leq 1$ if it satisfies the below condition for $z \in \Delta$.*

$$\left| \left(\frac{zf'(z) + az^2 f''(z)}{(1-a)f(z) + azf'(z)} \right)^2 - 1 \right| < 1$$

which gives

$$Re \left(\frac{zf^{'}(z) + az^2 f''(z)}{(1-a)f(z) + azf'(z)} \right) > 0.$$

Functions with positive real part are the members of the class denoted by $\mathcal{P}$ and are of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \ . \tag{1.2}$$

The pivotal moment in the exploration of univalent functions occurred in 1985, when Louis de Branges successfully proved the renowned Bieberbach conjecture, $|a_n| = n$ for $n = 2$ [2]. While this marked the conclusion of an era, numerous unresolved issues persist, including the notable Zalcman conjecture, which pertains to the coefficients a_n is as follows

$$|a_n - a_{2n-1}| \leq (n-1)^2, \qquad (n \geq 2).$$

Formulated in the early 1970s, Krushkal [4], made significant strides in this direction, employing the complex geometry of the universal Teichm üller space.We have

$$\left| a_n^c - a_2^{c(n-1)} \right| \leq 2^{c(n-1)} - 2^c, \qquad (n \geq 2).$$

over the class $\mathcal{S}$ for the cases $n = 4$, $c = 1$ and $n = 5$, $c = 1$. This inequality was introduced by Krushkal and proven for the whole class of univalent functions [4]. In 1999, a broader notion, generalized Zalcman conjecture was introduced by Ma [7]. The generalized Zalcman conjecture is

$$|a_m a_n - a_{m+n-1}| \leq (m-1)(n-1), \qquad (m, n \geq 2).$$

Ma [7] successfully resolved the open problem within the realm of starlike functions and univalent functions with real coefficients. Ravichandran and Verma [10], also tackled and closed the issue for starlike and convex functions of specified order, as well as for functions characterized by bounded turning. Ozaki and Nunokawa [9], established the univalence of functions within this class, deviating from the conventional characteristics observed in other univalent functions.

2. Lemmas and Preliminaries

Lemma 1 *[11] Let $h \in \mathcal{P}$, be given by (1.2), then*

$$|c_n| \leq 2, \ \forall n \in \mathbb{N}$$

and

$$\left| c_2 - \frac{\mu}{2} c_1^2 \right| \leq 2max\{1, |\mu - 1|\}.$$

Lemma 2 *[11] Let $h \in \mathcal{P}$, be given by (1.2), then for some complex valued x with $|x| \leq 1$, some complex valued ϑ with $|\vartheta| \leq 1$ and some complex valued θ with $|\theta| \leq 1$. We have*

$$\begin{aligned}
2c_2 &= c_1^2 + x\left(4 - c_1^2\right) \quad , \\
4c_3 &= c_1^3 + 2\left(4 - c_1^2\right) c_1 x - c_1 \left(4 - c_1^2\right) x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right) \vartheta \quad , \\
8c_4 &= c_1^4 + \left(4 - c_1^2\right) x \left[c_1^2(x^2 - 3x + 3) + 4x\right] \\
&\quad -4\left(4 - c_1^2\right)\left(1 - |x|^2\right)\left[c(x-1)\vartheta + \bar{x}\vartheta^2 - 1 - |\vartheta|^2\theta\right].
\end{aligned}$$

3. Main results

3.1. Zalcman Conjecture for the class $N_V^R(a)$

Theorem 1 *Let f given by (1.1), be in the class $N_V^R(a)$; $(0 \le a \le 1)$. Then we have the sharp bound*

$$\left|a_2^2 - a_3\right| \le max\left\{\frac{1}{(1+2a)}, \frac{\eta_1(a)}{(1+a)^2(1+2a)}\right\}$$

where,

$$\eta_1(a) = -3a^2 + 2a + 1.$$

Proof: First note that by equating the corresponding coefficients in the equation

$$\frac{zf^{'}(z) + az^2 f''(z)}{(1-a)f(z) + azf'(z)} = h(z), \tag{3.1}$$

we obtain

$$\begin{aligned}
a_2 &= \frac{c_1}{a+1}\ , \\
a_3 &= \frac{c_1^2}{2(1+2a)} + \frac{c_2}{2(1+2a)}\ , \\
a_4 &= \frac{c_1^3}{6(1+3a)} + \frac{c_1c_2}{2(1+3a)} + \frac{c_3}{3(1+3a)}\ , \\
a_5 &= \frac{c_1^4}{24(1+4a)} + \frac{c_1^2c_2}{4(1+4a)} + \frac{c_1c_3}{3(1+4a)} + \frac{c_2^2}{8(1+4a)} + \frac{c_4}{4(1+4a)}\ , \\
a_6 &= \frac{c_1^2}{120(1+5a)} + \frac{c_1^3c_2}{12(1+5a)} + \frac{c_1^2c_3}{6(1+5a)} + \frac{c_1c_2^2}{40(1+5a)} + \frac{c_1c_4}{4(1+5a)}\ . \\
&+ \frac{c_1c_2}{10(1+5a)} + \frac{c_2c_3}{2(1+5a)} + \frac{c_5}{5(1+5a)}.
\end{aligned}$$

Therefore,

$$a_2^2 - a_3 = \frac{c_1^2}{(1+a)^2} - \frac{c_1^2}{2(1+2a)} - \frac{c_2}{2(1+2a)}.$$

Note that, by Lemma 2 , we may write $2c_2 = c_1^2 + x\left(4 - c_1^2\right)$ and can easily obtain

$$a_2^2 - a_3 = \left[\frac{-a^2+2a+1}{2(1+a)^2(1+2a)}\right]c_1^2 - \frac{(4-c_1^2)x}{4(1+2a)}. \tag{3.2}$$

Without loss of generality, we let $0 \le c_1 = c \le 2$. Substituting this in (3.2), we obtain the following equation in terms of $|x|$.

$$\begin{aligned}
\left|a_2^2 - a_3\right| &= \frac{4-c^2}{4(1+2a)}|x| + \left[\frac{-3a^2+2a+1}{4(1+a)^2(1+2a)}\right]c_1^2 \\
&= \Upsilon(c, |x|).
\end{aligned}$$

We are required to obtain the maximum value of $\Upsilon(c, |x|)$ on $[0,2] \times [0,1]$. First, assume that there is a maximum at an interior point $\Upsilon(c_0, |x_0|)$ of $[0,2] \times [0,1]$. Differentiating $\Upsilon(c, |x|)$ with respect to $|x|$ and equating it to 0 implies that $c = c_0 = 2$ which is a contradiction. Thus for the maximum of $\Upsilon(c, |x|)$, we have to consider the end points of $[0,2] \times [0,1]$.

For $c = 0$, we obtain

$$\Upsilon(0, |x|) = \frac{4}{4(1+2a)}|x| \le \frac{1}{1+2a}.$$

For $c = 2$, we get

$$\Upsilon(2, |x|) = \left[\frac{-3a^2 + 2a + 1}{(1+a)^2(1+2a)}\right].$$

For $|x| = 0$, we get

$$\Upsilon(c, 0) = \left[\frac{-3a^2 + 2a + 1}{(1+a)^2(1+2a)}\right] c^2$$

which has the maximum value $\dfrac{|\eta_1(a)|}{(1+a)^2(1+2a)}$ attained at the end point $c = 2$.
For $|x| = 1$, we obtain

$$\Upsilon(c, 1) = \left[\frac{-3a^2 + 2a + 1}{4(1+a)^2(1+2a)}\right] c^2 + \frac{4 - c^2}{4(1+2a)}$$

which is maximum value of $\Upsilon(c, 1) = \dfrac{1}{1+2a}$ at $c = 0$ and $\dfrac{|\eta_1(a)|}{(1+a)^2(1+2a)}$ at $c = 2$. Hence,

$$\left|a_2^2 - a_3\right| \leq max \left\{\frac{1}{1+2a}, \frac{|\eta_1(a)|}{(1+a)^2(1+2a)}\right|$$

where,

$$\eta_1 = -3a^2 + 2a + 1.$$

□

Theorem 2 *Let f given by (1.1), be in the class $N_V^R(a)$; $(0 \leq a \leq 1)$. Then we have the sharp bound*

$$\left|a_3^2 - a_5\right| \leq max \left\{\frac{-4a^2 + 12a + 1}{8(1+2a)^2(1+4a)} + \frac{1}{2(1+4a)}, \frac{|\eta_2(a)|}{(1+2a)^2(1+4a)}\right\}$$

where,

$$\eta_2(a) = -20a^2 + 16a + 4.$$

Proof: First note that by equating the corresponding coefficients in the equation (3.1), using the fact that $2c_2 = c_1^2 + x\left(4 - c_1^2\right)$ and letting $X = (4 - c_1^2)$, we get

$$\begin{aligned}
a_3^2 - a_5 &= \left[\frac{-20a^2 + 16a + 4}{16(1+2a)^2(1+4a)}\right] c_1^4 \\
&+ \left[\frac{36(1+4a) - 43(1+2a)^2}{96(1+2a)^2(1+4a)}\right] c_1^2 x X - \left[\frac{7}{24(1+4a)}\right] c_1^2 x^2 X \\
&- \left[\frac{(p+4)}{32(1+4a)}\right] c_1 X x^3 + \left[\frac{-4a^2 + 12a + 1}{32(1+2a)^2(1+4a)}\right] X x^2 + \frac{11}{96(1+4a)} c_1^2 X x^2 \\
&+ \frac{7c_1 X x^2}{24(1+4a)} + \frac{X\overline{x}}{8(1+4a)} - \frac{X x^2 \overline{x}}{8(1+4a)} - \frac{X}{8(1+4a)} + \frac{c_1 X x}{8(1+4a)}. \qquad (3.3)
\end{aligned}$$

Without loss of generality, we let $0 \leq c_1 = c \leq 2$. Substituting this in (3.3) and using triangle inequality, we obtain the following polynomial in terms of $|x|$.

$$\begin{aligned}\left|a_3^2-a_5\right| &\leq \left[\frac{(c+4)(4-c^2)}{32(1+4a)c}\right]|x|^3\\ &+ \left[\frac{\left(-4a^2+12a+1\right)(4-c^2)}{32(1+2a)^2(1+4a)}+\frac{11(4-c^2)c^2}{96(1+4a)}+\frac{7(4-c^2)c}{24(1+4a)}-\frac{(4-c^2)\overline{x}}{8(1+4a)}\right]|x|^2\\ &+ \left[\frac{(36(1+4a)-43(1+2a)^2)(4-c^2)c^2}{96(1+2a)^2(1+4a)}+\frac{c(4-c^2)}{8(1+4a)}.\right]|x|\\ &+ \frac{(4-c^2)\overline{x}}{8(1+a)}+\frac{(4-c^2)}{8(1+a)}+\frac{7c(4-c^2)}{8(1+a)}\\ &+ \left[\frac{-20a^2+16a+4}{16(1+2a)^2(1+4a)}\right]c^4\\ &= \rho(c,|x|).\end{aligned}$$

On similar lines of Theorem 1, here we see that the maximum of $\rho(c,|x|)$ is attained at the end points of $[0,2]\times[0,1]$. Therefore,

for $c=0$, we obtain

$$\begin{aligned}\rho(0,|x|) &= \frac{(-4a^2+12a+1)|x|^2}{8(1+2a)^2(1+4a)}+\frac{1}{2(1+4a)}\\ &\leq \frac{(-4a^2+12a+1)}{8(1+2a)^2(1+4a)}+\frac{1}{2(1+4a)}.\end{aligned}$$

For $c=2$, we get

$$\rho(2,|x|)=\frac{|\eta_2|}{(1+2a)^2(1+4a)}$$

where,

$$\eta_2=-20a^2+16a+4.$$

For $|x|=0$, we get

$$\rho(c,0)=\frac{(4-c^2)}{8(1+a)}+\frac{7c(4-c^2)}{8(1+a)}\left[\frac{-20a^2+16a+4}{16(1+2a)^2(1+4a)}\right]c^4.$$

For $|x|=1$, we get

$$\begin{aligned}\left|a_3^2-a_5\right| &\leq \left[\frac{(c+4)(4-c^2)}{32(1+4a)c}\right]\\ &+ \left[\frac{\left(-4a^2+12a+1\right)(4-c^2)}{32(1+2a)^2(1+4a)}+\frac{11(4-c^2)c^2}{96(1+4a)}+\frac{7(4-c^2)c}{24(1+4a)}-\frac{(4-c^2)}{8(1+4a)}\right]\\ &+ \left[\frac{(36(1+4a)-43(1+2a)^2)(4-c^2)c^2}{96(1+2a)^2(1+4a)}+\frac{c(4-c^2)}{8(1+4a)}.\right]\\ &+ \frac{(4-c^2)}{8(1+a)}+\frac{(4-c^2)}{8(1+a)}+\frac{7c(4-c^2)}{8(1+a)}\\ &+ \left[\frac{-20a^2+16a+4}{16(1+2a)^2(1+4a)}\right]c_1^4\\ &= \rho(c,|x|)\end{aligned}$$

which has the maximum value $\dfrac{|-20a^2+16a+4|}{(1+2a)^2(1+4a)}$ attained at the end point $c=2$ and

$$\frac{(-4a^2+12a+1)}{8(1+2a)^2(1+4a)}+\frac{1}{2(1+4a)}$$

at $c=0$. The result follows. □

3.2. Generalized Zalcman Conjecture for the class $N_V^R(a)$

Theorem 3 *Let f given by (1.1), be in the class $N_V^R(a)$; $(0 \le a \le 1)$. Then we have the sharp bound*

$$|a_2a_3 - a_4| \le max\left\{\frac{4}{3(1+3a)}, \frac{|\eta_3(a)|}{(1+a)(1+2a)(1+3a)}\right\}$$

where,

$$\eta_3(a) = 2\left(-4a^2+3a+1\right).$$

Proof: First note that by equating the corresponding coefficients in the equation (3.1), using the fact that $2c_2 = c_1^2 + x\left(4-c_1^2\right)$ and letting $X = (4-c_1^2)$, a simple computation leads to

$$\begin{aligned} a_2a_3 - a_4 &= \left[\frac{-4a^2+3a+1}{4(1+a)(1+2a)(1+3a)}\right]c_1^3 + \frac{c_1Xx^2}{12(1+3a)} \\ &- \frac{X}{6(1+3a)} + \left[\frac{Xx^2}{6(1+3a)}\right] - \left[\frac{5a^2+3a+1}{6(1+a)(1+2a)(1+3a)}\right]c_1xX. \end{aligned} \tag{3.4}$$

Without loss of generality, we let $0 \le c_1 = c \le 2$. Substituting this in (3.4) and using triangle inequality, we obtain the following equation in terms of $|x|$.

$$\begin{aligned} |a_2a_3 - a_4| &\le \left[\frac{p(4-c^2)}{12(1+3a)} + \frac{(4-c^2)}{6(1+3a)}\right]|x|^2 + \left[\frac{-5a^2-3a-1}{6(1+a)(1+2a)(1+3a)}\right]c(4-c^2)|x| \\ &+ \left[\frac{-4a^2+3a+1}{4\left((1+a)(1+2a)(1+3a)\right)}\right]c^3 + \frac{4-c^2}{6(1+3a)} \\ &= \sigma(c,|x|). \end{aligned}$$

We are required to obtain the maximum value of $\sigma(c,|x|)$ on $[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\sigma(c_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $\sigma(c,|x|)$ with respect to $|x|$ and equating it to 0 implies that $c = c_0 = 2$ which is a contradiction. Thus for the maximum of $\sigma(c,|x|)$, we have to consider the end points of $[0,2]\times[0,1]$.

For $c=0$, we obtain

$$\sigma(0,|x|) = \frac{4}{6(1+3a)}|x|^2 + \frac{4}{6(1+3a)} \le \frac{4}{3(1+3a)}.$$

For $c=2$, we obtain

$$\sigma(2,|x|) = \frac{2\left(-a^2+3a+1\right)}{(1+a)(1+2a)(1+3a)}.$$

For $|x|=0$, we get

$$\sigma(c,0) = \left[\frac{-4a^2+3a+1}{4\left((1+a)(1+2a)(1+3a)\right)}\right]c^3 + \frac{4-c^2}{6(1+3a)}$$

which has the maximum value $\dfrac{2|\eta_3(a)|}{(1+a)(1+2a)(1+3a)}$ attained at the end point $c=2$.

For $|x|=1$, we obtain

$$\begin{aligned} \sigma(c,1) &= \left[\frac{p(4-c^2)}{12(1+3a)} + \frac{(4-c^2)}{6(1+3a)}\right] + \left[\frac{-5a^2-3a-1}{6(1+a)(1+2a)(1+3a)}\right]c(4-c^2) \\ &+ \left[\frac{-4a^2+3a+1}{4\left((1+a)(1+2a)(1+3a)\right)}\right]c^3 + \frac{4-c^2}{6(1+3a)} \end{aligned}$$

which has the maximum value of $\sigma(c,1) = \dfrac{4}{3(1+a)}$ at $c = 0$ and $\sigma(c,1) = \dfrac{2|\eta_3(a)|}{(1+a)(1+2a)(1+3a)}$ at $c = 2$.
Hence,

$$|a_2a_3 - a_4| \leq max\left\{\frac{4}{3(1+a)}, \frac{|\eta_3(a)|}{(1+a)(1+2a)(1+3a)}\right\}$$

where,

$$\eta_3(a) = \left(-4a^2 + 3a + 1\right).$$

□

Theorem 4 *Let f given by (1.1), be in the class $N_V^R(a)$; $(0 \leq a \leq 1)$. Then we have sharp bound*

$$|a_2a_4 - a_5| \leq max\left\{\frac{2}{(1+4a)}, \frac{|\eta_4(a)|}{(1+a)(1+3a)(1+4a)}\right\}$$

where,

$$\eta_4(a) = -15a^2 + 12a + 3.$$

Proof: First note that by equating the corresponding coefficients in the equation (3.1), using the fact that $2c_2 = c_1^2 + x\left(4 - c_1^2\right)$ and letting $X = (4 - c_1^2)$, we get

$$\begin{aligned} a_2a_4 - a_5 &= \left[\frac{-15a^2+12a+3}{16(1+a)(1+3a)(1+4a)}\right]c_1^4 \\ &+ \left[\frac{32(1+4a)-43(1+3a)(1+a)}{96(1+a)(1+3a)(1+4a)}\right]c_1^2xX + \left[\frac{4(1+4a)-7(1+3a)(1+a)}{24(1+a)(1+3a)(1+4a)}\right]c_1X \\ &+ \left[\frac{-(c_1+4)c_1Xx^3}{32(1+4a)}\right] + \left[\frac{6(1+4a)+7(1+3a)(1+a)}{24(1+a)(1+3a)(1+4a)}\right]c_1x^2X - \frac{17c_1^2x^2X}{96(1+4a)} \\ &- \frac{X\overline{x}}{8(1+4a)} - \frac{Xx^2\overline{x}}{8(1+4a)} - \frac{X}{8(1+4a)}. \end{aligned} \tag{3.5}$$

Without loss of generality, we let $0 \leq c_1 = c \leq 2$. Substituting in (3.5), we obtain the following equation in terms of $|x|$.

$$\begin{aligned} |a_2a_4 - a_5| &\leq \left[\frac{(c+4)c(4-c^2)}{32(1+4a)}\right]|x|^3 \\ &+ \left[\frac{(6(1+4a)+7(1+3a)(1+4a))c(4-c^2)}{24(1+a)(1+3a)(1+4a)} + \frac{17c^2(4-c^2)}{96(1+4a)}\frac{(4-c^2)\overline{x}}{8(1+4a)} - \frac{(4-c^2)^2}{32(1+4a)}\right]|x|^2 \\ &+ \left[\frac{(32(1+4a)-43(1+3a)(1+a))(4-c^2)c^2}{96(1+a)(1+3a)(1+4a)}\right]|x| \\ &+ \left[\frac{4(1+4a)-7(1+3a)(1+a)(4-c^2)c}{24(1+a)(1+3a)(1+4a)}\right] \\ &+ \left[\frac{(4-c^2)\overline{x}}{8(1+4a)}\right] - \frac{(4-c^2)}{8(1+4a)} \\ &= \zeta(c,|x|). \end{aligned} \tag{3.6}$$

On similar lines of Theorem 3, here we see that the maximum of $\zeta(c,|x|)$ is attained at the end points of $[0,2] \times [0,1]$. Therefore, for $c = 0$, we obtain

$$\begin{aligned} \zeta(0,|x|) &= \left[\frac{4|x|^2}{8(1+4a)} + \frac{16|x|^2}{32(1+4a)} + \frac{4\ \overline{x}}{8(1+4a)} - \frac{4}{8(1+4a)}\right] \\ &\leq \frac{2}{1+4a}. \end{aligned}$$

For $c = 2$, we get

$$\zeta(2,|x|) = \frac{|\eta_4(a)|}{(1+a)(1+3a)(1+4a)}.$$

For $|x| = 0$, we get

$$\begin{aligned} \zeta(p,0) &= \left[\frac{-15a^2+12a+3}{16(1+a)(1+3a)(1+4a)}\right] c \\ &+ \left[\frac{4(1+4a)-7(1+3a)(1+a)}{24(1+a)(1+3a)(1+4a)}\right] c(4-c^2) - \frac{4-c^2}{8(1+4a)}. \end{aligned}$$

For $|x| = 1$, we obtain

$$\begin{aligned} |a_2a_4 - a_5| &\leq \left[\frac{(c+4)c(4-c^2)}{32(1+4a)}\right] \\ &+ \left[\frac{(6(1+4a)+7(1+3a)(1+4a))c(4-c^2)}{24(1+a)(1+3a)(1+4a)} + \frac{17c^2(4-c^2)}{96(1+4a)}\frac{(4-c^2)}{8(1+4a)} - \frac{(4-c^2)^2}{32(1+4a)}\right] \\ &+ \left[\frac{(32(1+4a)-43(1+3a)(1+a))(4-c^2)c^2}{96(1+a)(1+3a)(1+4a)}\right] \\ &+ \left[\frac{4(1+4a)-7(1+3a)(1+a)(4-c^2)c}{24(1+a)(1+3a)(1+4a)}\right] \\ &+ \left[\frac{(4-c^2)}{8(1+4a)}\right] - \frac{(4-c^2)}{8(1+4a)} \end{aligned}$$

which has the maximum value $\frac{|\eta_4(a)|}{(1+a)(1+3a)(1+4a)}$ attained at the end point $c = 2$ and $\frac{2}{(1+4a)}$ at $c = 0$ where,

$$\eta_4(a) = -15a^2 + 12a + 3.$$

□

3.3. Krushkal Inequality for the class $N_V^R(a)$

Theorem 5 *Let f given by (1.1), be in the class $N_V^R(a)$; $(0 \leq a \leq 1)$. Then we have the sharp bound*

$$\left|a_4 - a_2^3\right| = max\left\{\frac{4}{3(1+3a)}, \frac{4|\eta_5(a)|}{(1+a)^3(1+3a)}\right\}$$

where,

$$\eta_5(a) = a^3 + 3a^2 - 3a - 1.$$

Proof: *First note that by equating the corresponding coefficients in the equation (3.1), using the fact that $2c_2 = c_1^2 + x\left(4 - c_1^2\right)$ and letting $X = (4 - c_1^2)$, we get*

$$\begin{aligned} a_4 - a_2^3 &= \left[\frac{a^3+3a^2-3a-1}{2(1+a)^3(1+3a)}\right] c_1^3 + \left[\frac{5}{12(1+3a)}\right] c_1xX \\ &- \left[\frac{(c_1+2)}{12(1+3a)}\right] x^2X + \frac{X}{6(1+3a)}. \end{aligned} \tag{3.7}$$

Without loss of generality, we let $0 \leq c_1 = c \leq 2$. *Substituting this in (3.5), we obtain the following equation in terms of* $|x|$.

$$\begin{aligned} |a_4 - a_2^3| &\leq \left[\frac{(c+2)(4-c^2)}{12(1+3a)}\right]|x|^2 + \left[\frac{5cX}{12(1+3a)}\right]|x| \\ &+ \left[\frac{a^3+3a^2-3a-1}{2(1+a)^3(1+3a)}\right]c^3 + \frac{4-c^2}{6(1+3a)} \\ &= \mu(c,|x|). \end{aligned} \tag{3.8}$$

We are required to obtain the maximum value of $\mu(c,|x|)$ *on* $[0,2]\times[0,1]$. *First, assume that there is a maximum at an interior point* $\mu(c_0,|x_0|)$ *of* $[0,2]\times[0,1]$. *Differentiating* $\mu(c,|x|)$ *with respect to* $|x|$ *and equating it to 0 implies that* $c = c_0 = 2$ *which is a contradiction. Thus for the maximum of* $\mu(c,|x|)$, *we have to consider the end points of* $[0,2]\times[0,1]$.

For $c = 0$, *we obtain*

$$\begin{aligned} \mu(0,|x|) &= \left[\frac{8}{12(1+3a)}\right]|x|^2 + \frac{4}{6(1+3a)} \\ &\leq \frac{4}{3(1+3a)}. \end{aligned}$$

For $c = 2$, *we get*

$$\mu(2,|x|) = \frac{4|\eta_5|}{(1+a)^3(1+3a)}.$$

For $|x| = 0$, *we get*

$$\mu(c,0) = \left[\frac{a^3+3a^2-3a-1}{2(1+a)^3(1+3a)}\right]c_1^3 + \frac{4-c^2}{6(1+3a)}.$$

For $|x| = 1$, *we obtain*

$$\begin{aligned} \mu(p,1) &= \left[\frac{(c+2)(4-c^2)}{12(1+3a)}\right] + \left[\frac{5cX}{12(1+3a)}\right] \\ &+ \left[\frac{a^3+3a^2-3a-1}{2(1+a)^3(1+3a)}\right]c^3 + \frac{4-c^2}{6(1+3a)} \end{aligned}$$

which has the maximum value $\dfrac{4|\eta_5(a)|}{(1+a)^3(1+3a)}$ *attained at the end point* $c = 2$ *and* $\dfrac{4}{3(1+3a)}$ *at* $c = 0$ *where,*

$$\eta_5(a) = a^3 + 3a^2 - 3a - 1.$$

□

Theorem 6 *Let* f *given by (1.1), be in the class* $N_V^R(a)$; $(0 \leq a \leq 1)$. *Then we have sharp bound*

$$|a_5 - a_2^4| \leq max\left\{\frac{1}{(1+4a)}, \frac{|\eta_6(a)|}{(1+a)^4(1+4a)}\right\}$$

where,

$$\eta_6(a) = 5(1+a)^4 - 16(1+4a).$$

Proof: First note that by equating the corresponding coefficients in the equation (3.1), using the fact that $2c_2 = c_1^2 + x\left(4 - c_1^2\right)$ and letting $X = 4 - c_1^2$, a simple computation leads to

$$\begin{aligned} a_5 - a_2^4 &= \left[\frac{5(1+a)^4 - 16(1+4a)}{16(1+4a)(1+a^4}\right] c_1^4 + \left[\frac{43}{96(1+4a)}\right] c_1^2 x X + \left[\frac{7}{24(1+4a)}\right] c_1 X \\ &- \left[\frac{(c+4)}{32(1+4a)}\right] c_1 X x^3 + \left[\frac{-5}{24(1+4a)}\right] c_1 X x^2 + \frac{Xx^2}{32(1+4a)} + \frac{-3c_1^2 X x^2}{32(1+4a)} \\ &- \frac{-c_1 X x}{8(1+4a)} - \frac{-X\overline{x}}{8(1+4a)} + \frac{Xx^2\overline{x}}{8(1+4a)} + \frac{X}{8(1+4a)}. \end{aligned} \tag{3.9}$$

Without loss of generality, we let $0 \leq c_1 = c \leq 2$. Substituting in (3.9) this into the above equation, we obtain the following equation in terms of $|x|$.

$$\begin{aligned} |c_5 - c_2^4| &\leq \left[\frac{(c+4)cX}{32(1+4a)}\right] |x|^3 \\ &+ \left[\frac{5c(4-c^2)}{24(1+4a)} + \frac{3c^2(4-c^2)}{23(1+4a)} + \frac{(4-c^2)}{32(1+4a)} + \frac{(4-c^2)\overline{x}}{8(1+4a)}\right] |x|^2 \\ &+ \left[\frac{(43c-12)c(4-c^2)}{96(1+4a)}\right] |x| + \frac{(1-\overline{x})(4-c^2)}{8(1+4a)} \\ &+ \left[\frac{5(1+a)^4 - 16(1+4a)}{16(1+4a)(1+a)^4}\right] c^4 \\ &= \varsigma(c, |x|). \end{aligned}$$

On similar lines of Theorem 5, here we see that the maximum of $\varsigma(c, |x|)$ is attained at the end points of $[0, 2] \times [0, 1]$. Therefore,

for $c = 0$ we obtain

$$\begin{aligned} \varsigma(0, |x|) &= \left[\frac{16}{32(1+4a)} + \frac{4\overline{x}}{8(1+4a)}\right] |x|^2 - \frac{4}{8(1+4a)} - \frac{4\overline{x}}{8(1+4a)} \\ &\leq \frac{1}{(1+4a)}. \end{aligned}$$

For $c = 2$, we obtain

$$\varsigma(2, |x|) = \frac{|\eta_6(a)|}{(1+4a)(1+a)}.$$

For $|x| = 0$, we get

$$\varsigma(c, 0) = \left[\frac{5(1+a)^4 - 16(1+4a)}{16(1+4a)(1+a)^4}\right] c^4 + \left[\frac{(4-c^2)}{8(1+4a)}\right].$$

For $|x| = 1$, we get

$$\begin{aligned} \varsigma(c, 1) &= \left[\frac{(c+4)cX}{32(1+4a)}\right] | \\ &+ \left[\frac{5c(4-c^2)}{24(1+4a)} + \frac{3c^2(4-c^2)}{23(1+4a)} + \frac{(4-c^2)}{32(1+4a)} + \frac{(4-c^2)}{8(1+4a)}\right] \\ &+ \left[\frac{(43c-12)c(4-c^2)}{96(1+4a)}\right] \\ &+ \left[\frac{5(1+a)^4 - 16(1+4a)}{16(1+4a)(1+a)^4}\right] c^4 \end{aligned}$$

which has the maximum value $\dfrac{|\eta_6(a)|}{(1+4a)(1+a)^4}$ attained at the end point $c = 2$ and $\dfrac{1}{(1+4a)}$ at $c = 0$ where,

$$\eta_6(a) = 5(1+a)^4 - 16(1+4a).$$

□

4. Declaration Statements

Availability of data and material

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Funding

Not applicable.

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