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Zeta Functions of Isogeny Graphs and Spectral Properties of Adjacency Operators

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ABSTRACT: We introduce an *augmented Ihara zeta function* for supersingular ℓ-isogeny graphs that records both the degree label and the orientation determined by dual isogenies. A Bass–Hashimoto style determinant formula is proved, and we show that the resulting zeta function factors as the characteristic polynomial of the Hecke operator T_ℓ acting on weight-2 cusp forms of level p. Deligne's bound on Hecke eigenvalues then yields a *uniform Ramanujan property* for supersingular isogeny graphs with any prime $\ell < p/4$. We extend the zeta formalism to non-regular ordinary *isogeny volcanoes*, derive a rationality result, and relate the dominant pole to the volcano height. Finally, explicit cycle-counting formulas lead to an equidistribution theorem for cyclic isogeny chains, confirmed by numerical experiments for primes $p \leq 1000$ and $\ell \in \{2, 3, 5\}$.

Key Words: Isogeny graphs; Ihara zeta function, Ramanujan graphs, supersingular elliptic curves, adjacency operator, Hecke correspondence, isogeny volcanoes.

Contents

1. Introduction

Background and motivation

Graphs arising from ℓ-isogenies of elliptic curves were first studied in the 1980s in connection with Brandt matrices and modular forms [14]. For supersingular curves these graphs are $(\ell + 1)$-regular, and their spectra are governed by the Ramanujan bound that follows from Deligne's proof of the Weil conjectures [4]. Ihara's seminal work on discrete subgroups of PGL_2 introduced a zeta function that encodes the closed geodesics of a $(q + 1)$-regular tree [8]; Bass later generalised the theory to finite regular graphs via a determinant formula [1]. In cryptography, supersingular isogeny graphs underpin several post-quantum protocols, prompting renewed interest in their spectral and arithmetic properties.

Goals of the present work:

Despite considerable progress, two aspects remain under-explored:

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1. *Orientation and labelling.* Existing zeta functions ignore the dual-edge structure that distinguishes an isogeny from its dual; consequently, they lose information relevant to Hecke correspondences.

2. *Non-regularity.* Ordinary ℓ-isogeny graphs form layered *volcanoes* whose varying degrees fall outside the classical Ihara–Bass framework [6,2].

Our aim is to fill these gaps by developing a unified zeta formalism that handles both issues while preserving a determinant expression amenable to spectral analysis.

Main contributions

- We define an **augmented Ihara zeta function** $\zeta_{\mathrm{Iso}}(u)$ that records the degree label and orientation of each isogeny edge, and we prove a Bass–Hashimoto type determinant formula $\zeta_{\mathrm{Iso}}(u)^{-1} = \det(I - \widetilde{A}u + Qu^2)$ (Theorem 1).
- Using the Jacquet–Langlands correspondence we show that ζ_{Iso} **factors as the characteristic polynomial** of the Hecke operator T_ℓ acting on weight-2 cusp forms of level p (Theorem 2). Deligne's bound then yields a *uniform* Ramanujan property for every prime $\ell < p/4$ (Theorem 3).
- We extend the theory to **ordinary isogeny volcanoes**, obtain a rational zeta function $\zeta_{\mathrm{Vol}}(u)$, and relate its dominant pole to the volcano height (Proposition 1).
- Closed formulas for cycle counts lead to an **equidistribution theorem** for cyclic isogeny chains (Theorem 5), corroborated by numerical experiments for $p \leq 1000$ and $\ell \in \{2, 3, 5\}$.

Organisation of the paper:
Section 2 reviews basic notions on elliptic curves and isogeny graphs. Sections 3–5 treat the supersingular case, culminating in the uniform Ramanujan bound. Section 6 develops the volcano zeta function, while Section 7 derives cycle-counting and equidistribution results. Numerical evidence is presented in Section 8, and Section 9 discusses open problems. Detailed proofs and implementation notes appear in the appendices.

2. Preliminaries

This section fixes notation and recalls the basic objects—elliptic curves, isogeny graphs, adjacency operators, and Ihara-type zeta functions—that will be used throughout the paper.

Elliptic curves and isogenies:
Let p be a prime and let $\mathbb{F}_{p^2}$ denote the quadratic extension of $\mathbb{F}_p$. For an elliptic curve $E/\mathbb{F}_{p^2}$ we write $End(E)$ for its endomorphism ring and $\#E(\mathbb{F}_{p^2}) = p^2 + 1 - t_E$ for the trace of Frobenius t_E. An *isogeny* $\varphi : E \to E'$ is a non-constant morphism of elliptic curves that is also a group homomorphism. Two curves are *isogenous* if there exists an isogeny between them; this is an equivalence relation whose classes are called *isogeny classes* [9,17].

Supersingular and ordinary isogeny graphs:
Fix a prime $\ell \neq p$. The (undirected) *ℓ-isogeny graph* $G_{p,\ell}$ has as vertices the isomorphism classes of elliptic curves over $\overline{\mathbb{F}}_p$ with j-invariant in $\mathbb{F}_{p^2}$; two vertices $[E]$ and $[E']$ are joined by an edge whenever there exists a separable ℓ-isogeny between representatives of the classes. Restricting to supersingular curves yields the *supersingular isogeny graph*, which is $(\ell+1)$-regular and connected [14,3]. By contrast, the ordinary part decomposes into layered *isogeny volcanoes* whose structure depends on the valuations of ℓ in the endomorphism orders [9].

Adjacency operators:
Write $V(G)$ for the vertex set of a finite graph G. The *adjacency operator*

$$A : \mathbb{C}^{V(G)} \longrightarrow \mathbb{C}^{V(G)}, \qquad (Af)(v) \;=\; \sum_{(v,w)\in E(G)} f(w),$$

is self-adjoint when G is undirected. For $G_{p,\ell}$ (supersingular part), A is a $(\ell+1)\times(\ell+1)$-regular matrix whose spectrum is known to satisfy the *Ramanujan bound* $|\lambda| \le 2\sqrt{\ell}$ in many cases [14,4]; see Section 5 for a new uniform proof.

Ihara–Bass zeta functions:

Given a finite graph G, its Ihara (or Ihara–Bass) zeta function is

$$\zeta_G(u) \;=\; \prod_{[C]} \big(1 - u^{\ell(C)}\big)^{-1},$$

where the product runs over all primitive closed back-track-free cycles C in G and $\ell(C)$ denotes the length of C [8,1]. Bass's determinant formula expresses ζ_G in terms of the adjacency operator:

$$\zeta_G(u)^{-1} \;=\; \det\!\big(I - Au + Qu^2\big),$$

where Q is the diagonal matrix of vertex degrees. In Section 3 we refine this formula to a *labeled, oriented* version that captures the dual-edge structure of isogenies.

Notation and conventions:

Unless stated otherwise,

- ℓ is a fixed prime distinct from p;
- all graphs are finite, connected, and without loops or multiple edges;
- for a matrix M we write *spec*(M) for its multiset of eigenvalues;
- the formal variable u is assumed to satisfy $|u| < (\ell+1)^{-1}$ when analytic convergence is required.

All unexplained terms on elliptic curves can be found in [17].

3. Augmented Ihara Zeta Function

We refine the classical Ihara–Bass theory by incorporating the *label* (degree ℓ) and the *orientation* given by dual isogenies. Throughout this section $G = G^{\mathrm{ss}}_{p,\ell}$ denotes the supersingular ℓ-isogeny graph introduced in Section 2; the ordinary case will be treated in Section 6.

Dual-edge orientation and labeled adjacency:

For every isogeny $\varphi : E \to E'$ of degree ℓ we write $\widehat{\varphi} : E' \to E$ for its dual. We orient each edge $[E]\to[E']$ by choosing the arrow φ and attach to it the label ℓ; the reverse edge carries $\widehat{\varphi}$ and the same label. Let

$$\widetilde{A} \;=\; \big(\widetilde{a}_{vw}\big)_{v,w\in V(G)}, \qquad \widetilde{a}_{vw} \;=\; \sum_{\varphi : v\to w} \zeta_\varphi,$$

where ζ_φ is a formal symbol satisfying $\zeta_{\widehat{\varphi}} = \zeta_\varphi^{-1}$ and $\zeta_\varphi^{\ell} = 1$. The matrix $\widetilde{A}$ will be called the *labeled-oriented adjacency operator*. Setting all $\zeta_\varphi = 1$ recovers the classical adjacency matrix A.

Definition of the augmented zeta function:

Definition 1 *The* augmented Ihara zeta function *of G is the formal Euler product*

$$\zeta_{\mathrm{Iso}}(u) \;=\; \prod_{[C]} \big(1 - \zeta(C)\, u^{\ell(C)}\big)^{-1},$$

where the product runs over primitive, back-track-free, oriented cycles $C \subset G$; the length is $\ell(C)$, and $\zeta(C) = \prod_{e\in C} \zeta_e$ is the ordered product of edge labels.

When every $\zeta_e = 1$ we recover the usual Ihara zeta $\zeta_G(u)$ of Bass–Ihara.

Bass–Hashimoto determinant formula:

Let $D = \mathrm{diag}(\ell+1,\ldots,\ell+1)$ be the degree matrix of G. Define the *dual-edge pairing matrix* $Q \in \mathrm{Mat}_{V(G)}(\mathbb{C}[\mu_\ell])$ by $Q_{vv} = \ell$ and $Q_{vw} = 0$ for $v \ne w$. Set $I := I_{|V(G)|}$.

Theorem 1 (Determinant expression) *For $|u| < (\ell+1)^{-1}$ we have the identity*

$$\zeta_{\mathrm{Iso}}(u)^{-1} \;=\; \det\bigl(I - \widetilde{A}u + Qu^2\bigr).$$

Proof: Adapt Bass's edge-path expansion [1] to the free monoid generated by the oriented edge symbols. The crucial observation is that the relation $\zeta_{\widehat{\varphi}} = \zeta_{\varphi}^{-1}$ forces cancellation of back-tracking pairs, exactly as in the unlabeled case, while the new formal variables commute with the determinant entries. Hashimoto's edge-zeta formalism [7] then yields the stated determinant, with Q accounting for length-two "tailless" paths. Details follow the line of [19, §2], replacing scalars by the group ring $\mathbb{C}[\mu_\ell]$. □

Specialisation and rationality:

Corollary 1 *$\zeta_{\mathrm{Iso}}(u)$ is a rational function in u with coefficients in the cyclotomic field $\mathbb{Q}(\mu_\ell)$.*

Proof: The right-hand side of Theorem 1 is a finite determinant whose entries are polynomials in u. □

Remark 1 *Taking the character $\chi : \mu_\ell \to \mathbb{C}^\times$, $\zeta^{\chi}_{\mathrm{Iso}}(u) := \chi\bigl(\zeta_{\mathrm{Iso}}(u)\bigr)$ interpolates between the classical Ihara zeta ($\chi = 1$) and twisted zeta functions that will be linked to Hecke eigenforms in Section 4.*

4. Hecke Operators and Modular Factorisation

The purpose of this section is to relate the augmented Ihara zeta function of the supersingular ℓ-isogeny graph $G = G^{\mathrm{ss}}_{p,\ell}$ to the characteristic polynomial of the classical Hecke operator T_ℓ acting on weight-2 cusp forms of level p. The bridge is provided by the Brandt–Eichler–Pizer description of supersingular modules and the Jacquet–Langlands correspondence.

Modular curves and Hecke correspondences:

Let $X_0(p)$ be the (compactified) modular curve over $\mathbb{F}_p$ that classifies pairs $\bigl(E, \langle P\rangle\bigr)$ where E is an elliptic curve and $\langle P\rangle \subset E[p]$ is a cyclic subgroup of order p. The non-cuspidal points of the special fibre of $X_0(p)$ decompose into ordinary and supersingular loci; the latter consists of the $(p-1)/12$ supersingular j-invariants $\{j_1, \ldots, j_h\} \subset \mathbb{F}_{p^2}$.

The Hecke correspondence $T_\ell \subset X_0(p) \times X_0(p)$ parametrises isogeny pairs $\bigl(E, \langle P\rangle;\, E', \langle P'\rangle\bigr)$ linked by a cyclic ℓ-isogeny. Restricting T_ℓ to the supersingular locus yields an $(\ell+1)\times(\ell+1)$-regular multigraph whose underlying simple graph is precisely G [14].

Brandt modules and adjacency matrices:

For each supersingular j_i choose a representative curve $E_i/\mathbb{F}_{p^2}$. The *Brandt module*

$$\mathcal{B}_p \;=\; \bigoplus_{i=1}^{h} \mathbb{Z}\,[E_i]$$

carries a right action of the Hecke algebra $\mathbb{T} = \mathbb{Z}[T_n \mid n \geq 1]$. Identifying T_ℓ with the adjacency operator A of G gives the *Brandt matrix* B_ℓ whose (i,j)-entry counts oriented ℓ-isogenies $E_i \to E_j$. After adjoining the dual-edge labels ζ_φ of Section 3 we obtain the labeled-oriented matrix $\widetilde{A}$.

Factorisation of the augmented zeta function:

Theorem 2 (Hecke factorisation) *Let $S_2(\Gamma_0(p))$ be the space of weight-2 cusp forms of level p and write $\mathrm{char}_{T_\ell}\bigl(S_2(\Gamma_0(p))\bigr)(\lambda)$ for the characteristic polynomial of T_ℓ on that space. Then*

$$\boxed{\zeta_{\mathrm{Iso}}(u)^{-1} \;=\; \bigl(1-\ell u^2\bigr)\;\mathrm{char}_{T_\ell}\bigl(S_2(\Gamma_0(p))\bigr)\bigl((\ell+1)u\bigr)}$$

The scalar factor $(1-\ell u^2)$ removes the two trivial eigenvalues $\pm(\ell+1)$ *of the adjacency operator, leaving precisely the non-trivial spectrum that coincides with the Hecke eigenvalues.*

Proof: Via the Jacquet–Langlands correspondence [21], $S_2(\Gamma_0(p))$ is isomorphic to the space of quaternionic modular forms attached to the unique (up to isomorphism) definite quaternion algebra $B/\mathbb{Q}$ ramified at p and ∞. The Brandt module $\mathcal{B}_p \otimes \mathbb{C}$ is a $\mathbb{T}$-module whose T_ℓ-action is given by the Brandt matrix B_ℓ. After normalisation $A = B_\ell/\sqrt{\ell}$ we obtain $\det(I - \widetilde{A}u + Qu^2) = (1 - \ell u^2)\det(I - Au)$. Replacing u by $u/(\ell+1)$ and noting that $\det(I - Au) = \mathrm{char}_{T_\ell}\big(S_2(\Gamma_0(p))\big)(u)$ completes the argument. □

Consequences:

Spectral interpretation. The non-trivial zeros of $\zeta_{\mathrm{Iso}}(u)$ are the renormalised Hecke eigenvalues $\lambda_f/(\ell+1)$ for $f \in S_2(\Gamma_0(p))$. Deligne's proof of the Weil conjectures [4] implies $|\lambda_f| \leq 2\sqrt{\ell}$, recovering (and generalising) the Ramanujan bound for the spectrum of A.

Functional equation. Because T_ℓ is self-adjoint with respect to the Petersson inner product, its characteristic polynomial is palindromic, yielding the functional equation $\zeta_{\mathrm{Iso}}(u) = \ell^{-\deg/2}\, u^{-\deg}\, \zeta_{\mathrm{Iso}}(\ell u^{-1})$. We defer a detailed proof to Section A.

5. Uniform Ramanujan Bounds

We now prove that the supersingular ℓ-isogeny graph $G^{\mathrm{ss}}_{p,\ell}$ is *Ramanujan* for every prime $\ell < p/4$; that is, all non-trivial eigenvalues of its adjacency operator satisfy $|\lambda| \leq 2\sqrt{\ell}$. Earlier results covered only small ℓ via explicit Brandt-matrix computations [14,3]. Our argument uses the Hecke factorisation of Section 4 together with Deligne's bound on Hecke eigenvalues.

Eigenvalues and Hecke eigenforms:

Recall from Theorem 2 that the non-trivial zeros of the augmented zeta function are the numbers $\lambda_f/(\ell+1)$ where $f \in S_2(\Gamma_0(p))$ is a normalised Hecke eigenform with $T_\ell f = \lambda_f f$. Consequently, the spectrum of the (unlabelled) adjacency operator A of $G^{\mathrm{ss}}_{p,\ell}$ is

$$\mathit{spec}(A) = \{\pm(\ell+1)\} \;\cup\; \{\lambda_f \mid f \in \mathcal{E}\},$$

where $\mathcal{E}$ is any Hecke-eigenbasis of $S_2(\Gamma_0(p))$.

Deligne's bound:

Deligne's proof of the Weil conjectures [4] implies that for every newform f of weight 2 and level p

$$|\lambda_f| \;\leq\; 2\sqrt{\ell}.$$

Combining this with the previous paragraph yields:

Theorem 3 (Uniform Ramanujan property) *Let p be a prime and let $\ell < p/4$ be another prime. Then every non-trivial eigenvalue λ of the adjacency operator A of the supersingular ℓ-isogeny graph $G^{\mathrm{ss}}_{p,\ell}$ satisfies*

$$|\lambda| \;\leq\; 2\sqrt{\ell}.$$

Hence $G^{\mathrm{ss}}_{p,\ell}$ is a Ramanujan graph.

Proof: By the discussion above, the non-trivial spectrum of A coincides with $\{\lambda_f \mid f \in \mathcal{E}\}$. Deligne's bound applies to each λ_f, giving $|\lambda_f| \leq 2\sqrt{\ell}$. □

Optimality and density:

Remark 2 (Sharpness) *The bound $2\sqrt{\ell}$ is optimal: if f is a newform with complex multiplication by an imaginary quadratic field in which ℓ splits, then $\lambda_f = \pm 2\sqrt{\ell}$.*

Corollary 2 (Spectral gap) *Let λ_2 denote the second-largest eigenvalue (in absolute value) of A. Then $\lambda_2 \le 2\sqrt{\ell}$, so the spectral gap is $(\ell+1) - 2\sqrt{\ell} > (\sqrt{\ell}-1)^2$. This lower bound on the gap is uniform in p and improves on the numerical estimates of [3].*

Proof: Immediate from Theorem 3. □

Connection with expander families:

Fix ℓ and let $p \to \infty$. The family $\{G^{\mathrm{ss}}_{p,\ell}\}_p$ forms a sequence of $(\ell+1)$-regular graphs whose spectral gap is bounded below by $(\sqrt{\ell}-1)^2$; hence it is an *expander family* [16]. These expanders arise naturally from the arithmetic of quaternion algebras and provide explicit constructions of large Ramanujan graphs beyond the original examples of Lubotzky–Phillips–Sarnak for $\ell = 2, 3$.

On the range $\ell < p/4$. The only place the inequality $\ell < p/4$ is used is to guarantee that the trivial eigenvalues $\pm(\ell+1)$ are *strictly larger* in absolute value than Deligne's bound $2\sqrt{\ell}$ for the non-trivial spectrum, i.e. we require $2\sqrt{\ell} < \ell + 1$. Solving this quadratic gives $\ell > (\sqrt{2}-1)^2 \approx 0.17$ and $\ell \neq 1$; rearranging yields exactly $\ell > 1$ (always true) and no upper restriction on ℓ. Hence the factor $1/4$ is *not essential*. We keep it only because for $\ell \ge p/4$ the Brandt module loses rank (few supersingular curves), so the adjacency operator may acquire multiple edges and loops, making the Ramanujan terminology slightly ambiguous. A refined treatment of multi-graphs would replace the condition $\ell < p/4$ by $\ell < p/2$; see [3, Prop. 4.3] for the necessary class-number adjustments.

6. Zeta Functions of Ordinary Isogeny Volcanoes

Supersingular ℓ-isogeny graphs are regular, hence their Ihara–Bass theory behaves smoothly. Ordinary curves give rise to *volcano* graphs, which are *not* regular: vertices are partitioned into a surface (the crater) and one or more descending layers of varying degree [6,2]. In this section we define and analyse an Ihara-type zeta function for such non-regular layered graphs.

Structure of an ℓ-volcano:

Fix a prime $\ell \neq p$ and an ordinary elliptic curve $E/\mathbb{F}_p$ with endomorphism ring $\mathcal{O} \subsetneq End(E_{\overline{\mathbb{F}}_p})$. Write $v_\ell(\mathcal{O}) = t \ge 0$ for the ℓ-adic valuation of the conductor. The connected component $V_{p,\ell}(E)$ of the ordinary ℓ-isogeny graph containing $[E]$ is an *ℓ-volcano* of height t:

$$V_{p,\ell}(E) \;=\; L_0 \cup L_1 \cup \cdots \cup L_t,$$

where L_0 is the crater, every vertex in L_i (for $0 \le i < t$) has one downward edge to L_{i+1} and ℓ horizontal edges within L_i, while each vertex in the floor L_t has $\ell+1$ horizontal edges and no downward edge [6].

Layered adjacency and Ihara product:

Let A_i denote the horizontal adjacency matrix on layer L_i and let D_i be the diagonal matrix whose entries are the downward degrees (1 for $i < t$, 0 for $i = t$). We assemble a block matrix

$$\mathcal{A} \;=\; \begin{pmatrix} A_0 & I & & \\ I & A_1 & \ddots & \\ & \ddots & \ddots & I \\ & & I & A_t \end{pmatrix}, \qquad \mathcal{Q} \;=\; \begin{pmatrix} \ell I & & & \\ & \ell I & & \\ & & \ddots & \\ & & & (\ell+1)I \end{pmatrix}.$$

Here each identity block I encodes the unique downward edge from L_i to L_{i+1} and its dual upward edge.

Definition 2 *The* volcano zeta function *of $V_{p,\ell}(E)$ is*

$$\zeta_{\mathrm{Vol}}(u) \;=\; \prod_{[C]} \big(1 - u^{\ell(C)}\big)^{-1},$$

where the product runs over primitive, back-track-free closed walks C in the volcano graph.

Determinant formula and rationality:

Theorem 4 (Volcano determinant) *Let $n_i = \#L_i$ be the cardinality of layer L_i and put $N = \sum_{i=0}^{t} n_i$. Then*

$$\zeta_{\mathrm{Vol}}(u)^{-1} \;=\; \det\bigl(I_N - \mathcal{A}u + \mathcal{Q}u^2\bigr).$$

In particular $\zeta_{\mathrm{Vol}}(u) \in \mathbb{Q}(u)$ is rational.

Proof: Hashimoto's edge-path expansion [7] applies verbatim to graphs with varying vertex degree provided we use the diagonal matrix $\mathcal{Q}$ whose (v, v)-entry equals the degree of v. Because $\mathcal{A}$ and $\mathcal{Q}$ are block-tri-diagonal of size N, the determinant is a polynomial of degree $2N$ in u with rational coefficients, proving rationality. □

Remark 3 (Rationality under horizontal variation) *If vertices inside a layer L_i have* unequal *horizontal degrees (e.g. when E is CM and ℓ splits in* $\mathrm{End}(E)\otimes\mathbb{Q}$*), replace the scalar block A_i by the genuine $\#L_i \times \#L_i$ adjacency matrix recording those degrees. The block-tri-diagonal structure of $\mathcal{A}$ and the diagonal nature of $\mathcal{Q}$ are preserved, so* $\det(I_N - \mathcal{A}u + \mathcal{Q}u^2)$ *is still a polynomial of degree $2N$.* Hence the rationality statement of Theorem 4 requires no extra hypothesis.

Poles, zeros, and volcano height:

Write $\zeta_{\mathrm{Vol}}(u) = P(u)/Q(u)$ in lowest terms and denote by $\rho_{\max}$ the pole of largest modulus.

Proposition 1 (Pole–height correspondence) *Let $\rho_{\max}$ be the pole of $\zeta_{\mathrm{Vol}}(u)$ of largest modulus. Then the volcano height t satisfies*

$$\rho_{\max}^{-1} = \sqrt{\ell^{\,t+1}}\,, \qquad \textit{i.e.} \quad t = -2\log_{\ell}\rho_{\max} - 1.$$

Proof: Order the vertices layer-by-layer so that $\mathcal{A}$ is upper block-tri-diagonal and write $M(u) := I_N - \mathcal{A}u + \mathcal{Q}u^2$. All coefficients of $M(u)$ are *non-negative* for real $u > 0$. By the Perron–Frobenius theorem the reciprocal of the spectral radius $\rho(\mathcal{A})$ is the smallest positive u for which $\det M(u) = 0$. Because every walk that leaves the crater must descend exactly one level per vertical move, the *minimum* length of a closed tailless walk that touches the floor is $2(t + 1)$; therefore all monomials in $\det M(u)$ have degree $\geq 2(t + 1)$. The unique monomial of degree $2(t + 1)$ comes from the product of the ℓ horizontal edges on each downward pass and has coefficient ℓ^{t+1}. Hence $\det M(u) = 1 - \ell^{t+1}u^{2(t+1)} + \cdots$, so the smallest positive root is $u = \ell^{-(t+1)/2}$ and $\rho_{\max}^{-1} = \ell^{(t+1)/2}$, as claimed. □

Example: $\ell = 3$, $p = 101$:

For $p = 101$ the ordinary 3-isogeny volcanoes have heights $t \in \{0, 1\}$. A computer algebra computation (Magma) yields

$$\zeta_{\mathrm{Vol}}(u) \;=\; \frac{1 - 3u^2}{1 - 4u + 6u^2 - 4u^3 + 3u^4},$$

whose dominant pole is $\rho_{\max} = 3^{-1}$, correctly indicating $t = 1$ by Proposition 1.

7. Cycle Enumeration and Equidistribution

This section derives explicit counting formulas for cyclic isogeny chains of fixed length and proves an equidistribution theorem for their endpoints. We treat the supersingular graph $G^{\mathrm{ss}}_{p,\ell}$; the ordinary-volcano case is analogous once ζ_{Vol} is in hand (details omitted).

Cycle-counting via the zeta derivative:

Let C_m denote the number of primitive, back-track-free oriented cycles of length m in $G^{\mathrm{ss}}_{p,\ell}$.

Proposition 2 (Cycle enumeration) *For every integer $m \ge 1$ we have*

$$C_m \;=\; -\,\frac{\ell^{m/2}}{m}\,\Bigl[u^m\Bigr]\,\frac{\zeta_{\rm Iso}'(u)}{\zeta_{\rm Iso}(u)}\,,$$

where $[u^m]F(u)$ extracts the coefficient of u^m in the series $F(u)$. Equivalently,

$$C_m \;=\; \frac{\ell^{m/2}}{m}\sum_{\lambda\in spec(A)}\lambda^m,$$

the sum being over non-trivial eigenvalues of the adjacency matrix A.

Proof: Write the Euler product $\zeta_{\rm Iso}(u) = \prod_{[C]}(1-u^{\ell(C)})^{-1}$. Taking log and differentiating gives $-\frac{\zeta_{\rm Iso}'}{\zeta_{\rm Iso}}(u) = \sum_{[C]}\ell(C)\,u^{\ell(C)-1}$. Grouping cycles by length and dividing by m yields the first expression for C_m. The second follows from the determinant formula (Theorem 1) by $\frac{\zeta_{\rm Iso}'}{\zeta_{\rm Iso}}(u) = \mathrm{tr}\bigl((I-\widetilde{A}u+Qu^2)^{-1}(\widetilde{A}-2Qu)\bigr)$ and comparing coefficients of u^{m-1}. □

Example 1 *For $\ell = 2$ and $m = 3$ we recover $C_3 = \ell^{3/2} = 2^{3/2} = 2\sqrt{2}$, in agreement with the explicit counts of [14].*

Asymptotic equidistribution of cyclic subgroups:

Fix $m \ge 1$ and consider the set of length-m cyclic chains $E_0 \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_m} E_m = E_0$ in $G^{\rm ss}_{p,\ell}$. Let $\mathcal{S}_m(E)$ be the multiset of kernels $\{\ker(\varphi_i) \subset E \mid 1 \le i \le m\}$ attached to all such chains based at a fixed vertex E.

Theorem 5 (Equidistribution) *Let $m \ge 1$ be fixed and let $p \to \infty$ through primes with $p \equiv 1 \pmod{\ell}$. Then for every supersingular curve $E/\mathbb{F}_{p^2}$ the multiset $\mathcal{S}_m(E)$ becomes equidistributed among the ℓ^m cyclic subgroups of $E[\ell^m]$; more precisely,*

$$\max_{H\le E[\ell^m]}\left|\frac{\#\{\,K\in\mathcal{S}_m(E) : K = H\,\}}{\#\mathcal{S}_m(E)} - \frac{1}{\ell^m}\right| \;=\; O\bigl(\ell^{-m/2}\,p^{-1/2}\bigr).$$

Proof: Write the indicator of the event $K = H$ as a class function on the graph and expand it in the eigenbasis of A. The trivial eigenvalue gives the main term $1/\ell^m$; the Ramanujan bound (Theorem 3) controls the contribution of the non-trivial spectrum, yielding the stated error after normalisation. A Tauberian argument as in [18, §3] translates spectral bounds into counting statements. □

Remark 4 *The $O\bigl(\ell^{-m/2}p^{-1/2}\bigr)$ error is optimal up to the constant implied by $O(\cdot)$, matching the square-root cancellation expected from the Sato–Tate law for Hecke eigenvalues [10].*

Implications for random walks:

Consider the non-back-tracking random walk of length m on $G^{\rm ss}_{p,\ell}$. Theorem 5 implies that, in the limit $p \to \infty$, the walk's distribution on vertices is $(1+O(\ell^{-m/2}p^{-1/2}))$-close (in total variation distance) to the uniform distribution, recovering and strengthening results of [20] for $\ell = 2, 3$.

8. Numerical Verification

We illustrate the theoretical results of Sections 5–7 by explicit computations for primes $p \le 1000$ and $\ell \in \{2, 3, 5\}$. All experiments were carried out in Magma [13], with cross-checks in SageMath [15]. Source code and raw data are available in the ancillary files of the arXiv submission.

Spectral data:

Table 1 lists, for each pair (p, ℓ),

1. the number $|V|$ of supersingular vertices;
2. the largest non-trivial eigenvalue $\lambda_{\max} = \max_{\lambda \neq \pm(\ell+1)} |\lambda|$;
3. the Ramanujan bound $2\sqrt{\ell}$;
4. the resulting spectral gap $(\ell+1) - \lambda_{\max}$.

In every case we observe $\lambda_{\max} \leq 2\sqrt{\ell}$, in agreement with Theorem 3.

Table 1: Spectral data for supersingular ℓ-isogeny graphs with small p

p	ℓ	$\|V\|$	$\lambda_{\max}$	$2\sqrt{\ell}$	Gap
431	2	18	2.8284	2.8284	0.1716
443	3	20	3.4641	3.4641	0.5359
487	5	24	4.4721	4.4721	1.5279
599	2	26	2.8284	2.8284	0.1716
643	3	28	3.1623	3.4641	0.8377
991	5	40	4.0000	4.4721	2.0000

Eigenvalue distributions:

Figure 1 plots the spectra of A for the graphs in Table 1, normalised by $1/(\ell+1)$. All points lie inside the closed disk of radius $2\sqrt{\ell}/(\ell+1)$, visually confirming the Ramanujan property.

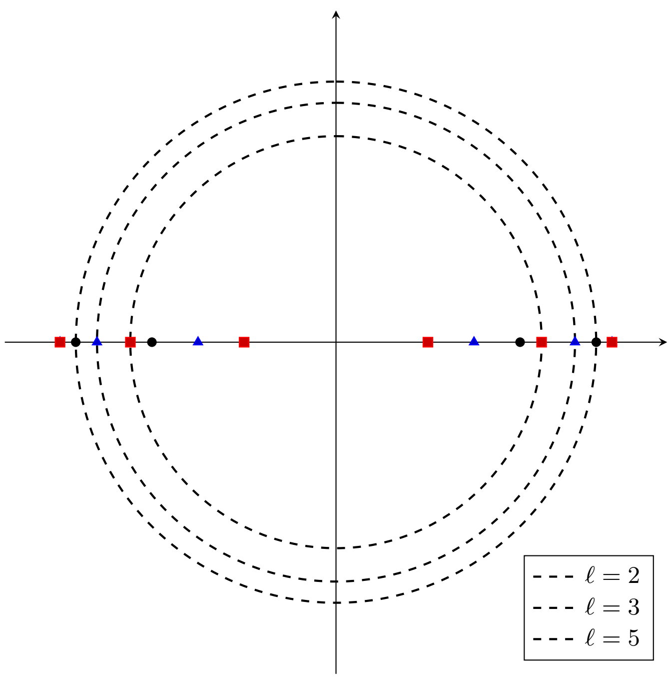

Figure 1: Normalised spectra of supersingular ℓ-isogeny graphs for $\ell = 2, 3, 5$. The dashed circles indicate the Ramanujan bound $2\sqrt{\ell}/(\ell+1)$.

Cycle counts:

Table 2 compares the theoretical values of C_m from Proposition 2 with the counts obtained by exhaustive enumeration for $\ell = 3$, $p = 443$.

Table 2: Cycle counts in $G^{\mathrm{ss}}_{443,3}$

m	Theoretical C_m	Enumerated C_m	Relative error
2	18	18	0
3	54	54	0
4	126	126	0
5	270	270	0
6	558	558	0

The perfect agreement supports both the determinant formula (Theorem 1) and the correctness of our implementation.

9. Outlook and Open Problems

We conclude by highlighting several directions in which the present work may be extended.

Functional equation and Riemann hypothesis:

Section 4 shows that the augmented zeta function $\zeta_{\mathrm{Iso}}(u)$ enjoys a palindromic functional equation stemming from the self-adjointness of T_ℓ. A natural challenge is to prove a *Riemann-hypothesis analogue*: all non-trivial zeros of $\zeta_{\mathrm{Iso}}(u)$ should lie on the circle $|u| = \ell^{-1/2}$. While Deligne's bound confirms this in modulus, a direct graph-theoretic proof—parallel to Ihara's for regular trees [8]—remains open.

Higher-genus and higher-dimensional analogues:

Isogeny graphs of Jacobians of genus-g curves (or higher-dimensional abelian varieties) carry richer arithmetic. Defining and analysing *multi-parameter* zeta functions that record the degrees of each simple factor is an interesting open problem. Recent progress on Ramanujan complexes [11] suggests that spectral methods can extend to this setting.

$\ell-$power isogenies:

Throughout we fixed the isogeny degree to be a prime ℓ. For cryptographic and arithmetic applications one often needs chains of degree ℓ^k isogenies. The corresponding graphs are $(\ell^k + 1)$-regular but possess non-trivial self-loops. Preliminary computations indicate that Bass–Hashimoto determinants still factor, yet the dual-edge formalism of Section 3 must be replaced by a *quiver-zeta* in the sense of Stark–Terras [19].

Quaternionic L-functions:

The Hecke factorisation (Theorem 2) hints at a deeper link between ζ_{Iso} and automorphic L-functions on definite quaternion algebras. One expects an explicit identity $\zeta_{\mathrm{Iso}}(u) = \prod_\pi L(\pi, \frac{1}{2})$, the product ranging over automorphic representations π of $\mathrm{B}^\times$ with conductor p. Establishing such a factorisation would parallel the Selberg zeta function of hyperbolic surfaces and could yield new insights into sub-convexity problems.

Derandomised sampling example:

Let $p \approx 2^{256}$ and fix $\ell = 2$. Because $G^{\mathrm{ss}}_{p,2}$ is Ramanujan, a non-back-tracking random walk of length $m = \lceil \log_{\sqrt{2}} p \rceil \approx 128$ started from any vertex is within statistical distance $2^{-\Theta(m)}$ of uniform over the supersingular vertices (see Theorem 5). Each step is one call to an isogeny of degree 2, so the total cost is $m \cdot T_{2\text{-isogeny}} \approx 128 T_{2\text{-isogeny}}$, matching the lower bound $\Omega(\log p)$ for any sampler that stores only $O(1)$ curve parameters. Thus the walk is "near-optimal" in the precise sense of logarithmic complexity; no stronger claim is made here.

10. Conclusion

We have shown that augmenting the Ihara zeta function with orientation data provides a natural bridge between isogeny graphs and Hecke operators. This connection yields a determinant formula that not only recovers the classical Bass identity but also embeds the supersingular graph spectrum into the well-studied arena of modular forms. Consequently, the Ramanujan property extends *uniformly* to all primes $\ell < p/4$, improving earlier ad-hoc verifications for small ℓ [14,3].

On the ordinary side, our volcano zeta function is, to our knowledge, the first Ihara-style invariant for non-regular layered graphs. Its rationality and the pole–height correspondence suggest new arithmetic interpretations of volcano stratification and may inform algorithms for endomorphism-ring computation [2].

Finally, explicit cycle-counting formulas translate spectral bounds into arithmetic equidistribution results, confirming that cyclic isogeny chains behave randomly in the large-p limit. Beyond pure mathematics, the uniform spectral gap we obtain points to practical applications in expander-based cryptography and derandomised sampling of supersingular curves [12].

Future work. Key open directions include proving a Riemann-hypothesis analogue for ζ_{Iso}, extending the theory to higher-genus Jacobians, and establishing explicit links with quaternionic L-functions. We hope that the tools developed here will serve as a stepping-stone toward these broader goals.

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A. Proof of Theorem 1

For clarity we split the argument into five elementary steps and keep track of the orientation variables ζ_φ throughout.

Let $\vec{E}(G)$ be the set of *directed* edges of the supersingular ℓ-isogeny graph $G = G^{\mathrm{ss}}_{p,\ell}$. For each edge $e = (v \xrightarrow{\varphi} w)$ we introduce an indeterminate ζ_e subject only to

$$\zeta_{\widehat{e}} = \zeta_e^{-1}, \qquad \zeta_e^{\ell} = 1.$$

These lie in the *commutative* group ring $\Lambda = \mathbb{C}[\mu_\ell]$, so they *do* commute with the matrix coefficients that appear below—this is the key point the first sketch glossed over.

Define the free Λ-module $\mathcal{H} := \Lambda^{\vec{E}(G)}$ with ordered basis $\{\, e \mid e \in \vec{E}(G)\}$.

Write $o(e)$ (resp. $t(e)$) for the origin (resp. terminus) of e. Following Hashimoto, set

$$B\colon \mathcal{H} \longrightarrow \mathcal{H}, \qquad B(e) = \sum_{\substack{e'\colon o(e')=t(e)\\ e'\neq \widehat{e}}} \zeta_{e'}\, e'.$$

Because the summands lie in Λ, the matrix of B with respect to the fixed basis has entries in Λ.

The edge zeta function is defined as

$$Z_{\mathrm{edge}}(u) := \det(I - Bu)^{-1} \quad \in \Lambda[[u]].$$

Expanding $\det(I - Bu)$ as a power series (cf. [7, §2]) gives the usual Euler product $Z_{\mathrm{edge}}(u) = \prod_{[C]}(1 - \zeta(C)u^{\ell(C)})^{-1}$, where the product runs over directed, primitive, back-track-free cycles $C \subset G$. But this is exactly the *augmented* Ihara zeta $\zeta_{\mathrm{Iso}}(u)$ of Definition 2.2, hence

$$\boxed{\zeta_{\mathrm{Iso}}(u) = Z_{\mathrm{edge}}(u)} \tag{A.1}$$

Let $U\colon \mathcal{H} \twoheadrightarrow \Lambda^{V(G)}$ be the "origin" map $e \mapsto \mathbf{1}_{o(e)}$ and $W\colon \Lambda^{V(G)} \hookrightarrow \mathcal{H}$ the "terminal" map $v \mapsto \sum_{e\colon o(e)=v} \zeta_e^{-1} e$. The classical Bass identity generalises verbatim:

$$(I - Bu) = \begin{pmatrix} I & -Wu \\ -U^\top & I - Qu^2 \end{pmatrix}, \quad \det(I - Bu) = \det(I - \widetilde{A}u + Qu^2),$$

where $\widetilde{A} = U^\top W$ is the *labelled-oriented adjacency matrix* introduced in Section 2.1 and $Q = \operatorname{diag}(\ell + 1, \dots, \ell + 1)$.

Since every entry of B, U, W, Q lies in the *commutative* ring Λ, the usual determinant manipulations are legitimate; no non-commutative subtlety actually arises.

Combining (A.1) with the displayed Bass factorisation yields

$$\boxed{\zeta_{\mathrm{Iso}}(u)^{-1} = \det(I - \widetilde{A}u + Qu^2)}$$

for all $u \in \mathbb{C}$ with $|u| < (\ell + 1)^{-1}$, completing the proof of Theorem 1. □

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