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The generalized fixed point theorem in fuzzy metric spaces and its application to an integral equation

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ABSTRACT: This paper presents a generalized fixed-point theorem in fuzzy metric spaces using an implicit relation to unify different contraction types. Based on continuous t-norms, the result extends previous work and includes corollaries demonstrating its generality. The approach simplifies analysis by eliminating separate proofs for each contraction type, while an application to integral equations demonstrates its practical utility, guaranteeing existence and uniqueness of solutions under specific conditions.

Key Words: fuzzy metric spaces, fixed point, common fixed point, implicit relation, integral equation.

Contents

1. Introduction

Fuzzy metric spaces, introduced by Kramosil and Michalek in 1994 [5], represent a significant advancement in the study of metric spaces. These spaces combine the flexibility of fuzzy sets and metric spaces. Later George and Veeramani in 1994 [1] modified the definition and obtain a Hausdorff topology for this kind of fuzzy metric spaces.

A key component of fuzzy metric spaces is the concept of a continuous t-norm. Recall that a binary operation $T : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following properties:

- Associativity and commutativity,
- Continuity,
- $T(a,1) = a$ for all $a \in [0,1]$,
- $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$.

Classic examples of continuous t-norms include:

- The product t-norm: $T_P(a,b) = a \cdot b$,
- The minimum t-norm: $T_{\min}(a,b) = \min\{a,b\}$,
- The Lukasiewicz t-norm: $T_L(a,b) = \max\{a+b-1, 0\}$.

Building on this foundation, the authors of [3] introduced the notion of an H-type t-norm. This is defined using the operator $T_n : [0,1] \to [0,1]$, where:

$$T_1(x) = T(x,x), \quad T_{n+1}(x) = T\left(T_n(x), x\right), \quad n \in \mathbb{N},\ x \in [0,1].$$

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A t-norm T is said to be of H-type if the family $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$. While $T_{\min}$ serves as a simple example of an H-type t-norm, more complex examples can be found in [3].

Furthermore, Klement et al. [7] demonstrated that any t-norm T can be uniquely extended to an n-ary operation. For $(x_1, \dots, x_n) \in [0,1]^n$, this extension is defined as:

$$T_{i=1}^{1} x_i = x_1, \quad T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) = T\left(x_1, x_2, \dots, x_n\right).$$

Moreover, they proved that T can be extended to a countable infinite operation. For any sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$, the value is given by:

$$T_{i=1}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{n} x_i.$$

The authors of [4] further established that for a sequence $(x_n)_{n\in\mathbb{N}} \subset [0,1]$ with $\lim\limits_{n\to\infty} x_n = 1$, and for an H-type t-norm T, the following holds:

$$\lim_{n\to\infty} T_{i=n}^{\infty} x_i = \lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1.$$

A fuzzy metric space is defined as a 3-tuple (X, M, T), where:

- X is a non-empty set,
- T is a continuous t-norm,
- M is a fuzzy set on $X \times X \times (0, \infty)$.

The fuzzy metric M satisfies the following axioms for all $x, y, z \in X$ and $t, u > 0$:

(M_1) $M(x, y, t) > 0$,

(M_2) $M(x, y, t) = 1$ if and only if $x = y$,

(M_3) $M(x, y, t) = M(y, x, t)$,

(M_4) $M(x, z, t + u) \geq T\left(M(x, y, t), M(y, z, u)\right)$,

(M_5) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

In [2] Grabiec proved that $M(x, y, .)$ is nondecreasing for all $x, y \in X$.

In 2004, the authors of [9] established that the fuzzy metric is a continuous function on $X \times X \times (0, \infty)$.

As explored in [6], it is often assumed that the fuzzy metric satisfies the additional property:

$$(M_6) : \lim_{t\to\infty} M(x, y, t) = 1.$$

Using the property (M_6) the same authors obtained the following Lemma.

Lemma 1.1 *If for some $k \in (0, 1)$ and $x, y \in X$,*

$$M(x, y, t) \geq M\left(x, y, \frac{t}{k}\right), \quad \forall t > 0, \tag{1.1}$$

then $x = y$.

This paper introduces a generalized fixed-point theorem in fuzzy metric spaces, using an implicit relation to unify different contraction types. The approach simplifies analysis by eliminating the need for separate proofs for each contraction type.

Based on continuous t-norms (particularly H-type), the result extends previous work and is accompanied by corollaries demonstrating its generality. An application to an integral equation demonstrates its practical utility, guaranteeing the existence and uniqueness of solutions under specific conditions.

In the rest of this paper, we will use the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$.

The authors of [8] established the following key lemma:

Lemma 1.2 *Let (x_n) be a sequence in a fuzzy metric space (X, M, T). If there exists $k \in (0,1)$ such that*

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{k}\right), \quad t > 0,\ n \in \mathbb{N},$$

and if the following condition holds:

$$\lim_{n\to\infty} T_{i=n}^{\infty} M\left(x_0, x_1, \frac{1}{\mu^i}\right) = 1, \quad \mu \in (0,1), \tag{1.2}$$

then (x_n) is a Cauchy sequence.

2. Main results

We begin by stating assumptions about the implicit relation, which generalize the specific types of contractions.

Let T be a continuous t-norm and $k \in (0,1)$. Consider a function $\phi_{T,k} : (0,1]^6 \to \mathbb{R}$, defined as:

$$(t_1, t_2, t_3, t_4, t_5, t_6) \mapsto \phi_{T,k}(t_1, t_2, t_3, t_4, t_5, t_6).$$

We impose the following conditions on $\phi_{T,k}$:

(H_1) $\phi_{T,k}$ is continuous with respect to t_1, t_2, t_4, t_5, and t_6.

(H_2) $\phi_{T,k}$ is nonincreasing in the variables t_5 and t_6.

(H_3) There exists $\alpha \in (0,1)$ such that: for all nondecreasing functions $u, v : (0,\infty) \to (0,1]$, we have:

$$\phi_{T,k}\left(u(kt), v(t), u(t), v(t), T\left(u\left(\frac{t}{2}\right), v\left(\frac{t}{2}\right)\right), 1\right) \geq 0$$

or

$$\phi_{T,k}\left(u(kt), v(t), v(t), u(t), 1, T\left(u\left(\frac{t}{2}\right), v\left(\frac{t}{2}\right)\right)\right) \geq 0$$
$$\implies u(\alpha t) \geq \min\{u(t), v(t)\}.$$

(H_4) For all function $u : (0,\infty) \to (0,1]$, we have

$$\phi_{T,k}\left(u(kt), u(t), 1, 1, u(t), u(t)\right) \geq 0 \implies u(kt) \geq u(t).$$

Example 2.1 $\phi_{T,k}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$, *where $T = T_{min}$ and $k \in (0, \frac{1}{2})$.*

Lemma 2.1 *Let (X, M, T) be a fuzzy metric space, and let $f, g : X \to X$ be two mappings. Assume that there exists $k \in (0,1)$ such that:*

$$\phi_{T,k}\left(M(fx, gy, kt), M(x, y, t), M(fx, x, t), M(gy, y, t), M(fx, y, t), M(gy, x, t)\right) \geq 0, \quad \forall (x,y) \in X^2. \tag{2.1}$$

Furthermore, suppose that the conditions (H_2) and (H_3) hold. Then, if $fx = x$, it follows that $gx = x$, and conversely.

Proof. Assume that $fx = x$. By (2.1), we have

$$\phi_{T,k}\left(M(fx, gx, kt), M(x, x, t), M(fx, x, t), M(gx, x, t), M(fx, x, t), M(gx, x, t)\right) \geq 0.$$

This simplifies to:

$$\phi_{T,k}\left(M(x, gx, kt), 1, 1, M(gx, x, t), 1, M(gx, x, t)\right) \geq 0.$$

Using (H_2), we deduce that:

$$\phi_{T,k}\left(M(x,gx,kt),1,1,M(gx,x,t),1,M\left(gx,x,\frac{t}{2}\right)\right)\geq 0.$$

Now, by using (H_3) with $u(t) = M(gx,x,t)$ and $v(t) = 1$, we infer that $M(gx,x,\alpha t)\geq \min\{M(gx,x,t),1\} = M(gx,x,t)$, then from Lemma 1.1, it follows that $gx = x$.

The same reasoning applies if $gx = x$.

Theorem 2.1 *Let (X,M,T) be a complete fuzzy metric space, and Let $f,g : X\to X$ be two mappings satisfying (2.1) with $\phi_{T,k} : (0,1]^6\to\mathbb{R}$ is a function satisfying the conditions $(H_1),(H_2)$, and (H_3). Assume that there exists $x_0\in X$ such that:*

$$\lim_{n\to\infty} T_{i=n}^{\infty} M\left(x_0,fx_0,\frac{1}{\mu^i}\right) = 1,\quad \mu\in(0,1). \tag{2.2}$$

Then, f and g have a unique common fixed point $x\in X$.

Proof. Existence: According to Lemma 2.3, it is enough to establish that f or g has a fixed point. Let $x_0\in X$, $x_{2n+1} = fx_{2n}$, and $x_{2n+2} = gx_{2n+1}$, $n\in\mathbb{N}_0$.

For each $n\in\mathbb{N}$, using the hypothesis (2.1), with $x = x_{2n}$ and $y = x_{2n-1}$, we get the following:

$$\phi_{T,k}\left(\begin{array}{c} M\left(fx_{2n},gx_{2n-1},kt\right),M\left(x_{2n},x_{2n-1},t\right),M\left(fx_{2n},x_{2n},t\right), \\ M\left(gx_{2n-1},x_{2n-1},t\right),M\left(fx_{2n},x_{2n-1},t\right),M\left(gx_{2n-1},x_{2n},t\right)\end{array}\right)\geq 0.$$

This simplifies to:

$$\phi_{T,k}\left(\begin{array}{c} M\left(x_{2n+1},x_{2n},kt\right),M\left(x_{2n},x_{2n-1},t\right),M\left(x_{2n+1},x_{2n},t\right), \\ M\left(x_{2n},x_{2n-1},t\right),M\left(x_{2n+1},x_{2n-1},t\right),M\left(x_{2n},x_{2n},t\right)\end{array}\right)\geq 0.$$

Using (H_2), we deduce that:

$$\phi_{T,k}\left(\begin{array}{c} M\left(x_{2n+1},x_{2n},kt\right),M\left(x_{2n},x_{2n-1},t\right),M\left(x_{2n+1},x_{2n},t\right), \\ M\left(x_{2n},x_{2n-1},t\right),T(M(x_{2n+1},x_{2n},\frac{t}{2}),M(x_{2n},x_{2n-1},\frac{t}{2})),1\end{array}\right)\geq 0.$$

By hypothesis (H_3), it follows that:

$$M\left(x_{2n+1},x_{2n},\alpha t\right)\geq\min\{M\left(x_{2n+1},x_{2n},t\right),M\left(x_{2n},x_{2n-1},t\right)\}. \tag{2.3}$$

On the other hand, in the same way, by using (2.1), with $x = x_{2n}$ and $y = x_{2n+1}$, we obtain for each $n\in\mathbb{N}_0$:

$$\phi_{T,k}\left(\begin{array}{c} M\left(fx_{2n},gx_{2n+1},kt\right),M\left(x_{2n},x_{2n+1},t\right),M\left(fx_{2n},x_{2n},t\right), \\ M\left(gx_{2n+1},x_{2n+1},t\right),M\left(fx_{2n},x_{2n+1},t\right),M\left(gx_{2n+1},x_{2n},t\right)\end{array}\right)\geq 0.$$

This simplifies to:

$$\phi_{T,k}\left(\begin{array}{c} M\left(x_{2n+1},x_{2n+2},kt\right),M\left(x_{2n},x_{2n+1},t\right),M\left(x_{2n+1},x_{2n},t\right), \\ M\left(x_{2n+2},x_{2n+1},t\right),M\left(x_{2n+1},x_{2n+1},t\right),M\left(x_{2n+2},x_{2n},t\right)\end{array}\right)\geq 0.$$

Using (H_2), we deduce that:

$$\phi_{T,k}\left(\begin{array}{c} M\left(x_{2n+1},x_{2n+2},kt\right),M\left(x_{2n},x_{2n+1},t\right),M\left(x_{2n+1},x_{2n},t\right), \\ M\left(x_{2n+2},x_{2n+1},t\right),1,T(Mx_{2n+2},x_{2n+1},\frac{t}{2}),M(x_{2n+1},x_{2n},\frac{t}{2})\end{array}\right)\geq 0.$$

By hypothesis (H_3), it follows that:

$$M\left(x_{2n+1},x_{2n+2},\alpha t\right)\geq\min\{M\left(x_{2n},x_{2n+1},t\right),M\left(x_{2n+1},x_{2n+2},t\right)\}. \tag{2.4}$$

We distinguish three cases:

Case 1: If, for some n, we have $M\left(x_{2n}, x_{2n-1}, t\right) \geq M\left(x_{2n+1}, x_{2n}, t\right)$, then by (2.3) we have:

$$M\left(x_{2n+1}, x_{2n}, \alpha t\right) \geq M\left(x_{2n+1}, x_{2n}, t\right),$$

which, by Lemma 1.1, implies that $x_{2n} = x_{2n+1} = fx_{2n}$, and therefore x_{2n} is a fixed point of f.

Case 2: If, for some n, we have $M\left(x_{2n}, x_{2n+1}, t\right) \geq M\left(x_{2n+1}, x_{2n+2}, t\right)$, then by (2.4) we have:

$$M\left(x_{2n+1}, x_{2n+2}, \alpha t\right) \geq M\left(x_{2n+1}, x_{2n+2}, t\right),$$

which, by Lemma 1.1, implies that $x_{2n+1} = x_{2n+2} = gx_{2n+1}$, and hence x_{2n+1} is a fixed point of g.

Case 3: For each n we have $M\left(x_{2n}, x_{2n-1}, t\right) < M\left(x_{2n+1}, x_{2n}, t\right)$ and $M\left(x_{2n}, x_{2n+1}, t\right) < M\left(x_{2n+1}, x_{2n+2}, t\right)$, then by (2.3) and (2.4) we have:

$$M\left(x_{2n+1}, x_{2n}, \alpha t\right) \geq M\left(x_{2n}, x_{2n-1}, t\right), \qquad \forall n \in \mathbb{N}.$$

$$M\left(x_{2n+1}, x_{2n+2}, \alpha t\right) \geq M\left(x_{2n}, x_{2n+1}, t\right), \quad \forall n \in \mathbb{N}_0.$$

Therefore,

$$M(x_n, x_{n+1}, \alpha t) \geq M(x_{n-1}, x_n, t), \quad \forall n \in \mathbb{N}.$$

From Lemma 1.2, we deduce that (x_n) is a Cauchy sequence. Since (X, M, T) is a complete space, then, the sequence (x_n) converges to some point $x \in X$. Next, we show that

$$fx = x = gx.$$

According to (2.1), with $y = x_{2n-1}$, we have:

$$\phi_{T,k}\left(M\left(fx, gx_{2n-1}, kt\right), M\left(x, x_{2n-1}, t\right), M\left(fx, x, t\right),\right.$$
$$\left.M\left(gx_{2n-1}, x_{2n-1}, t\right), M\left(fx, x_{2n-1}, t\right), M\left(gx_{2n-1}, x, t\right)\right) \geq 0.$$

This simplifies to:

$$\phi_{T,k}\left(M\left(fx, x_{2n}, kt\right), M\left(x, x_{2n-1}, t\right), M\left(fx, x, t\right), M\left(x_{2n}, x_{2n-1}, t\right), M\left(fx, x_{2n-1}, t\right), M\left(x_{2n}, x, t\right)\right) \geq 0.$$

Using (H_2), we deduce that:

$$\phi_{T,k}\left(\begin{array}{c} M\left(fx, x_{2n}, kt\right), M\left(x, x_{2n-1}, t\right), M\left(fx, x, t\right), \\ M\left(x_{2n}, x_{2n-1}, t\right), T\left(M\left(fx, x, \frac{t}{2}\right), M\left(x, x_{2n-1}, \frac{t}{2}\right)\right), M\left(x_{2n}, x, t\right) \end{array}\right) \geq 0.$$

Taking limits as $n \to \infty$ and using (H_1), we get:

$$\phi_{T,k}\left(M\left(fx, x, kt\right), 1, M\left(fx, x, t\right), 1, M\left(fx, x, \frac{t}{2}\right), 1\right) \geq 0.$$

Now, by applying (H_3), with $u(t) = M(fx, x, t)$, and $v(t) = 1$, we see that $M(fx, x, \alpha t) \geq \min\left\{M(fx, x, t), 1\right\} = M\left(fx, x, t\right)$, it follows from Lemma 1.1 that $fx = x$, and from Lemma 2.1, $x = gx$. That is, x is a common fixed point of f and g.

Unicity: Suppose that there exists $y \in X$, another common fixed point of f and g. Then, we have:

$$\phi_{T,k}\left(M(fx, gy, kt), M(x, y, t), M(fx, x, t), M(gy, y, t), M(fx, y, t), M(gy, x, t)\right) \geq 0.$$

This simplifies to:

$$\phi_{T,k}\left(M(x, y, kt), M(x, y, t), 1, 1, M(x, y, t), M(y, x, t)\right) \geq 0.$$

Applying (H_4), we obtain $M(x, y, kt) \geq M(x, y, t)$. Then, by Lemma 2.1, $x = y$.

Corollary 2.1 *Let (X, M, T) be a complete fuzzy metric space, and let $f, g : X \to X$ be two mappings. Assume that there exists $k \in (0, 1)$ such that for all $x, y \in X$:*

$$M(fx, gy, kt) \geq M(x, y, t),$$

and the condition (2.2) is satisfied. Then, f and g have a unique common fixed point $x \in X$.

Proof. Apply Theorem 2.1, with

$$\phi_{T,k}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - t_2.$$

Example 2.2 *Let $X = [0, 1]$, and $M : X \times X \times (0, \infty) \to (0, 1]$ by:*

$$M(x, y, t) = \frac{t}{t+|x - y|}.$$

Then (X, M, T_p) is a complete fuzzy metric. Define $f, g : X \to X$ by $fx = gx = \frac{3}{4}x$, $x \in X$. We have

$$M(fx, gy, t) \geq M\left(x, y, \frac{t}{k}\right),$$

where $0 < k = \frac{3}{4} < 1$. Since $\sum\limits_{i\geq 1}\left(1 - \frac{1}{1+|x_0-fx_0|\mu^i}\right) < \infty$, for all $x_0 \in X$. Therefore, all conditions of Corollary 2.1 is fulfilled, then, f and g have a unique common fixed point in X.

Remark 2.1 *By taking $f = g$ in Corollary 2.1, we obtain the Banach fixed-point theorem in the setting of a fuzzy metric space.*

Corollary 2.2 *Let (X, M, T) be a complete fuzzy metric space, and let $f, g : X \to X$ be two mappings. Assume that there exists $k \in (0, 1)$ such that for all $x, y \in X$:*

$$M(fx, gy, kt) \geq \min\{M(x, y, t), M(fx, x, t), M(gy, y, t)\},$$

and the condition (2.2) is satisfied. Then, f and g have a unique common fixed point $x \in X$.

Proof. Apply Theorem 2.1, with

$$\phi_{T,k}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4\}.$$

The following Corollary generalizes Theorem 2 in [8].

Corollary 2.3 *Let $(X, M, T = T_{min})$ be a complete fuzzy metric space , and let $f, g : X \to X$ be two mappings. Assume that there exists $k \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$:*

$$M(fx, gy, kt) \geq \min\{M(x, y, t), M(fx, x, t), M(gy, y, t), M(fx, y, t), M(gy, x, t)\},$$

and the condition (2.2) is satisfied. Then, f and g have a unique common fixed point $x \in X$.

Proof. Apply Theorem 2.1, with

$$\phi_{T,k}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}.$$

Example 2.3 *Let $(X = [0, 1], M, T_{min})$ be a complete fuzzy metric space with $M(x, y, t) = \frac{t}{t+|x-y|}$. We define $f, g : X \to X$ by $fx = \frac{x}{3}$, and $gx = \frac{x}{6}$, $x \in X$. In this case, we have*

$$M(fx, x, t) = \frac{t}{t + |\frac{x}{3} - x|} \quad ; \quad M(gy, y, t) = \frac{t}{t + |\frac{y}{6} - y|} \quad ; \quad M(fx, gy, t) = \frac{t}{t + |\frac{x}{3} - \frac{y}{6}|},$$

for all $x, y \in X$. We prove that:

$$M(fx, gy, kt) \geq \min\{M(x, y, t), M(fx, x, t), M(gy, y, t), M(fx, y, t), M(gy, x, t)\}$$

holds for each $x, y \in X$, where $k = \frac{1}{3}$.

1. If $x \leq \frac{y}{2}$, then

$$\left|\frac{x}{3}-\frac{y}{6}\right| = \frac{y}{6}-\frac{x}{3} = \frac{1}{3}\left(\frac{y}{2}-x\right) \leq \frac{1}{3}(y-x) = \frac{1}{3}|y-x|.$$

Which implies that:

$$\frac{t}{t+\left|\frac{x}{3}-\frac{y}{6}\right|} \geq \frac{t}{t+\frac{1}{3}|y-x|}.$$

Consequently, we obtain:

$$\begin{aligned} M(fx,gy,t) &\geq M\left(x,y,\frac{t}{k}\right) \\ &\geq \min\left\{M\left(x,y,\frac{t}{k}\right), M\left(fx,x,\frac{t}{k}\right), M\left(gy,y,\frac{t}{k}\right), M\left(fx,y,\frac{t}{k}\right), M\left(gy,x,\frac{t}{k}\right)\right\}. \end{aligned}$$

2. If $x \geq \frac{y}{2}$, we have $M(gy,x,t) = \frac{t}{t+\left|\frac{y}{6}-x\right|}$. Then

$$\left|\frac{x}{3}-\frac{y}{6}\right| = \frac{x}{3}-\frac{y}{6} = \frac{1}{3}\left(x-\frac{y}{2}\right) \leq \frac{1}{3}\left(x-\frac{y}{6}\right) = \frac{1}{3}\left|x-\frac{y}{6}\right|.$$

Which implies that:

$$\frac{t}{t+\left|\frac{x}{3}-\frac{y}{6}\right|} \geq \frac{t}{t+\frac{1}{3}\left|\frac{y}{6}-x\right|}.$$

Consequently, we obtain:

$$\begin{aligned} M(fx,gy,t) &\geq M\left(gy,x,\frac{t}{k}\right) \\ &\geq \min\left\{M\left(x,y,\frac{t}{k}\right), M\left(fx,x,\frac{t}{k}\right), M\left(gy,y,\frac{t}{k}\right), M\left(fx,y,\frac{t}{k}\right), M\left(gy,x,\frac{t}{k}\right)\right\}. \end{aligned}$$

Since $k = \frac{1}{3} \in \left(0,\frac{1}{2}\right)$, and the condition (2.2) is satisfied (because T_{min} is of H-type). Therefore, all conditions of Corollary 2.3 are satisfied, then, f and g have a unique common fixed point in X.

Corollary 2.4 *Let $(X, M, T \geq T_p)$ be a complete fuzzy metric space, and let $f, g : X \to X$ be two mappings. Assume that there exists $k \in \left(0,\frac{1}{2}\right)$ such that for all $x, y \in X$:*

$$M(fx,gy,kt) \geq \min\left\{M(x,y,t), M(fx,x,t), M(gy,y,t), \sqrt{M(fx,y,t)}, \sqrt{M(gy,x,t)}\right\},$$

and the condition (2.2) is satisfied. Then, f and g have a unique common fixed point $x \in X$.

Proof. Apply Theorem 2.1, with

$$\phi_{T,k}(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \min\left\{t_2,t_3,t_4,\sqrt{t_5},\sqrt{t_6}\right\}.$$

3. Application

Consider $X = C([a,b],\mathbb{R})$, where $a, b \in \mathbb{R}$ and $a < b$ the space of all continuous functions defined on $I = [a,b]$, and define $M : X \times X \times (0,\infty) \to \mathbb{R}$ by

$$M(x,y,t) = e^{-\frac{\sup_{t\in[0,1]}|x(t)-y(t)|}{t}}, \quad \text{for all } x,y \in X, \quad t > 0.$$

Then (X, M, T_p) is a complete fuzzy metric space.
Now consider the integral equation:

$$x(u) = \lambda \int_a^b f(u,v,x(v))\,dv + g(u), \quad \text{for all } u \in I, \text{ and } x \in X, \tag{3.1}$$

where $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : I \to \mathbb{R}$, and $\lambda > 0$.

For the above integral equation, define the operator $F : X \to X$ by

$$Fx(u) = \lambda \int_a^b f(u, v, x(v))\, dv + g(u), \quad u \in I.$$

Our hypotheses are as follows:

(A) There exists a continuous function $\eta : I \times I \to (0, \infty)$ such that

$$|f(u, v, r) - f(u, v, r')| \leq \eta(u, v)|r - r'|, \quad \text{for all } u, v \in I, \text{ and } r, r' \in \mathbb{R}.$$

(B)

$$\sup_{u \in I} \int_a^b \eta(u, v)\, dv < \frac{1}{\lambda}.$$

Under assumptions **(A)** and **(B)**, the integral equation (3.1) has a unique solution in X.

It is clear that any fixed point of the operator F is a solution of (3.1). By conditions **(A)** and **(B)**, we have:

$$\begin{aligned}
\sup_{u \in [a,b]} |Fx(u) - Fy(u)| &\leq \sup_{u \in [a,b]} \int_a^b |\lambda f(u, v, x(v)) - \lambda f(u, v, y(v))|\, dv \\
&\leq \lambda \sup_{u \in [a,b]} \int_a^b \eta(u, v)|x(v) - y(v)|\, dv \\
&\leq \lambda \sup_{u \in [a,b]} \int_a^b \eta(u, v)\, dv \sup_{v \in [a,b]} |x(v) - y(v)| \\
&= k \sup_{v \in [a,b]} |x(v) - y(v)|,
\end{aligned}$$

where $k = \lambda \sup\limits_{u \in [a,b]} \int_a^b \eta(u, v)\, dv$.

Consequently, we obtain:

$$e^{-\frac{\sup\limits_{u \in [a,b]} |Fx(u) - Fy(u)|}{t}} \geq e^{-\frac{k \sup\limits_{v \in [a,b]} |x(v) - y(v)|}{t}}$$

That is

$$M(Fx, Fy, t) \geq M\left(x, y, \frac{t}{k}\right).$$

Since $0 < k < 1$, and the condition (2.2) is satisfied, indeed $\sum\limits_{i \geq 1} \left(1 - e^{-\sup\limits_{u \in [a,b]} |x_0 - Fx_0| \mu^i}\right)$ is convergent, for all $x_0 \in X$. Thus, all the conditions of Corollary 2.1 are satisfied with, $F = f = g$. Then F has a unique fixed point in X, which implies that the integral equation (3.1) has a unique solution in X.

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