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Fixed Point results in b-Metric Spaces Using Meir-Keeler Contraction with Application to the Dynamic Behavior of a Multispan Uniform Continuous Beam

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ABSTRACT: In this paper, we establish some new fixed-point results in the framework of b-metric spaces by utilizing Meir-Keeler type contractions. An illustrative example is provided to confirm the validity of the established theorems. Furthermore, to demonstrate the applicability of our findings, the obtained results are applied to solve a Volterra integral equation, which is used to analyze the dynamic behavior of a multispan uniformly continuous beam. The presented solution highlights the practical relevance of our theoretical results in mathematical modeling and engineering applications.

Key Words: Fixed point, Meir Keeler contraction, metric space, b-metric space, Volterra integral equation.

Contents

1. Introduction

Metric fixed point theory is a broad field of study with numerous applications. Many nonlinear problems in practical mathematics can be reduced for solving nonlinear functional equations, which can be expressed in terms of locating the fixed points of a given nonlinear operator of an infinite dimensional function space A into itself. One of the most renowned result in theory of fixed point is known as Banach contraction theorem (BCP) which was introduced by Banach [3] in 1922. Assuming a complete metric space A and $H : A \to A$ satisfying the following conditions for $0 < \delta < 1$, such that for every $\psi, \phi \in A$,

$$d(H\psi, H\phi) \leq \delta d(\psi, \phi).$$

This result has been extended in different types of metric space, see [5,6,7,14,18,21,27,28]. Bakhtin [2] generalised Banach's contraction principle in 1989. In fact, he introduced the notion of a b-metric space and generalize several fixed point theorems for contractive mappings in b-metric spaces. Czerwik [8] later proved the results of b-metric space in 1993.

Definition 1.1 [8] Consider, A be a non-empty set and $l \geq 1$ be a real number. A mapping $b_\sigma : A \times A \to [0, \infty)$ is considered as b-metric if it fulfills the following conditions for each $\psi, \phi, \rho \in A$:

1. $b_\sigma((\psi, \phi) = 0$ for each $\psi = \phi$;
2. $b_\sigma(\psi, \phi) = b_\sigma(\phi, \psi)$;
3. $b_\sigma(\psi, \rho) \leq l[b_\sigma(\psi, \phi) + b_\sigma(\phi, \rho)]$.

The pair (A, b_σ) is a b-metric space.

Example 1.1 *[8]*

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1. *Consider,* $A := l_p(\mathbb{R})$ *with* $0 < p < 1$, *where* $l_p(\mathbb{R}) = \{\{\psi_n\} \subset \mathbb{R} st. \sum_{n=1}^{\infty} \mid \psi_n{}^p \mid < \infty\}$.

   Define $b_\sigma : A \times A \to \mathbb{R}^+$ *as:*

$$b_\sigma(\psi, \phi) = (\sum_{n=1}^{\infty} \mid \psi_n - \phi_n \mid^p)^{\frac{1}{p}},$$

   where $\psi = \{\psi_n\}; \phi = \{\phi_n\}$ *(see [19,9][4]). Then* b_σ *is a b-metric space with coefficient* $l = 2^{\frac{1}{p}}$.

2. *Let* $A := L_p[0, 1]$ *be the space of all real functions* $\psi(t), t \in [0, 1]$ *such that all real functions* $\psi(t), t \in [0, 1]$ *for which* $\int_0^1 \mid \psi(t) \mid^p < \infty$ *with* $0 < p < 1$.
   Define $d : X \times X \to \mathbb{R}^+$ *as:*

$$b_\sigma(\psi, \phi) = (\int_0^1 \mid \psi(t) - \phi(t) \mid^p dt)^{\frac{1}{p}},$$

   then b_σ *is a b-metric space.*

Definition 1.2 [20] Let (A, b_σ) be a b-metric space. A sequence $\{\psi_n\}$ in A is:

1. Cauchy iff $b_\sigma(\psi_n, \psi_m) \to 0$ as $n, m \to \infty$;
2. Convergent iff there exist $\psi \in A$ such that $d(\psi_n, \psi) \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} \psi_n = x$;
3. complete if every Cauchy sequence is convergent.

In 1969, Meir-Keeler [26] obtained a new and generalised version of BCP known as Meir-Keeler contraction, which contributes the below definition;

Definition 1.3 [26] Let A be a complete metric space and let $h : A \to A$ fulfills the below conditions:

Given $\epsilon > 0$, there exists $\delta > 0$ for which

$$\epsilon \leq d(\psi, \phi) < \epsilon + \delta implies d(h\psi, h\phi) < \epsilon. \tag{1.1}$$

Then, h has a unique fixed point ζ. Moreover, for any $\psi \in A$,

$$lim_{n\to\infty} h^n\psi = \zeta, \tag{1.2}$$

where, $h^n\psi$ is the nth iteration of h at a point ψ.

The related results have been modified in a different manner, see [1,10,11,12,13,14,15,16,17,22,23,24, 25,29].

2. Main Result

First we will define the following lemma.

Lemma 2.1 *Consider* (A, b_σ) *be a complete b-metric space with* $l \geq 1$ *and let* $h, g : A \to A$ *be two self-maps such that* $h(\psi) \subseteq g(\psi)$. *If* $\phi_n = h\psi_n = g\psi_{n+1}$ *and* $\phi_n \neq \phi_{n+1}$ *for every* $n \in N$ *satisfies*

$$b_\sigma(\phi_n, \phi_{n+1}) < \lambda b_\sigma(\phi_{n-1}, \phi_n)$$

for all $n \in N$, *where* $\lambda \in (0, 1)$, *then* $\phi_n \neq \phi_m$, *whenever* $n \neq m$.

Now, we will prove our main result.

Theorem 2.1 *Suppose* (A, b_σ) *be a b-metric space with* $l > 1$ *and let* $h, g : A \to A$ *be two self-maps such that* $h(\psi) \subseteq g(\psi)$, *and one of the subset of* A *is complete. Consider the following conditions, holds for each* $\epsilon > 0$ *there exists* $\delta > 0$ *for which,*

$$\epsilon \leq b_\sigma(g\psi, g\phi) < \epsilon + \delta \text{ implies } lb_\sigma(h\psi, h\phi) < \epsilon \text{ and } h\psi = h\phi \text{ whenever } g\psi = g\phi. \tag{2.1}$$

Then h *and* g *have a unique point of coincidence, say* $\omega \in A$. *Moreover, if* h *and* g *are weakly compatible, then they have a unique common fixed point in* A

Proof: By using (2.1) for all $\psi, \phi \in A$ and $g\psi \neq g\phi$,

$$lb_\sigma(h\psi, h\phi) < b_\sigma(g\psi, g\phi). \tag{2.2}$$

Consider, $\psi_0 \in A$ be an arbitrary point, since $h(A) \subseteq g(A)$, choose sequence $\{\psi_n\}$ and $\{\phi_n\}$ in A such that $\phi_n = f\psi_n = g\psi_{n+1}, n = 0, 1, 2, ...$
if $\phi_{n+1} = \phi_n$ for some $n = p \in N$, then $g\phi_{p+1} = \phi_p = y_{p+1} = h\psi_{p+1}$, so h and g have a common coincidence point. Therefore, we consider $\phi_{n+1} \neq \phi_n$ for each $n \in N$.
Using (2.2) and taking $\psi = \psi_{n+1}, \phi = \phi_n$, we get,

$$lb_\sigma(\phi_n, \phi_{n+1}) < b_\sigma(\phi_{n-1}, \phi_n). \tag{2.3}$$

Since, $l > 1$, $\{b_\sigma(\phi_n, \phi_{n+1})\}$ is a decreasing sequence, therefore

$$\lim_{n\to\infty} b_\sigma(\phi_n, \phi_{n+1}) = 0. \tag{2.4}$$

By using (2.3) and(2.1), we get $\phi_n \neq \phi_m$, for $n \neq m$
Now using (2.2) and taking $\psi = \psi_{n+k}$, $\phi = \phi_{m+k}$, we get

$$l^k b_\sigma(\phi_{n+k}, \phi_{m+k}) < b_\sigma(\phi_n, \phi_m). \tag{2.5}$$

Next, to show $\{\phi_n\}$ is a Cauchy. For any $\epsilon > 0$, choose N such that, $n \geq N$, for which

$$b_\sigma(\phi_{n+1}, \phi_n) \leq \frac{\epsilon - \frac{\epsilon}{l}}{1+l}$$

Put $K(\phi_N, \epsilon) = \{\phi \in \{\phi_n\} : b_\sigma(\phi, \phi_N) \leq \epsilon\}$.
Now, define $Q : \{\phi_n\} \to \{\phi_n\}$ by $Q(\phi_n) = \phi_{n+1}$. If $\phi_m \in K(\phi_N, \epsilon)$ with $m > N$, then $\phi_m \neq \phi_N$,

$$\begin{aligned}
b_\sigma(Q^2\phi_m, \phi_N) &\leq l\big(b_\sigma(Q^2\phi_m, Q^2\phi_N) + b_\sigma(Q^2\phi_N, \phi_N)\big) \\
&= l\big(b_\sigma(\phi_{m+2}, \phi_{N+2}) + b_\sigma(\phi_{N+2}, \phi_N)\big) \\
&\leq l\big(b_\sigma(\phi_{m+2}, \phi_{N+2}) + b_\sigma(\phi_{N+2}, \phi_{N+1}) + b_\sigma(\phi_{N+1}, \phi_N)\big) \\
&\leq l\left(\frac{1}{l^2} b_\sigma(\phi_m, \phi_N) + \left(\frac{1}{l} + 1\right) b_\sigma(\phi_{N+1}, \phi_N)\right) \\
&= \frac{1}{l} b_\sigma(\phi_m, \phi_N) + l\left(\frac{1}{l} + 1\right) b_\sigma(\phi_{N+1}, \phi_N) \\
&\leq \frac{\epsilon}{l} + (1 + l) \cdot \frac{\epsilon - \frac{\epsilon}{l}}{1 + l} = \epsilon.
\end{aligned}$$

That is, Q^2 maps $K(\phi_N, \epsilon)$ into itself. Since, $\phi_{N+1} \in K(\phi_N, \epsilon)$, then, $\phi_{N+2} \in K(\phi_N, \epsilon)$.
Therefore, using (2.5) and triangular inequality,

$$b_\sigma(\phi_N, \phi_{N+2}) \leq l(b_\sigma(\phi_N, \phi_{N+1}) + b_\sigma(\phi_{N+1}, \phi_{N+2}))$$

$$\leq lb_\sigma\left(\left(\frac{\epsilon - \frac{\epsilon}{l}}{1+l}\right) + (1 + l)\frac{\epsilon - \frac{\epsilon}{l}}{1+l}\right)).$$

Therefore, $b_\sigma(\phi_N, \phi_{N+2}) \leq \epsilon(\frac{2l^2 - l - 1}{l(l+1)})$.
Put $\epsilon' = \epsilon(\frac{2l^2 - l - 1}{l(l+1)})$.
For $n > m > N$, since, $\phi_n, \phi_m \in K(\phi_N, \epsilon)$, we have

$$b_\sigma(\phi_n, \phi_m) \leq lb_\sigma(\phi_n, \phi_N) + b_\sigma(\phi_N, \phi_m)$$

$\leq lb_\sigma(\phi_n, \phi_{n+1}) + b_\sigma(\phi_{n+1}, \phi_N) + b_\sigma(\phi_N, \phi_m)) \leq l(\frac{\epsilon - \frac{\epsilon}{l}}{1+l} + \epsilon' + \epsilon') \leq \frac{-4l^2+3l-1}{l(l+1)}\epsilon.$
Thus, $\{\phi_n\}$ is a Cauchy. Since, $g(A)$ or $h(A)$ is complete, $h(A) \subseteq g(A)$, then $\{\phi_n\}$ converges to some point ω in $g(A)$. Thus, there exist a point $\rho \in A$ such that $g\rho = \omega$. To prove $h\rho = g\rho$, consider that

$h\rho \neq g\rho$.
By using triangular inequality, (2.2) and (2.4),

$$\begin{aligned} b_\sigma(h\rho, g\rho) &\leq l\big(b_\sigma(h\rho, h\psi_n) + b_\sigma(h\psi_n, g\rho)\big) \\ &\leq l\left(\frac{1}{l}b_\sigma(h\rho, h\psi_{n+1}) + b_\sigma(h\psi_{n+1}, h\psi_n) + b_\sigma(h\psi_n, g\rho)\right) \\ &\leq l\left(\frac{1}{l}b_\sigma(g\rho, g\psi_{n+1}) + b_\sigma(h\psi_n, h\psi_{n+1}) + b_\sigma(h\psi_{n+1}, g\rho)\right) \\ &= b_\sigma(g\rho, \phi_n) + l\, b_\sigma(\phi_n, \phi_{n+1}) + l\, b_\sigma(\phi_{n+1}, g\rho). \end{aligned}$$

putting $n \to \infty$, we have

$$b_\sigma(h\rho, g\rho) \leq 0,$$

which gives rise to a contradiction. Thus, $g\rho = \omega$.
Next to show that the point of coincidence of h and g is unique.
Suppose $u \neq \omega$ be the another coincidence point, so there exist $a \in A$ such that $ha = ga = \mu$.
Then,

$$b_\sigma(\omega, \mu) = b_\sigma(h\rho, ha) < \tfrac{1}{l}b_\sigma(g\rho, ga) = \tfrac{1}{l}b_\sigma(\omega, \mu),$$

here, contradiction arises. Thus, h and g have a unique point of coincidence and if h and g are weakly compatible, it is easy to prove that ω is the unique common fixed point. □

Example 2.1 *Let $R = \{0, 2\}, S = \{\frac{1}{n} \in N\}$, and $A = \{R \bigcup S\}$. Define $d : A \times A \to [0, \infty)$ as follows:*

$$b_\sigma(x, y) = \begin{cases} 0, & \text{if } \psi = \phi, \\ 1, & \text{if } \psi \neq \phi \text{ and } \{\psi, \phi\} \subset R \text{ or } \{\psi, \phi\} \subset S, \\ |\psi - \phi|^2, & \text{if } \psi \in R \text{ and } \phi \in S. \end{cases} \tag{2.6}$$

Then, (A, b_σ) is a complete b-metric space with $l = 2$ Now, define

$$h(\psi) = \begin{cases} 5 & \text{if } \psi \in R \bigcup S, \\ 5 + \frac{\psi - 5}{6} & \text{if } \psi \in T \end{cases} \tag{2.7}$$

and

$$g(\psi) = \begin{cases} 5 & \text{if } \psi \in R \bigcup S, \end{cases} \tag{2.8}$$

then for $\epsilon > 0$, choose $\delta = \epsilon$. As all the conditions of the above theorem are satisfied. Therefore, unique point of coincidence found in h and g.

3. Application

Consider the integral equations

$$\psi(t) = \int_a^b p(t, s, \psi(s)ds, s \in I = [a, b] \tag{3.1}$$

and

$$\phi(t) = \int_a^b p(t, s, \phi(s)ds, s \in I = [a, b], \tag{3.2}$$

where $p : I \times I \times \mathbb{R} \to \mathbb{R}$ be a given function.
Now for $\psi \in A$, define,

$$\|\psi\|_\infty = max_{s \in I} \mid \psi \mid .$$

Define, $b_\rho(\psi,\phi)$ as follows;

$$b_\rho(\psi,\phi) = \mid \psi - \phi \mid^2 = max_{s\in I} \mid \psi(s) - \phi(s) \mid^2 .$$

Now, consider the following assumptions;

1. $\psi \leq \phi$, then
$$p(t,s,\psi(s)ds \leq p(t,s,\phi(s)dss \in I = [a,b].$$
2. for all $\psi,\phi \in A$ with $\psi \leq \phi$ for all $s \in$
$$\mid p(t,s,\psi(s)ds - p(t,s,\phi(s)ds \mid \leq \|\psi - \phi\|.$$
3. there exist a continuous function $\psi_0 : I \to \mathbb{R}$ such that
$$\psi_0(t) = \int_a^b p(t,s,\psi_0(s)ds, s \in I = [a,b].$$

Then, the Volterra equations (3.1) (3.2) have a solution using Theorem 2.1.

The issue of the dynamic behaviour of a multispan uniform continuous beam supported arbitrarily on its edges arises in the dynamics of bridges, highways, railways, runways, missiles, aircrafts, and other structures. The Volterra integral equations, which are consistent deformation equations corresponding to each redundant force, are solved to produce redundant forces that are located in the positions of the Volterra integral equations, which are multispan structures replaced by single-span beams loaded with specific moving loads.

4. Conclusion

By using Meir-Keeler contraction, new results in fixed point theory in setting of b-metric space have been proved. Further, an example is also provided. Moreover, by taking more than two self-maps, this result can further be modified to give new results in metric fixed point theory. Further a solution to Volterra Integral equation is also provided as an applications which can be used further to study the dynamic behaviour of a multispan uniform continuous beam.

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