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General Decay for the Wave Model with Nonlinear Dissipation and a Localized Memory *

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ABSTRACT: In this paper, we study the asymptotic behavior as well as the global existence of the solution of a dissipative wave equation. The exponential decay results of the energy are established via suitable Lyapunov functionals in a bounded domain Ω of $\mathbb{R}^n$.

Key Words: Faedo-Galerkin method, Viscoelastic wave equations, Variable coefficients, Nonlinear localized damping, Multiplier method, Exponential stabilization.

Contents

1. Introduction

This paper is concerned with the existence and uniform decay rates of solutions of the viscoelastic problem with nonlinear damping terms

$$\begin{cases} u_{tt}(t) - div(A(x)\nabla u(t)) + \int_0^t g(t-s) div(A(x)\nabla u(s))\, ds + C(x)K(u_t) = 0, \\ u = 0 \qquad \text{on } \Gamma_1 \times (0,\infty), \\ \frac{\partial u}{\partial \nu_A} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu_A} ds = 0 \qquad \text{on } \Gamma_0 \times (0,\infty), \\ u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x) \quad \text{on } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n, n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint and $A(x) = (a_{ij}(x))_{1\leq i,j\leq n}$ is a symmetric matrix function, the coefficients a_{ij} are C^∞ function in $\mathbb{R}^n$, $\frac{\partial u}{\partial \nu_A} = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial u}{\partial x_i} v_j, \nu = (v_1, v_2, \ldots, v_n)$ is the unit normal vector of Γ oriented towards the exterior of Ω and $\nu_A = A.\nu$. We assume that the second order differential operator A satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x)\zeta_i\zeta_j > \lambda \sum_{i=1}^n \zeta_i^2, \quad x \in \Omega \quad ,0 \neq \zeta = (\zeta_1; \ldots; \zeta_n) \in \mathbb{R}^n, \tag{1.2}$$

where $\lambda > 0$ is a positive constant. Let $x^0 \in \mathbb{R}^n$ be an arbitrary point of $\mathbb{R}^n$ and we set

$$\Gamma(x_0) = \{x \in \Gamma; m(x).\nu(x) > 0\},$$

where $m(x) = x - x^0$ for $x \in \Gamma$.
Let ω be a neighborhood of Γ in Ω and consider $\alpha > 0$ sufficiently small such that

$$Q_0 = \{x_0 \in \Omega; d(x_0, \Gamma(x_0)) < \alpha\},$$
$$Q_1 = \{x_0 \in \Omega; d(x_0, \Gamma(x_0)) < 2\alpha\},$$

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where $d(x_0, \Gamma(x_0))$ is the distance between the point x_0 and the set $\Gamma(x_0)$. The system is subjected to a locally distributed visco-elastic effect controlled by a symmetric matrix positive-definite $A(x)$ and completed by friction damping $C(x)K(u_t)$ acting on a region ω of Ω, considering that the well-known geometric control condition (ω, T_0) is true and assuming that the relaxation function g is bounded by a function which decreases to zero. We will prove that the solutions of the corresponding partial viscoelastic model generally decrease.

Afterwards, we make some remarks about previous works in connection with problem (1.1).
When $C(x) = 0$ and $A(x)$ is the Identity matrix of $\mathbb{R}^n$ problem (1.1) was studied by F-Cavalcanti et al [3]. They proved the existence by means of the Faedo-Galerkin method and uniform decay rates of the energy by making use of the multiplier technique combined with integral inequalities due to Komornik.
Now, for $g = 0, C(x) = 0$ and $A(x)$ is the Identity matrix of $\mathbb{R}^n$, problem (1.1) was studied by M. Aassila [1], G. Chen and H. Wong [11], I. Lasiecka and D. Tataru [7] and also by E. Zuazua [12].
In [4], Cavalcanti et al. proved exponential decay of solutions of the following viscoelastic nonlinear wave equation with strong localized damping:

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t = 0 & \text{in } \Omega \times \mathbf{R}_+, \\ u = 0 & \text{on } \Gamma \times \mathbf{R}_+, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{on } \Omega. \end{cases}$$

The authors proved the global existence of weak solutions by a perturbed energy functional.
In [13], when the feedback term depends on the velocity in a linear way, Zuazua proved exponential decay of the energy for the following problem

$$u_{tt} - \Delta u + \alpha u + f(u) + a(x)u_t = 0 \text{ in } \mathbf{R^n} \times (0, \infty)$$

with the damping term $a(x)u_t$ is effective in set $\mathbf{R^n} \backslash \{x \in \mathbf{R^n} :| x |\geq R\}$. Very recently in [10], Tebou proved the existence of global solution, as well as, the exponential stability result for a localized nonlinear strong damping.

$$\begin{cases} u_{tt} - \Delta u + a(x)g(\Delta u_t) = 0 & \text{in } \Omega \times \mathbf{R}_+, \\ u = 0 & \text{on } \Gamma \times \mathbf{R}_+, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{on } \Omega. \end{cases}$$

In [2] Cavalcanti et al. considered the wave equation with two types of locally distributed frictional damping and a Kelvin-Voigt type damping.

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t - div(b(x)\nabla u_t) = 0 & \text{in } \Omega \times \mathbf{R}_+, \\ u = 0 & \text{on } \Gamma \times \mathbf{R}_+, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{on } \Omega. \end{cases}$$

Using a combination of the multiplier techniques and the frequency domain method, they showed a exponential stability of the dynamical system, provided that the coefficient of the Kelvin-Voigt damping is smooth enough and satisfies a structural condition.
Recently in [9] Z. Sabbagh et al. proved the global existence of the solutions and exponential decay results of the energy for the following problem

$$\begin{cases} u_{tt} + \Delta^2 u - \int_0^t h(t-s)\Delta^2 u ds + a(x)g(u_t) = 0 & \text{in } \Omega \times \mathbf{R}_+, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma \times \mathbf{R}_+, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{on } \Omega. \end{cases}$$

The exponential decay results of the energy are established via suitable Lyapunov functionals and for existence they used the Faedo-Galerkin approximations together with some energy estimates.

2. Notation and Statement of Results

We consider the Hilbert space

$$V = \{v \in H^1(\Omega) : v = 0 \ \ on \ \ \Gamma_1\}$$

and define the following

$$(u,v)=\int_\Omega u(x)v(x)\,dx,\quad (u,v)_{\Gamma_0}=\int_{\Gamma_0} u(x)v(x)\,d\Gamma,$$

$$\|u(t)\|_2^2=\int_\Omega |u(x)|^2dx,\quad \|u(t)\|_{\Gamma_0}^2=\int_{\Gamma_0}|u(x)|^2d\Gamma,\quad \|u(t)\|_{(\Gamma_0,p)}^p=\int_{\Gamma_0}|u(x)|^pd\Gamma.$$

$$X^TA(x)Y=<X.Y>_A,$$

where X^T it is the transpose of the vector X and

$$<X.X>_A^{\frac{1}{2}}=|\,X\,|_A \text{ for all } X,Y\in\mathbb{R}^n.$$

$$(\nabla u,\nabla v)_A=\int_\Omega <\nabla u(x).\nabla v(x)>_A\,dx,\quad \|\nabla u(t)\|_A^2=\int_\Omega <\nabla u(x).\nabla u(x)>_A\,dx.$$

Let ω be an open non-empty subset of Ω. First assume that C and K satisfy the following conditions

(H1) The function $C:\Omega\longrightarrow\mathbb{R}$ is a non-negative and bounded such that

$$\left\{\begin{array}{l}\exists C_0>0, C(x)\geq C_0\ a.e\ in\ \Omega\\ \text{with } C(x)\in W^{1,\infty}(\Omega)\end{array}\right\}.$$

(H2) We assume that $K\in C^1(\mathbb{R},\mathbb{R})$ is a non-decreasing globally Lipschitz function with $K(0)=0$ and there exist c_1,c_2 satisfuing .

$$\begin{array}{c}c_1\mid s\mid\leq K(s)\leq c_2\mid s\mid,\\ \exists\tau,c_3>0,c_3\leq K'(s)\leq\tau,\forall s\in\mathbb{R}.\end{array}$$

(H3) The function $g:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is in $W^{2,1}(0,\infty)\cap W^{1,\infty}(0,\infty)$ and satisfies

$$1-\int_0^{+\infty}g(s)ds=l>0$$

and there exist η_1,η_2,η_3 positive constants such that $g(0)>0$ and

$$-\eta_1 g(s)\leq g'(s)\leq-\eta_2 g(s)\ for\ all\ s\geq 0,$$
$$0\leq g''(s)\leq\eta_3 g(t)\ for\ all\ s\geq 0.$$

The first equation below implies that

$$g(0)\mathrm{e}^{-\eta_1 s}\leq g(s)\leq g(0)\mathrm{e}^{-\eta_2 s}.$$

Throughout this paper, we denote the operator defined by

$$(g\ o\ \nabla u)_A(t)=\int_0^t g(t-s)\int_\Omega|\,\nabla u(s)-\nabla u(t)\,|_A^2\,dxds.$$

To obtain the global existence for strong solutions, the following assumptions are made on the initial data

(A) assumption on the initial data

$$\{u^0,u^1\}\in(V\cap H^2(\Omega)).$$

The energy related to the problem (1.1) is given by

$$e(t)=\frac{1}{2}\int_\Omega|\,u_t(t)\,|^2\,dx+\frac{1}{2}\int_\Omega|\,\nabla u(t)\,|_A^2\,dx.$$

Define the modified energy as

$$E(t)=\frac{1}{2}\int_\Omega|\,u_t(t)\,|^2\,dx+\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\int_\Omega|\,\nabla u(t)\,|_A^2\,dx+\frac{1}{2}(g\ o\ \nabla u)_A. \tag{2.1}$$

Lemma 2.1 *Let u be a solution to the problem (1.1), then the energy functional defined by (2.1) is a non-increasing function for all $t \in \mathbb{R}_+$ and satisfies*

$$E'(t) \leq -\int_\Omega C(x)K(u_t(x))u_t(x)dx - \frac{\eta_2}{2}(g\ o\ \nabla u)_A(t) - \frac{1}{2}g(t)\int_\Omega |\ \nabla u(t)\ |_A^2\ dx.$$

Proof: Multiplying the first equation by u_t, integrating over Ω , using Green formula and the boundary conditions, we get

$$\begin{gathered}\tfrac{1}{2}\frac{d}{dt}\left[\int_\Omega u_t(x)dx + \int_\Omega <\nabla u(s), \nabla u(t)>_A dx\right] - \int_0^t g(t-s)\int_\Omega <\nabla u(t), \nabla u_t(t)>_A dxds \\ + \int_\Omega C(x)K(u_t(x))u_t(x)dx = 0.\end{gathered} \tag{2.2}$$

For the third term of the left hand side of the equation (2.2) we have

$$\begin{aligned}&\int_0^t g(t-s)\int_\Omega <\nabla u(s).\nabla u_t(t)>_A dxds \\ &= \int_0^t g(t-s)\int_\Omega <\nabla u_t(t).\nabla u(s) - \nabla u(t)>_A dxds + \int_0^t g(t-s)\int_\Omega <\nabla u_t(t).\nabla u(t)>_A dxds \\ &= -\tfrac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_\Omega |\ \nabla u(s) - \nabla u(t)\ |_A^2\ dxds + \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_\Omega |\ \nabla u(t)\ |_A^2\ dxds \\ &= -\tfrac{1}{2}\tfrac{d}{dt}\int_0^t g(t-s)\int_\Omega |\ \nabla u(s) - \nabla u(t)\ |_A^2\ dxds + \frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)\int_\Omega |\ \nabla u(t)\ |_A^2\ dxds \\ &+\tfrac{1}{2}\int_0^t g^{'}(t-s)\int_\Omega |\ \nabla u(s) - \nabla u(t)\ |_A^2\ dxds - \frac{1}{2}g(t)\int_\Omega |\ \nabla u(t)\ |_A^2\ dx.\end{aligned} \tag{2.3}$$

Inserting (2.3) into (2.2), we obtain

$$\begin{aligned}E'(t) &= \tfrac{d}{dt}\left[\tfrac{1}{2}\int_\Omega |\ u_t(t)\ |^2\ dx + \frac{1}{2}(1 - \int_0^t g(s)ds)\int_\Omega |\ \nabla u(t)\ |_A^2\ dx + \frac{1}{2}(g\ o\ \nabla u)_A(t)\right] \\ &= -\int_\Omega C(x)K(u_t(x))u_t(x)dx + \frac{1}{2}(g^{'}\ o\ \nabla u)_A(t) - \frac{1}{2}g(t)\int_\Omega |\ \nabla u(t)\ |_A^2\ dx.\end{aligned} \tag{2.4}$$

Then from the assumptions (H_1) and (H_2), we have

$$-\int_\Omega C(x)K(u_t(x))u_t(x)dx \leq 0. \tag{2.5}$$

By considering the assumption (H_3) we obtain

$$\frac{1}{2}(g'\ o\ \nabla u)_A(t) \leq -\frac{\eta_2}{2}(g\ o\ \nabla u)_A(t) \leq 0.$$

Then $E'(t) \leq 0$ and consequently $E(t)$ is a non-increasing functions. This completes the proof. □

3. Well-Posedness of the Problem

Theorem 3.1 *We assume that (H_1)-(H_3) and (A) hold.Then the problem (1.1) possesses a unique strong solution $u : \Omega \times (0;\infty) \to \mathbf{R}$, such that $u \in L^\infty(0,\infty;V)$, $u' \in L^\infty(0,\infty;V)$ and $u'' \in L^\infty(0,\infty;L^2(\Omega))$. Also, assuming that $\|g(t)\|_1$ is sufficiently small, the energy determined by the strong solution u has the exponential decay rates.*

Theorem 3.2 *Let $\{u^0;u^1\} \in V \times L^2(\Omega)$ and assuming that(H_1)-(H_3) and (A) hold. Then (1.1) has a unique weak solution $u : \Omega \times (0;\infty) \to \mathbf{R}$ in the space $C^0([0,\infty);V) \cap C^1([0,\infty);L^2(\Omega))$.*

3.1. Existence of solutions

In this section we prove the existence and regularity of solution to problem (1.1). We consider strong solutions and then, using a density argument, we extend the same result for weak solution. we will announce the variational formulation of the problem (1.1), then transform it to an equivalent problem with a zero like initial value.
A variational formulation of problem (1.1) leads to the equation :

$$(u_{tt}(t), w) + (\nabla u(t), \nabla w)_A - \int_0^t g(t-s)(\nabla u(s), \nabla w)_A ds + (c(x)K(u_t), w) = 0 \tag{3.1}$$

for all $w \in V$, by using the following change of variables $v(x,t) = u(x,t) - \Psi(x,t)$, where $\Psi(x,t) = u^0(x) + tu^1(x)$ for all $(x,t) \in \Omega \times (0,\infty)$. We obtain the following equivalent problem

$$\begin{cases} v_{tt}(t) - div(A(x)\nabla v(t)) + \int_0^t g(t-s) div(A(x)\nabla v(s) ds + C(x)K(v_t + u^1(x)) = H(t), \\ v = 0 \text{ on } \Gamma_1 \times (0,\infty), \\ \frac{\partial v}{\partial \nu_A} - \int_0^t g(t-s)\frac{\partial v}{\partial \nu_A} ds = -\frac{\partial \Psi}{\partial \nu_A} + \int_0^t g(t-s)\frac{\partial \Psi}{\partial \nu_A} ds \text{ on } \Gamma_0 \times (0,\infty), \\ v(x,0) = 0, \quad v_t(x,0) = 0 \text{ on } \Omega, \end{cases} \tag{3.2}$$

where

$$H(t) = div(A(x)\nabla \Psi(t)) - \int_0^t g(t-s) div(A(x)\nabla \Psi(s) ds.$$

We put

$$Y(t) = -\frac{\partial \Psi}{\partial \nu_A} + \int_0^t g(t-s)\frac{\partial \Psi}{\partial \nu_A} ds.$$

If v is solution of (3.2) on $[0,T]$. Then $u = v + \Psi$ is solution of (1.1).
Let $(w_j)_{j\in\mathbb{N}}$ be a basis in $V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$ and $V_m = vect\{w_1, w_2, ..., w_m\}$ and let $v_m(t) = \sum_{j=1}^m \alpha_j(t) w_j$ be the solution to the Cauchy problem.

$$\begin{aligned} &(v_m''(t), w) + (\nabla v_m(t), \nabla w)_A - \int_0^t g(t-s)(\nabla v_m(s), \nabla w)_A ds \\ &\qquad + (c(x)K(v_m'(t) + \Psi'(t), w) = (H(t), w) + (Y(t), w)_{\Gamma_0}, \\ &v_m(0) = v_m'(0) = 0 \end{aligned} \tag{3.3}$$

for all $w \in V_m$.
we can prove the existence of solution of the problem (3.3) with the standard methods in differential equations on some interval $[0, t_m]$. Then, this solution can be extended to the closed interval $[0,T]$ by using the first estimate (3.10) below.
Firstly we multiplying (3.3) by $\alpha_j'(t)$, summing over j and noting that $v_m(0) = 0$, we obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left[\|v_m'(t)\|_2^2 + \|\nabla v_m(t)\|_A^2\right] + (c(x)K(v_m'(t) + \Psi'(t)), v_m'(t)) = \\ &(H(t), v_m'(t)) + \tfrac{d}{dt}(Y(t), v_m(t))_{\Gamma_0} - (Y'(t), v_m(t))_{\Gamma_0} - g(0)\|\nabla v_m(t)\|_A \\ &- \int_0^t g'(t-s)(\nabla v_m(s), \nabla v_m(t))_A ds + \tfrac{d}{dt}(\int_0^t g(t-s)(\nabla v_m(s), \nabla v_m(t))_A ds) \\ &\qquad + (c(x)K(v_m'(t) + \Psi'(t), \Psi'(t)). \end{aligned} \tag{3.4}$$

Having in mind (H_1) and (H_2) and using the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, for an arbitrary $\eta > 0$ and by (3.4) we have

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left[\|v_m'(t)\|_2^2 + \|\nabla v_m(t)\|_A^2\right] + C_0c_1\|v_m'(t) + \Psi'(t)\|_2^2 \\
\leq\ & (H(t), v_m'(t)) + \tfrac{d}{dt}(Y(t), v_m(t))_{\Gamma_0} - (Y'(t), v_m(t))_{\Gamma_0} - g(0)\|\nabla v_m(t)\|_A \\
&+N_1(\eta) + \eta c_2\|v_m'(t) + u^1\|_2^2 + \tfrac{d}{dt}(\textstyle\int_0^t g(t-s)(\nabla v(s), \nabla v_m(t))_A ds) \\
&+\|\nabla v_m(t)\|_A \textstyle\int_0^t -g'(t-s)\|\nabla v_m(s)\|_A ds,
\end{aligned} \tag{3.5}$$

where $N_1(\eta) = \frac{1}{4\eta}\int_\Omega C(x)^2 u^1(x)^2 dx$.

Estimate for $K_1 =: \|\nabla v_m(t)\|_A \int_0^t g'(t-s)\|\nabla v_m(s)\|_A ds$. From the Cauchy-Schwartz inequality and taking assumption (H_3) into accunt we obtain

$$\begin{aligned}
\mid K_1 \mid\ &\leq\ \eta_1\|\nabla v_m(t)\|_A \textstyle\int_0^t g(t-s)\|\nabla v_m(s)\|_A ds \\
&\leq\ \tfrac{\eta_1^2}{2}\|\nabla v_m(t)\|_A^2 + \tfrac{1}{2}(\textstyle\int_0^t g(t-s)^{\frac{1}{2}} g(t-s)^{\frac{1}{2}}\|\nabla v_m(s)\|_A ds)^2 \\
&\leq\ \tfrac{\eta_1^2}{2}\|\nabla v_m(t)\|_A^2 + \tfrac{1}{2}\|g\|_{L^1(0,\infty)}\textstyle\int_0^t g(t-s)\|\nabla v_m(s)\|_A^2 ds.
\end{aligned} \tag{3.6}$$

Estimate for $K_2 =: -(Y'(t), v_m(t))_{\Gamma_0}$. Using the Cauchy-Schwartz inequality and having in mind (1.2) we obtain

$$\begin{aligned}
\mid K_2 \mid\ &\leq\ \tfrac{C^2}{2}\|Y'(t)\|_{\Gamma_0}^2 + \tfrac{1}{2}\|\nabla v_m(t)\|_2^2 \\
&\leq\ \tfrac{C^2}{2}\|Y'(t)\|_{\Gamma_0}^2 + \tfrac{1}{2\lambda}\|\nabla v_m(t)\|_A^2,
\end{aligned} \tag{3.7}$$

where $C > 0$ satisfies the following inequality $\|v\|_2 \leq c\|\nabla v\|_2$ for all $v \in V$.

Then, combining (3.5)-(3.7) we obtain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left[\|v_m'(t)\|_2^2 + \|\nabla v_m(t)\|_A^2\right] + (C_0c_1 - \eta c_2)\|v_m'(t) + \Psi'(t)\|_2^2 \\
\leq\ & \tfrac{1}{2}\|H(t)\|_2^2 + \tfrac{1}{2}\|v_m'(t)\|_2^2 + (\tfrac{\eta_2^2}{2} + \tfrac{1}{2\lambda} - g(0))\|\nabla v_m(t)\|_A^2 \\
&+\tfrac{C^2}{2}\|Y'(t)\|_{\Gamma_0}^2 + N_1(\eta) + \tfrac{1}{2}\|g\|_{L^1(0,\infty)}\textstyle\int_0^t g(t-s)\|\nabla v_m(s)\|_A^2 ds \\
&+\tfrac{d}{dt}(Y(t), v_m(t))_{\Gamma_0} + \tfrac{d}{dt}(\textstyle\int_0^t g(t-s)(\nabla v(s), \nabla v_m(t))_A ds).
\end{aligned} \tag{3.8}$$

We integrate (3.8) over $(0, t)$ and having in mind $v_m(0) = v'(0) = 0$ and $(Y(t), v_m(t))_{\Gamma_0} \leq \frac{C^2}{4\eta}\|Y(t)\|_{\Gamma_0}^2 + \frac{\eta}{\lambda}\|\nabla v_m(t)\|_A^2$ we obtain

$$\begin{aligned}
&\tfrac{1}{2}\left[\|v_m'(t)\|_2^2 + \|\nabla v_m(t)\|_A^2\right] + (C_0c_1 - \eta c_2)\textstyle\int_0^t \|v_m'(\tau) + \Psi'(\tau)\|_2^2 d\tau \\
\leq\ & \tfrac{1}{2}\textstyle\int_0^t \|H(\tau)\|_2^2 d\tau + \int_0^t (\tfrac{1}{2}\|v_m'(\tau)\|_2^2 + (\tfrac{\eta_2^2}{2} + \tfrac{1}{2\lambda} - g(0))\|\nabla v_m(\tau)\|_A^2) d\tau \\
&+\tfrac{C^2}{2}\textstyle\int_0^t \|Y'(\tau)\|_{\Gamma_0}^2 d\tau + N_2(T, \eta) + \tfrac{1}{2}\|g\|_{L^1(0,\infty)}^2 \int_0^t \|\nabla v_m(s)\|_A^2 ds \\
&\tfrac{C^2}{4\eta}\|Y(t)\|_{\Gamma_0}^2 + \tfrac{\eta}{\lambda}\|\nabla v_m(t)\|_A^2 + \tfrac{1}{4\eta}\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}\textstyle\int_0^t \|\nabla v_m(s)\|_A^2 ds \\
&+\eta\|\nabla v_m(t)\|_A^2,
\end{aligned} \tag{3.9}$$

where $N_2(T, \eta) = \int_0^T N_1(\eta) ds$. Choosing $\eta > 0$ sufficiently small, using Grönwall's lemma integral form and from (3.8) and (3.9) we obtain

$$\frac{1}{2}\|v_m'(t)\|_2^2 + \frac{1}{2}\|\nabla v_m(t)\|_A^2 + \int_0^t \|v_m'(\tau) + \Psi'(\tau)\|_2^2 d\tau \leq N_3, \tag{3.10}$$

where N_3 is a positive constant independent of $m \in \mathbb{N}, \lambda > 0$ and $t \in [0,T]$. From (3.10) and considering assumptions (H_2) we obtain

$$\int_0^t \|v_m'(\tau) + \Psi'(\tau)\|_2^2 d\tau \leq \int_0^t \|k(v_m'(\tau) + \Psi'(\tau))\|_2^2 d\tau \leq N_4, \tag{3.11}$$

where N_4 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0,T]$. Then from the first estimate (3.10), (3.11) and (H_1) we obtain that the solution v_m exist globally in $[0;+\infty[$ and

$$v_m \textit{ is bounded in } L^\infty(0,T;V), \tag{3.12}$$
$$v_m' \textit{ is bounded in } L^\infty(0,T;L^2(\Omega)), \tag{3.13}$$
$$\sqrt{C(x)}K(v_m' + u^1) \textit{ is bounded in } L^2(\Omega \times (0;T)) \tag{3.14}$$

and then from the first estimate (3.10) and (3.11) we obtain that the solution v_m exist globally in $[0;+\infty[$ and

$$v_m \textit{ is bounded in } L^2(0,T;V), \tag{3.15}$$
$$v_m' \textit{ is bounded in } L^2(0,T;L^2(\Omega)), \tag{3.16}$$
$$\sqrt{C(x)}K(v_m' + u^1) \textit{ is bounded in } L^2(\Omega \times (0;T)). \tag{3.17}$$

Then we using Aubin-Lion's theorem there is a subsequence of $\{v_m\}_n$ that from now on we still represent by the same notation, such as

$$v_m \ \rightarrow \ v \textit{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.18}$$

The second estimate

Now, we are going to estimate $v_m''(0)$ in the L^2-norm. we Take $w = v_m''(0)$ in (3.3) and noticing that $v_m(0) = v_m'(0) = 0$,$H(0) = div(A(x)\nabla u^0(x))$ and $Y(0) = -\frac{\partial u^0}{\partial \nu_A}$ we get

$$\|v_m''(0)\|_2^2 + (C(x)K(u^1(x), v_m''(0)) + (\frac{\partial u^0}{\partial \nu_A}, v_m''(0))_{\Gamma_0} = (div(A\nabla u^0), v_m''(0)). \tag{3.19}$$

We assume that we have the compatibility condition

$$\frac{\partial u^0(x)}{\partial \nu_A} = 0 \ \ on \ \Gamma_0. \tag{3.20}$$

Then, considering the Cauchy-Schwartz inequality and taking assumptions (3.20), **(H1)** and **(H2)** in to account, by (3.19), we obtain

$$\begin{gathered}\|v_m''(0)\|_2^2 \leq \|div(A\nabla u^0)\|_2 \|v_m''(0)\|_2 + c_2\|C\|_{L^\infty(\Omega)}\|u^1\|_2\|v_m''(0)\|_2,\\ \|v_m''(0)\|_2 \leq \|div(A\nabla u^0)\|_2 + c_2\|C\|_{L^\infty(\Omega)}\|u^1\|_2,\\ \|v_m''(0)\|_2^2 \leq N_5^2,\end{gathered} \tag{3.21}$$

where N_5 is a positive constant that depends on the initial conditions.
We take the derivative of (3.3) and we multiplying by $\alpha_j''(t)$ and summing over j we infer that

$$\begin{gathered}\tfrac{1}{2}\tfrac{d}{dt}[\|v_m''(t)\|_2^2 + \|\nabla v_m'(t)\|_A^2] - g(0)(\nabla v_m(t), \nabla v_m''(t))_A\\ - \int_0^t g'(t-s)(\nabla v_m(s), \nabla v_m''(t))_A ds + (C(x)K'(v_m'(t) + u^1(x)), v_m''(t)^2)\\ = (H'(t), v_m''(t)) + \tfrac{d}{dt}(Y'(t), v_m'(t))_{\Gamma_0} - (Y''(t), v_m'(t))_{\Gamma_0}.\end{gathered} \tag{3.22}$$

Estimate for $k_3 := g(0)(\nabla v_m(t), \nabla v''_m(t))_A$.

$$k_3 = g(0)\frac{d}{dt}(\nabla v_m(t), \nabla v'_m(t))_A - g(0)\|\nabla v'_m(t)\|_A^2. \tag{3.23}$$

Estimate for $k_4 := \int_0^t g'(t-s)(\nabla v_m(s), \nabla v''_m(t))_A ds$.

$$\begin{aligned} k_4 &= \tfrac{d}{dt}\int_0^t g'(t-s)(\nabla v_m(s), \nabla v'_m(t))_A ds - g(0)(\nabla v_m(s), \nabla v'_m(t))_A \\ &\quad - \int_0^t g''(t-s)(\nabla v_m(s), \nabla v'_m(t))_A ds. \end{aligned} \tag{3.24}$$

Combining (3.22), (3.23) and (3.24) we have

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}[\|v''_m(t)\|_2^2 + \|\nabla v'_m(t)\|_A^2] + g(0)\|\nabla v'_m(t)\|_A^2 + (C(x)K'(v'_m(t)+u^1(x)), v''_m(t)^2) \\ &= (H'(t), v''_m(t)) + \tfrac{d}{dt}(Y'(t), v'_m(t))_{\Gamma_0} - (Y''(t), v'_m(t))_{\Gamma_0} + g(0)\tfrac{d}{dt}(\nabla v_m(t), \nabla v'_m(t))_A \\ &+ \tfrac{d}{dt}\int_0^t g'(t-s)(\nabla v_m(s), \nabla v'_m(t))_A ds - g(0)(\nabla v_m(s), \nabla v'_m(t))_A \\ &- \int_0^t g''(t-s)(\nabla v_m(s), \nabla v'_m(t))_A ds. \end{aligned} \tag{3.25}$$

From the Cauchy-Schwartz inequality, employing the inequality $ab \le \frac{1}{4\eta}a^2 + \eta b^2$ and taking $C > 0$ like (3.7) and **(H3)**, (1.2) into account, from (3.22)-(3.25) we obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}[\|v''_m(t)\|_2^2 + \|\nabla v'_m(t)\|_A^2] + (g(0) - \eta(\tfrac{1}{\lambda}+2))\|\nabla v'_m(t)\|_A^2 \\ &+(C(x)K'(v'_m(t)+u^1(x)), v''_m(t)^2) \le \tfrac{1}{2}\|H'(t)\|_2^2 + \tfrac{1}{2}\|v''_m(t)\|_2^2 + \tfrac{d}{dt}(Y'(t), v'_m(t))_{\Gamma_0} \\ &+\tfrac{C^2}{4\eta}\|y''(t)\|_{\Gamma_0}^2 + \tfrac{g'(0)^2}{4\eta}\|\nabla v_m(t)\|_A^2 + \tfrac{\eta_3^2}{4\eta}\|g\|_{L^1(0,\infty)}\int_0^t g(t-s)\|\nabla v_m(s)\|_A^2 ds \\ &\qquad +g(0)\tfrac{d}{dt}(\nabla v_m(t), \nabla v'_m(t))_A + \tfrac{d}{dt}\int_0^t g'(t-s)(\nabla v_m(s), \nabla v'_m(t))_A ds. \end{aligned} \tag{3.26}$$

We integrate (3.26) over $(0,t)$, noting that $v_m(0) = v'(0) = 0$ and having in mind (3.21) we obtain

$$\begin{aligned} &\tfrac{1}{2}\|v''_m(t)\|_2^2 + \tfrac{1}{2}\|\nabla v'_m(t)\|_A^2 + (g(0) - \eta(\tfrac{1}{\lambda}+2))\int_0^t \|\nabla v'_m(s)\|_A^2 ds \\ &+ \int_0^t (C(x)K'(v'_m(s)+u^1(x)), v''_m(s)^2) ds \\ &\le \tfrac{N_5^2}{2} + \tfrac{1}{2}\int_0^t \|H'(s)\|_2^2 ds + \tfrac{1}{2}\int_0^t \|v''_m(s)\|_2^2 ds + (Y'(t), v'_m(t))_{\Gamma_0} \\ &+\tfrac{C^2}{4\eta}\int_0^t \|Y''(s)\|_2^2 ds + \tfrac{g'(0)^2}{4\eta}\int_0^t \|\nabla v_m(s)\|_A^2 ds \\ &+\tfrac{\eta_3^2}{4\eta}\|g\|_{L^1(0,\infty)}^2\int_0^t \|\nabla v_m(s)\|_A^2 ds + g(0)(\nabla v_m(t), \nabla v'_m(t))_A \\ &\qquad + \int_0^t g'(t-s)(\nabla v_m(s), \nabla v'_m(t))_A ds. \end{aligned} \tag{3.27}$$

Estimate for $k_5 := \int_0^t g'(t-s)(\nabla v_m(s), \nabla v'_m(t))_A ds$. We use the Cauchy-Schwartz inequality, employing the inequality $ab \le \frac{1}{4\eta}a^2 + \eta b^2$.

$$\begin{aligned} |\, k_5 \,| &\le \eta_1 \int_0^t |\, g(t-s)\,|\, \|\nabla v_m(s)\|_A \|\nabla v'_m(t)\|_A ds \\ &\le \tfrac{\eta_1^2}{4\eta}(\int_0^t |\, g(t-s)\,|\, \|\nabla v_m(s)\|_A ds)^2 + \eta\|\nabla v'_m(t)\|_A^2 \\ &\le \tfrac{\eta_1^2}{4\eta}\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}\int_0^t \|\nabla v_m(s)\|_A^2 ds + +\eta\|\nabla v'_m(t)\|_A^2. \end{aligned} \tag{3.28}$$

Estimate for $k_6 := (Y'(t), v'_m(t))_{\Gamma_0}$ and $k_7 := g(0)(\nabla v_m(t), \nabla v'_m(t))_A$. We use the Cauchy-Schwartz inequality, employing the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, we get

$$\begin{aligned} | k_6 | &\leq \tfrac{C^2}{4\eta} |Y'(t)\|_{\Gamma_0} + \tfrac{\eta}{\lambda} |\nabla v'_m(t)\|_A, \\ | k_7 | &\leq \tfrac{g(0)^2}{4\eta} |\nabla v_m(t)\|_A + \eta |\nabla v'_m(t)\|_A. \end{aligned} \tag{3.29}$$

Then combining (3.27) (3.28) and (3.28), choosing $\eta > 0$ sufficiently small and using Grönwall's lemma we obtain the second estimate.

$$\begin{aligned} &\|v''_m(t)\|_2^2 + \|\nabla v'_m(t)\|_A^2 + \textstyle\int_0^t \|\nabla v'_m(s)\|_A^2 ds \\ &+ \textstyle\int_0^t (C(x)K'(v'_m(s) + u^1(x)), v''_m(s)^2) ds \leq N_6, \end{aligned} \tag{3.30}$$

where N_6 is a positive constant independent of $m \in \mathbb{N}$ and $t \in [0, T]$. We conclude that

$$v'_m \textit{ is bounded in } L^\infty(0, T; V), \tag{3.31}$$
$$v''_m \textit{ is bounded in } L^\infty(0, T; L^2(\Omega)). \tag{3.32}$$

Then, applying Dunford-Petti's theorem, we obtain from (3.13)-(3.14), (3.32) and (3.32), after replacing the sequences v_m by subsequence if necessary, that

$$v_m \ \rightarrow \ v \ weak-star \ in \ L^\infty(0, T; H^2(\Omega)), \tag{3.33}$$
$$v'_m \ \rightarrow \ v' \ weak-star \ in \ L^\infty(0, T; V), \tag{3.34}$$
$$v''_m \ \rightarrow \ v'' \ weak-star \ in \ L^\infty(0, T; L^2(\Omega)). \tag{3.35}$$

From (3.21) ,(3.30) and (3.10),(3.11) and we conclude that

$$v'_m \textit{ is bounded in } L^2(0, T; H^1(\Omega)), \tag{3.36}$$
$$v''_m \textit{ is bounded in } L^2(0, T; L^2(\Omega)) \tag{3.37}$$

and we using Aubin-Lion's theorem there is a subsequence of $\{v_m\}_n$ that from now on we still represent by the same notation, such as

$$v'_m \ \rightarrow \ v' \ strongly \ in \ L^2(0, T; L^2(\Omega)). \tag{3.38}$$

From (3.21) and (3.30) we also have

$$v''_m \ \rightarrow \ v'' \ weakly \ in \ L^2(0, T; L^2(\Omega)). \tag{3.39}$$

It remain now to prove

$$\begin{gathered} \textstyle\int_0^T \int_\Omega C(x) K(v'_m(s) + u^1(x)) dx dt \longrightarrow \int_0^T \int_\Omega C(x) K(v'(s) + u^1(x)) dx dt \\ \forall v \in L^2(0, T; L^2(\Omega)). \end{gathered} \tag{3.40}$$

Let us prove the convergence (3.40), to this end, it suffices to prove

$$\sqrt{C(x)} K(v'_m(s) + u^1(x)) \longrightarrow \sqrt{C(x)} K(v'(s) + u^1(x)). \tag{3.41}$$

Let m and n be positive integers with $m > n$, set $V_{mn} = v_m - v_n$ the function V_{mn} satisfies

$$\begin{aligned} &(V_{mn}^{\prime\prime(t)} - div(A(x)\nabla V_{mn}) + \textstyle\int_0^t g(t-s) div(A(x)\nabla V_{mn}(s)) ds, v) \\ &+ (C(x)(K(v'_m + u^1(x))) - K(v'_n + u^1(x)), v) = 0. \end{aligned} \tag{3.42}$$

We take $v = V'_{mn}$ and using Green's formula, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\int_\Omega \mid V'_{mn}(t)\mid^2 dx + \frac{1}{2}(1-\int_0^t g(s)ds)\int_\Omega \mid \nabla V_{mn}\mid_A^2 dx + \frac{1}{2}(g\ o\ \nabla V_{mn})_A(t)\right]. \\ &+\int_\Omega C(x)(K(v'_m+u^1(x))) - K(v'_n+u^1(x))V'_{mn}dx - \frac{1}{2}(g^{'}\ o\ \nabla V_{mn})_A(t) \\ &+\frac{1}{2}g(t)\int_\Omega \mid \nabla V_{mn}\mid_A^2 dx = 0. \end{aligned} \tag{3.43}$$

We integrate (3.43) over $(0,t)$ and having in mind $V_{mn}(0) = V'_{mn}(0) = 0$ we obtain

$$\begin{aligned} &\|V'_{mn}(t)\|_2^2 + (1-\int_0^t g(s)ds)\|\nabla V_{mn}(t)\|_A^2 + (g\ o\ \nabla V_{mn})_A(t) - \int_0^T (g^{'}\ o\ \nabla V_{mn})_A(s)ds \\ &+2\int_0^T\int_\Omega C(x)(K(v'_m+u^1(x))) - K(v'_n+u^1(x))V'_{mn}dxds \\ &+\int_0^T g(s)\|\nabla V_{mn}(s)\|_A^2 = \|V'_{mn}(0)\|_2^2 + \|\nabla V_{mn}(0)\|_A^2. \end{aligned} \tag{3.44}$$

Using **(H2)**, we obtain

$$\begin{aligned} &c_3\int_0^T\int_\Omega C(x)\mid v'_m - v'_n\mid^2 dxds \leq \int_0^T\int_\Omega C(x)(K(v'_m+u^1(x))) - K(v'_n+u^1(x))V'_{mn}dxds \\ &\frac{1}{\tau}\int_0^T\int_\Omega C(x)\mid K(v'_m+u^1(x))) - K(v'_n+u^1(x)\mid^2 dxds \\ &\leq \int_0^T\int_\Omega C(x)(K(v'_m+u^1(x))) - K(v'_n+u^1(x))V'_{mn}dxds \\ &\int_0^T\int_\Omega C(x)\mid K(v'_m+u^1(x))) - K(y'+u^1(x))\mid^2 dxds \leq \tau^2\int_0^T\int_\Omega C(x)\mid v'_m - y'\mid^2 dxds. \end{aligned} \tag{3.45}$$

Now, combining the convergence (3.33)-(3.35) with (3.44) and (3.45), give us that the sequences $(\sqrt{C}v'_m)_m$ and $(\sqrt{C}g(v'_m+u^1))_m$ are Cauchy sequence in $L^2(0,T;L^2(\Omega))$.
Frome (3.35), we obtain

$$\sqrt{C}v_m \to \sqrt{C}v\ in\ L^2(0,T;L^2(\Omega)). \tag{3.46}$$

Then we take $y = v$ in the inequality (3.45) and letting go to infinity we derive (3.41) and we are done.

4. Uniform Decay

In this section, we use multiplier techniques to show that the energy required to solve the problem (1.1) decreases exponentially.
(H4) We consider $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} 0 \leq \psi(x) &\leq 1, \\ \psi(x) = 1\ &,in\ \Omega/Q_1, \\ \psi(x) = 0\ &,in\ Q_0. \end{aligned}$$

For an arbitrary ϵ, define the perturbed energy

$$E_\epsilon(t) = E(t) + \epsilon\rho(t), \tag{4.1}$$

$$\begin{aligned}
&\rho(t) = J_1 + J_2,\\
&\text{where } J_1 =: 2\int_\Omega u_t(x)h.\nabla z(x)dx,\ J_2 =: \theta\int_\Omega u_t(x)z(x)dx \text{ and } \theta \in]n-2;n[,\\
&h = m\psi, \text{where } m(x) = x - x_0 \text{ and } \psi \text{ is defined in } \textbf{(H4)},\\
&z(t) = u(t) - \int_0^t g(t-s)u(s)ds,\\
&\nabla z(t) = \nabla u(t) - \int_0^t g(t-s)\nabla u(s)ds.
\end{aligned} \tag{4.2}$$

We put

$$(\nabla u(t).\nabla v(t))_A = \int_\Omega < \nabla u(x,t).\nabla v(x,t) >_A dx,$$

$$\|\nabla u(t)\|_A^2 = (\nabla u(t).\nabla u(t))_A = \int_\Omega \mid \nabla u(x,t) \mid_A^2 dx.$$

Proposition 4.1 *: There exists $\delta > 0$ such that*

$$\mid E_\epsilon(t) - E(t) \mid \leq \epsilon\delta E(t).$$

Proof: Estimate for $J_1 := 2\int_\Omega u_t(t)h^T.\nabla z(t)dx$.
From the Cauchy-Schwartz inequality and $2ab \leq a^2 + b^2$ and having in mind (1.2), we obtain

$$\begin{aligned}
\mid J_1 \mid &\leq 2\int_\Omega \mid u_t(t) \mid \sum_{i=1}^n \mid h_i(x)\frac{\partial z}{\partial x_i}(t)) \mid dx\\
&\leq 2R(x_0)\|u_t(t)\|_2(\int_\Omega \nabla z(t)^T\nabla z(x)dx)^{\frac{1}{2}}\\
&\leq 2\frac{R(x_0)}{\lambda^{\frac{1}{2}}}\|u_t(t)\|_2(\int_\Omega < \nabla z(t).\nabla z(t) >_A dx)^{\frac{1}{2}}\\
&\leq \frac{R(x_0)}{\lambda^{\frac{1}{2}}}\left[\|u_t(t)\|_2^2 + \|\nabla z(t)\|_A^2\right],
\end{aligned} \tag{4.3}$$

where

$$R(x_0) = \max_{x\in\overline{\Omega}} \mid x - x^0 \mid .$$

Estimate for $J_2 := \theta\int_\Omega u_t(t)z(t)dx$, from the Cauchy-Schwartz inequality and Considering (1.2) we have

$$\begin{aligned}
\mid J_2 \mid &\leq \theta[\tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{c^2}{2}\int_\Omega \nabla z(t)^T\nabla z(t)dx]\\
&\leq \theta[\tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{c^2}{2\lambda}\int_\Omega < \nabla z(t).\nabla z(t) >_A dx]\\
&\leq \theta\max(1,\tfrac{c^2}{\lambda})[\tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla z(t)\|_A^2 dx],
\end{aligned} \tag{4.4}$$

where we used $c > 0$ such that $\|v\|_2 \leq c\|\nabla v\|_2$ for all $v \in V$.
From (4.2), (4.3) and (4.4), we get

$$\mid \rho(t) \mid \leq \left(\frac{R(x_0)}{\lambda^{\frac{1}{2}}} + \theta\max\left(1,\frac{c^2}{\lambda}\right)\right)\left[\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla z(x)\|_A^2\right]. \tag{4.5}$$

We find that :

$$\begin{aligned}
\|\nabla z(t)\|_A^2 &= \|\nabla u(t)\|_A^2 - 2\int_0^t g(t-s)(\nabla u(s).\nabla u(t))_A ds\\
&\quad + \int_0^t g(t-s)\left(\int_0^t g(t-\tau)(\nabla u(s).\nabla u(\tau))_A d\tau\right)ds.
\end{aligned} \tag{4.6}$$

On the right side of the equation, we make estimations for certain terms.
Now, estimate for $J_3 = -2\int_0^t g(t-s)(\nabla u(s).\nabla u(t))_A ds$, considering the Cauchy-Schwartz inequality and $2ab \leq a^2+b^2$, we obtain

$$\begin{aligned}
\mid J_3 \mid &\leq 2\int_0^t g(t-s)\|\nabla u(t)\|_A\|\nabla u(s)\|_A ds \\
&\leq \|\nabla u(t)\|_A^2 + (\int_0^t g(t-s)\|\nabla u(s)\|_A ds)^2 \\
&\leq \|\nabla u(t)\|_A^2 + (\int_0^t g(t-s)^{\frac{1}{2}} g(t-s)^{\frac{1}{2}}\|\nabla u(s)\|_A ds)^2 \\
&\leq \|\nabla u(t)\|_A^2 + \|g\|_{L_1(0,\infty)}\int_0^t g(t-s)(\|\nabla u(s)-\nabla u(t)\|_A + \|\nabla u(t)\|_A)^2 ds \\
&\leq \|\nabla u(t)\|_A^2 + 2\|g\|_{L_1(0,\infty)}\int_0^t g(t-s)(\|\nabla u(s)-\nabla u(t)\|_A^2 + \|\nabla u(t)\|_A^2) ds.
\end{aligned} \tag{4.7}$$

Considering $\|g\|_{L_1(0,\infty)} \leq 1$, we get

$$\mid J_3 \mid \leq 3\|\nabla u(t)\|_A^2 + 2(g\ o\ \nabla u)_A(t). \tag{4.8}$$

Estimate for $J_4 = \int_0^t g(t-s)(\int_0^t g(t-\tau)(\nabla u(s).\nabla u(\tau))_A d\tau)ds$.

$$\begin{aligned}
\mid J_4 \mid &\leq (\int_0^t g(t-s)\|\nabla u(s)\|_A ds)^2 \\
&\leq 2\|g\|_{L_1(0,\infty)}\int_0^t g(t-s)(\|\nabla u(s)-\nabla u(t)\|_A^2 + \|\nabla u(t)\|_A^2) ds \\
&\leq 2(g\ o\ \nabla u)_A(t) + 2\|\nabla u(t)\|_A^2.
\end{aligned} \tag{4.9}$$

Using (4.8), (4.9) and (H_3) in (4.6) and replace everything on (4.5)

$$\begin{aligned}
\mid \rho(t) \mid &\leq (\tfrac{R(x_0)}{\lambda^{\frac{1}{2}}} + \theta\max(1,\tfrac{c^2}{\lambda}))\left[\tfrac{1}{2}\|u_t(t)\|_2^2 + 10l^{-1}(\tfrac{1}{2}(1-\int_0^\infty g(s)ds)\|\nabla u(t)\|_A^2)\right. \\
&\qquad \left. +8(\tfrac{1}{2}(g\ o\ \nabla u(t))_A(t))\right] \\
&\leq (\tfrac{R(x_0)}{\lambda^{\frac{1}{2}}} + \theta\max(1,\tfrac{c^2}{\lambda}))(10l^{-1}+9)E(t).
\end{aligned} \tag{4.10}$$

The proof is now complete. □

Proposition 4.2 *Choosing $\|g\|_{L^1(0,\infty)}$ sufficiently small. Then, there exists a positive constant $\delta_1 = \delta_1(\epsilon)$ such that $E_\epsilon'(t) \leq -\delta_1 E_\epsilon(t)$.*

Proof: We take the derivative of $\rho(t)$ given in (4.2) with respect to t and injecting $u_{tt}(t) = div(A\nabla z(t)) - c(x)K(u_t(t))$ in $\rho'(t)$ we obtain the following result

$$\begin{aligned}
\rho'(t) &= 2\int_\Omega div(A\nabla z(t))h^T.\nabla z(t)dx - 2\int_\Omega c(x)K(u_t(t))h^T.\nabla z(t)dx + 2\int_\Omega u_t(t)h^T.\nabla z_t(t)dx \\
&\quad +\theta\int_\Omega div(A\nabla z(t))z(t)dx - \theta\int_\Omega c(x)K(u_t(t))z(t)dx + \theta\int_\Omega u_t(t)z_t(t)dx.
\end{aligned} \tag{4.11}$$

Next, we will evaluate some terms on the right-hand side of the equality(4.11).
Estimate for $I_1 := \theta\int_\Omega u_t(t)z_t(t)dx$, substituting $z_t(t)$ in I_1 we get

$$I_1 = \theta\int_\Omega u_t(t)^2 dx - \theta g(0)\int_\Omega u_t(t)u(t)dx - \theta\int_0^t g'(t-s)(\int_\Omega u_t(t)u(s)dx)ds. \tag{4.12}$$

Estimate for $I_2 := 2\int_\Omega div(A\nabla z(t))h^T.\nabla z(t)dx$. Using Green formulas we deduce

$$\begin{aligned} I_2 &= -2\sum_{i,k=1}^{n}\int_\Omega (\sum_{j=1}^{n} a_{ij}\frac{\partial z}{\partial x_j}(t))\frac{\partial}{\partial x_i}(h_k.\frac{\partial z(t)}{\partial x_k})dx \\ &\quad +2\int_\Gamma \frac{\partial z}{\partial \nu_A}(t)h^T.\nabla z(t)d_\Gamma \\ &= -2\sum_{i,k=1}^{n}\int_\Omega (\sum_{j=1}^{n} a_{ij}\frac{\partial z}{\partial x_j}(t))\frac{\partial h_k}{\partial x_i}\frac{\partial z}{\partial x_k}(t))dx \\ &\quad +2\int_\Gamma \frac{\partial z}{\partial \nu_A}(t)h^T.\nabla z(t)d_\Gamma - 2\sum_{i,k=1}^{n}\int_\Omega (\sum_{j=1}^{n} a_{ij}\frac{\partial z}{\partial x_j}(t))\frac{\partial}{\partial x_k}(\frac{\partial z}{\partial x_i}(t))h_k dx, \end{aligned} \tag{4.13}$$

where $\frac{\partial z}{\partial \nu_A}(t) =< \nabla z(x,t).\nu >_A$ and ν represents the unit normal vector pointing towards the exterior of Ω.

For the last term $I_{2.1} = -\sum_{i,k=1}^{n}\int_\Omega(\sum_{j=1}^{n} a_{ij}2\frac{\partial z}{\partial x_j}(t))\frac{\partial}{\partial x_k}(\frac{\partial z}{\partial x_i}(t))h_k dx$ we have

$$\begin{aligned} I_{2.1} &= -2\sum_{k=1}^{n}\int_\Omega \nabla(\frac{\partial z}{\partial x_k}(t))^T A\nabla z(t)h_k dx \\ &= -2\sum_{k=1}^{n}\int_\Omega < \nabla(\frac{\partial z}{\partial x_k}(t)).\nabla z(t) >_A h_k dx \\ &= -\sum_{k=1}^{n}\int_\Omega \frac{\partial}{\partial x_k}(< \nabla z(t).\nabla z(t) >_A)h_k dx + \sum_{k=1}^{n}\int_\Omega \nabla z(t)^T B_k \nabla z(t)h_k dx, \end{aligned} \tag{4.14}$$

where $B_k(x)$ is a matrix $(\frac{\partial}{\partial x_k}a_{i,j})_{1\leq i,j\leq n}$,then we have

$$\begin{aligned} I_2 &= -2\sum_{i,k=1}^{n}\int_\Omega (\sum_{j=1}^{n} a_{ij}\frac{\partial z}{\partial x_j}(t))\frac{\partial h_k}{\partial x_i}\frac{\partial z}{\partial x_k}(t))dx \\ &\quad +\sum_{k=1}^{n}\int_\Omega \nabla z(t)^T\frac{\partial}{\partial x_k}(A)\nabla z(t)h_k dx \\ &\quad +\int_\Gamma \tfrac{\partial z}{\partial \nu_A}(t)2h^T.\nabla z(t)d_\Gamma - \sum_{k=1}^{n}\int_\Omega \frac{\partial}{\partial x_k}(< \nabla z(t).\nabla z(t) >_A)h_k dx. \end{aligned} \tag{4.15}$$

For the term $I_{2.2} = -\sum_{k=1}^{n}\int_\Omega \frac{\partial}{\partial x_k}(< \nabla z(t).\nabla z(t) >_A)h_k dx$, we use the Green formula to obtain

$$\begin{aligned} I_{2.2} &= \int_\Omega < \nabla z(t).\nabla z(t) >_A div(h)dx - \int_\Gamma < \nabla z(t).\nabla z(t) >_A h^T.\nu d_\Gamma \\ &= \int_\Omega \mid \nabla z(t) \mid_A^2 div(h)dx - \int_\Gamma \mid \nabla z(t) \mid_A^2 h^T.\nu d_\Gamma. \end{aligned} \tag{4.16}$$

Inserting (4.16) in (4.15) we obtain

$$\begin{aligned} I_2 &= -2\sum_{i,k=1}^{n}\int_\Omega (\sum_{j=1}^{n} a_{ij}\frac{\partial z}{\partial x_j}(t))\frac{\partial h_k}{\partial x_i}\frac{\partial z}{\partial x_k}(t))dx \\ &\quad +\int_\Gamma \tfrac{\partial z}{\partial \nu_A}(t)2h^T.\nabla z(t)d_\Gamma + \int_\Omega \mid \nabla z(t) \mid_A^2 div(h)dx \\ &\quad -\int_\Gamma \mid \nabla z(t) \mid_A^2 h^T.\nu d_\Gamma + \sum_{k=1}^{n}\int_\Omega \nabla z(t)^T\frac{\partial}{\partial x_k}(A)\nabla z(t)h_k dx. \end{aligned} \tag{4.17}$$

Estimate for $I_3 := 2\int_\Omega u_t(t)h^T\nabla z_t(t)dx$. Use the Green formula and note that $u_t = 0$ on Γ

$$\begin{aligned} I_3 &= -\int_\Omega u_t(t)^2 div(h)dx - 2g(0)\int_\Omega u_t(t)h^T\nabla u(t)dx \\ &\quad -2\int_0^t g^{'}(t-s)\int_\Omega u_t(t)h^T\nabla u(s)dxds. \end{aligned} \tag{4.18}$$

Estimate for $I_4 := \theta \int_\Omega div(\nabla z(t)^T A) z(t) dx$. we use the Green formula and we notice that $z = 0$ on Γ

$$I_4 = -\theta \int_\Omega \nabla z(t)^T A \nabla z(t) dx = -\theta \|\nabla z(t)\|_A \tag{4.19}$$

Estimate for $I_5 := -\theta \int_\Omega c(x) K(u_t(t)) 2h^T \nabla z(t) dx$.

$$I_5 = -2\theta \int_\Omega c(x) K(u_t(t)) h^T \nabla u(t) dx + 2\theta \int_0^t g(t-s) (\int_\Omega c(x) K(u_t(t)) h^T \nabla u(s) dx) ds.$$

Estimate for $I_6 := -\theta \int_\Omega c(x) K(u_t(t)) z(t) dx$.

$$I_6 = -\theta \int_\Omega c(x) K(u_t(t)) u(t) dx + \theta \int_0^t g(t-s) (\int_\Omega c(x) K(u_t(t)) u(s) dx) ds. \tag{4.20}$$

Then, combining (4.11)-(4.20) we arrive at

$$\begin{aligned}
\rho'(t) \quad = \quad & -2 \sum_{i,k=1}^n \int_\Omega (\sum_{j=1}^n a_{ij} \frac{\partial z}{\partial x_j}(t)) \frac{\partial h_k}{\partial x_i} \frac{\partial z}{\partial x_k}(t)) dx + \int_\Gamma \frac{\partial z}{\partial \nu_A}(t) 2h^T \nabla z(t) d_\Gamma \\
& + \int_\Omega \mid \nabla z(t) \mid_A^2 div(h) dx - \int_\Gamma \mid \nabla z(t) \mid_A^2 h^T \nu d_\Gamma + \sum_{k=1}^n \int_\Omega \frac{\partial}{\partial x_k}(A) \nabla z(t)^T \nabla z(t) h_k dx \\
& - \int_\Omega c(x) K(u_t(t)) 2h^T \nabla u(t) dx + \int_0^t g(t-s) (\int_\Omega c(x) K(u_t(t)) 2h^T \nabla u(s) dx) ds \\
& - \int_\Omega u_t(t)^2 div(h) dx - 2g(0) \int_\Omega u_t(t) h^T \nabla u(t) - 2 \int_0^t g^{'}(t-s) \int_\Omega u_t(t) h^T \nabla u(s) dx ds \\
& -\theta \|\nabla z(t)\|_A - \theta \int_\Omega c(x) K(u_t(t)) 2h^T \nabla u(t) dx \\
& +\theta \int_0^t g(t-s) (\int_\Omega c(x) K(u_t(t)) 2h^T \nabla u(s) dx) ds - \theta \int_\Omega c(x) K(u_t(t)) u(t) dx \\
& +\theta \int_0^t g(t-s) (\int_\Omega c(x) K(u_t(t)) u(s) dx) ds + \theta \int_\Omega u_t(t)^2 dx - \theta g(0) \int_\Omega u_t(t) u(t) dx \\
& \qquad -\theta \int_0^t g^{'}(t-s) (\int_\Omega u_t(t) u(s) dx) ds.
\end{aligned} \tag{4.21}$$

Observe that since $z = 0$ on Γ we have $\frac{\partial z}{\partial x_k} = \frac{\partial z}{\partial \nu} \nu_k$ which implies

$$\begin{aligned}
\mid \nabla z(t) \mid_A^2 \quad &= \quad \sum_{i,j=1}^n a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial z}{\partial x_i} = \sum_{i,j=1}^n a_{ij} \frac{\partial z}{\partial \nu} \nu_j \frac{\partial z}{\partial \nu} \nu_i = (\frac{\partial z}{\partial \nu})^2 tr(A), \\
h^T \nabla z(t) \quad &= \quad \frac{\partial z}{\partial \nu} h^T \nu, \\
\frac{\partial z}{\partial \nu_A}(t) \quad &= \quad \nabla z(t)^T A \nu = \sum_{i,j=1}^n a_{ij} \frac{\partial z}{\partial x_j} \nu_i = \sum_{i,j=1}^n a_{ij} \frac{\partial z}{\partial \nu} \nu_j \nu_i = \frac{\partial z}{\partial \nu} tr(A),
\end{aligned}$$

where $tr(A)$ is the trace of the matrix A defined below, then we have

$$\begin{aligned}
& \textstyle\int_{\partial\Omega} \frac{\partial z}{\partial \nu_A}(t) 2h^T \nabla z(t) d_\Gamma - \int_\Gamma \mid \nabla z(t) \mid_A^2 h^T \nu d_\Gamma \\
& = \textstyle\int_{\Gamma/\Gamma(x_0)} (\frac{\partial z}{\partial \nu})^2 m^T \nu tr(A) d_\Gamma + \int_{\Gamma(x_0)} (\frac{\partial z}{\partial \nu})^2 m^T \nu tr(A) d_\Gamma \leq 0,
\end{aligned} \tag{4.22}$$

where $m(x) = x - x_0$, after that we substituting $\frac{\partial h_k}{\partial x_i}(x)$ in $-2\sum_{i,k=1}^{n}\int_\Omega(\sum_{j=1}^{n} a_{ij}\frac{\partial z}{\partial x_j}(t))\frac{\partial h_k}{\partial x_i}\frac{\partial z}{\partial x_k}(t))dx$ and having in mind (**H4**) we have

$$\begin{array}{rl} & -2\displaystyle\sum_{i,k=1}^{n}\int_\Omega(\sum_{j=1}^{n} a_{ij}\frac{\partial z}{\partial x_j}(t))\frac{\partial h_k}{\partial x_i}\frac{\partial z}{\partial x_k}(t))dx \\ = & -2\displaystyle\int_\Omega \nabla z(t)^T A\nabla z(t)dx + 2\int_{Q_1} \mid \nabla z(t) \mid_A^2 dx - \int_{Q_1} \mid \nabla z(t) \mid_A^2 \psi dx \\ & -2\displaystyle\int_\Omega \nabla z(t)^T A\nabla\psi\nabla z(t)^T m dx \\ = & -2\displaystyle\int_\Omega \nabla z(t)^T A\nabla\psi\nabla z(t)^T m dx - 2\|\nabla z(t)\|_A + 2\int_{Q_1}(1-\psi) \mid \nabla z(t) \mid_A^2 dx \end{array} \tag{4.23}$$

and we substituting $div(h) = n\psi + m^T\nabla\psi$ in $\int_\Omega \mid \nabla z(t) \mid_A^2 div(h)dx$ and $-\int_\Omega u_t(t)^2 div(h)dx$ we have

$$\begin{array}{rcl} \int_\Omega \mid \nabla z(t) \mid_A^2 div(h)dx & = & n\|\nabla z(t)\|_A^2 + n\int_{Q_1}(1-\psi) \mid \nabla z(t) \mid_A^2 dx \\ & & + \int_{Q_1} \mid \nabla z(t) \mid_A^2 \nabla\psi^T m dx \end{array} \tag{4.24}$$

and

$$\begin{array}{rcl} -\int_\Omega u_t(t)^2 div(h)dx & = & -n\|u_t(t)\|_2^2 + n\int_{Q_1}(\psi-1)u_t(t)^2dx \\ & & -\int_{Q_1/Q_0} \nabla\psi^T m u_t(t)^2dx. \end{array} \tag{4.25}$$

Then, from (4.21)-(4.25) and after some computation, we conclude that

$$\begin{array}{rcl} \rho'(t) & \leq & (n-2-\theta)\|\nabla z(t)\|_A^2 + (\theta-n)\|u_t(t)\|_2^2 + (n-2)\int_{Q_1}(\psi-1) \mid \nabla z(t) \mid_A^2 dx \\ & & + \int_{Q_1} \mid \nabla z(t) \mid_A^2 \nabla\psi^T m dx + n\int_{Q_1}(1-\psi)u_t(t)^2dx \\ & & +\displaystyle\sum_{k=1}^{n}\int_\Omega \nabla z(x,t)^T\frac{\partial}{\partial x_k}(A)\nabla z(t)h_k dx - 2\int_\Omega \nabla z(t)^T A\nabla\psi\nabla z(t)^T m dx \\ & & -\int_{Q_1/Q_0}\nabla\psi^T m u_t(t)^2dx - (1+\theta)\int_\Omega c(x)K(u_t(t))2h^T\nabla u(t)dx \\ & & +(1+\theta)\int_0^t g(t-s)(\int_\Omega c(x)K(u_t(t))2h^T\nabla u(s)dx)ds \\ & & -2g(0)\int_\Omega u_t(t)h^T\nabla u(t)dx - 2\int_0^t g'(t-s)\int_\Omega u_t(t)h^T\nabla u(s)dxds \\ & & -\theta\int_\Omega c(x)K(u_t(t))u(t)dx + \theta\int_0^t g(t-s)(\int_\Omega c(x)K(u_t(t))u(s)dx)ds \\ & & -\theta g(0)\int_\Omega u_t(t)u(t)dx - \theta\int_0^t g'(t-s)(\int_\Omega u_t(t)u(s)dx)ds. \end{array} \tag{4.26}$$

Putting in mind (1.2) and by using the the Cauchy-Schwartz inequality the following identity $I_7 = -2\int_\Omega \nabla z^T A\nabla\psi\nabla z(t)^T m dx$, we obtain

$$\begin{array}{rcl} \mid I_7 \mid & \leq & 2\int_\Omega(\nabla z^T A\nabla z)^{\frac{1}{2}}(\nabla\psi^T A\nabla\psi)^{\frac{1}{2}}(\nabla z^T\nabla z)^{\frac{1}{2}}(m^T m)^{\frac{1}{2}}dx \\ & \leq & 2\int_\Omega < \nabla z(t).\nabla z(t) >_A^{\frac{1}{2}} \frac{1}{\lambda^{\frac{1}{2}}} < \nabla z(t).\nabla z(t) >_A^{\frac{1}{2}} \mid \nabla\psi \mid_A \mid m \mid_2 dx \\ & \leq & \frac{2R(x_0)}{\lambda^{\frac{1}{2}}}\max_{x\in\overline{\Omega}} \mid \nabla\psi \mid_A \int_{Q_1} \mid \nabla z(t) \mid_A dx. \end{array} \tag{4.27}$$

Estimate for $I_8 = \int_{Q_1} \mid \nabla z(t) \mid_A^2 \nabla\psi^T m dx$. Analogously like (4.27) we have

$$\mid I_8 \mid \leq \frac{R(x_0)}{\lambda^{\frac{1}{2}}}\max_{x\in\overline{\Omega}} \mid \nabla\psi \mid_A \int_{Q_1} \mid \nabla z(t) \mid_A dx. \tag{4.28}$$

Then let $k_1 = min\{2(\theta - n + 2), 2(n - \theta)\}$ be a positive constant. Having in mind (4.6) and adding and subtracting $\frac{k_1}{2}\int_0^t g(s)ds\|\nabla u(t)\|_A^2 + \frac{k_1}{2}(g\ o\ \nabla u)_A(t)$ to obtain $-k_1E(t)$ from (4.26)-(4.28), with supposing without loss of generality that $g(0) \leq 1$, we infer

$$\begin{aligned}
\rho^{'}(t) \quad \leq \quad & -k_1E(t) - \tfrac{k_1}{2}\textstyle\int_0^t g(s)ds\|\nabla u(t)\|_A^2 + \tfrac{k_1}{2}(g\ o\ \nabla u)_A(t) \\
& +4\textstyle\int_0^t g(t-s)\|\nabla u(s)\|_A\|\nabla u(t)\|_A ds + 2(\int_0^t g(t-s)\|\nabla u(s)\|_A ds)^2 \\
& +\tfrac{3R(x_0)}{\lambda^{\frac{1}{2}}}\max_{\forall x\in\Omega} \mid \nabla\psi \mid_A \textstyle\int_{Q_1} \mid \nabla z(t) \mid_A^2 dx \\
& -2g(0)\textstyle\int_\Omega u_t(t)h^T\nabla u(t)dx - 2\int_0^t g^{'}(t-s)\int_\Omega u_t(t)h^T\nabla u(s)dxds \\
& -\theta g(0)\textstyle\int_\Omega u_t(t)u(t)dx - \theta\int_0^t g^{'}(t-s)(\int_\Omega u_t(t)u(s)dx)ds \\
& -(1+\theta)\textstyle\int_\Omega c(x)K(u_t(t))2h^T\nabla u(t)dx \\
& -\theta\textstyle\int_\Omega c(x)K(u_t(t))u(t)dx + (1+\theta)\int_0^t g(t-s)(\int_\Omega c(x)K(u_t(t))2h^T\nabla u(s)dx)ds \\
& +\theta\textstyle\int_0^t g(t-s)(\int_\Omega c(x)K(u_t(t))u(s)dx)ds \\
& +\tfrac{1}{\lambda}\max_{\forall x\in\Omega} \mid \nabla\psi \mid_A R(x_0)\textstyle\int_{Q_1/Q_0} u_t(t)^2dx + 2n\int_{Q_1} u_t(t)^2dx \\
& +\sum_{k=1}^n \int_\Omega \nabla z(t)^T \frac{\partial}{\partial x_k}(A)\nabla z(t)h_k dx.
\end{aligned} \tag{4.29}$$

Next, we analyze some terms on the right hand side in the above the inequality.
Estimate for $F_4 := 4\int_0^t g(t-s)\|\nabla u(s)\|_A\|\nabla u(t)\|_A ds$. Considering Cauchy-Schwartz inequality and also employing the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, for an arbitrary $\eta > 0$, we obtain

$$\begin{aligned}
\mid F_4 \mid \quad & \leq \quad 4[\tfrac{1}{4\eta}\|\nabla u(t)\|_A^2 + \eta(\textstyle\int_0^t g(t-s)\|\nabla u(s)\|_A ds)^2] \\
& \leq \quad \tfrac{1}{\eta}\|\nabla u(t)\|_A^2 + 4\eta\|g\|_{L^1(0,\infty)}\textstyle\int_0^t g(t-s)\|\nabla u(s)\|_A^2 ds \\
& \leq \quad \tfrac{1}{\eta}\|\nabla u(t)\|_A^2 + 8\eta(g\ o\ \nabla u)_A(t) + 4\eta(\textstyle\int_0^t g(s)ds)\|\nabla u(t)\|_A^2 \\
& \leq \quad \tfrac{1}{\eta}\|\nabla u(t)\|_A^2 + 16\eta E(t) + 4\eta(\textstyle\int_0^t g(s)ds)\|\nabla u(t)\|_A^2,
\end{aligned} \tag{4.30}$$

where we used $\|g\|_{L^1(0,\infty)} \leq 1$.
Estimate for $F_5 := 2(\int_0^t g(t-s)\|\nabla u(s)\|_A ds)^2$. Considering Cauchy-Schwartz inequality it holds that

$$\begin{aligned}
\mid F_5 \mid \quad & \leq \quad 2\|g\|_{L^1(0,\infty)}\textstyle\int_0^t g(t-s)\|\nabla u(s)\|_A^2 ds \\
& \leq \quad 4(g\ o\ \nabla u)_A(t) + 4\|\nabla u(t)\|_A^2,
\end{aligned} \tag{4.31}$$

where in the last inequality we used $\|g\|_{L^1(0,\infty)} \leq 1$.
Estimate for $F_6 := \frac{3}{\Lambda^{\frac{1}{2}}}\max_{\forall x\in\Omega} \mid \nabla\psi \mid_A R(x_0)\int_{Q_1} \mid \nabla z(x,t) \mid_A^2 dx$.
Setting $C_A := \frac{3}{\Lambda^{\frac{1}{2}}}\max_{\forall x\in\Omega} R(x_0) \mid \nabla\psi \mid_A$ and taking (4.6) into account, we obtain, as in (4.30) and

(4.31), the estimate

$$\begin{aligned} \mid F_6 \mid \;&\leq\; C_A\left[\|\nabla u(t)\|_A^2 + 2\int_0^t g(t-s)\|\nabla u(s)\|_A\|\nabla u(t)\|_A ds + (\int_0^t g(t-s)\|\nabla u(s)\|_A ds)^2\right] \\ &\leq\; C_A\Big[\|\nabla u(t)\|_A^2 + \tfrac{1}{2\eta}\|\nabla u(t)\|_A^2 + 4\eta(\int_0^t g(s)ds)\|\nabla u(t)\|_A^2 + 4(g\ o\ \nabla u)_A(t) \\ &\qquad\qquad + 4\|\nabla u(t)\|_A^2\Big] \\ &\leq\; C_A(5+\tfrac{1}{2\eta})\|\nabla u(t)\|_A^2 + 4C_A\eta(\int_0^t g(s)ds)\|\nabla u(t)\|_A^2 + 4C_A(g\ o\ \nabla u)_A(t). \end{aligned} \tag{4.32}$$

Estimate for $F_7 := -2g(0)\int_\Omega u_t(t)h^T\nabla u(t)dx$. Analogously we have done before, we arrive at

$$\begin{aligned} \mid F_7 \mid \;&\leq\; 2g(0)\int_\Omega \mid u_t(t)\mid (h^Th)^{\frac{1}{2}}(\nabla u(t)^T\nabla u(t))^{\frac{1}{2}}dx \\ &\leq\; g(0)\tfrac{2}{\lambda^{\frac{1}{2}}}\int_\Omega \mid u_t(t)\mid (h^Th)^{\frac{1}{2}} < \nabla u(t), \nabla u(t) >_A^{\frac{1}{2}} dx \\ &\leq\; g(0)\tfrac{2}{\lambda^{\frac{1}{2}}}\left[\eta\|u_t(t)\|_2^2 + \tfrac{1}{4\eta}R(x_0)^2\|\nabla u(t)\|_A^2\right] \\ &\leq\; \tfrac{4}{\lambda^{\frac{1}{2}}}g(0)\eta E(t) + \tfrac{1}{2\eta\lambda^{\frac{1}{2}}}g(0)R(x_0)^2\|\nabla u(t)\|_A^2. \end{aligned} \tag{4.33}$$

Estimate for $F_8 := -2\int_0^t g^{'}(t-s)\int_\Omega u_t(t)h^T\nabla u(s)dxds$. Analogously we have done before and taking the assumption **(H3)** into account and we used $2ab \leq \frac{1}{\eta}a^2 + \eta b^2$, we arrive at

$$\begin{aligned} \mid F_8 \mid \;&\leq\; \tfrac{2}{\lambda^{\frac{1}{2}}}R(x_0)\int_0^t \mid g^{'}(t-s)\mid \|\nabla u(s)\|_A\|u_t(t)\|_2 ds \\ &\leq\; \tfrac{2}{\lambda^{\frac{1}{2}}}\eta_1 R(x_0)\int_0^t \mid g(t-s)\mid \|\nabla u(s)\|_A ds\|u_t(t)\|_2 \\ &\leq\; 2\eta E(t) + \tfrac{2}{\eta\lambda}\eta_1^2 R(x_0)^2(g\ o\ \nabla u)_A(t) + \tfrac{2}{\eta\lambda}\eta_1^2 R(x_0)^2\|\nabla u(t)\|_A^2. \end{aligned} \tag{4.34}$$

Estimate for $F_9 := -\theta g(0)\int_\Omega u_t(t)u(t)dx$. Using $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$ also having in mind Poincare inequality, we have

$$\mid F_9 \mid \leq 2\eta E(t) + \frac{c^2\theta^2}{4\lambda\eta}\|\nabla u(t)\|_A^2, \tag{4.35}$$

where $c > 0$ and satisfies $\|v\|_2 \leq c\|\nabla v\|_2$.
Estimate for $F_{10} := -\theta\int_0^t g^{'}(t-s)(\int_\Omega u_t(t)u(s)dx)ds$.

$$\begin{aligned} \mid F_{10} \mid \;&\leq\; \tfrac{c\theta}{\lambda^{\frac{1}{2}}}\int_0^t \mid g^{'}(t-s)\mid \|u_t(t)\|_2\|\nabla u(t)\|_A ds + \tfrac{c^2\theta^2}{4\lambda\eta}\|\nabla u(t)\|_A^2 \\ &\leq\; 2\eta E(t) + \tfrac{\eta_1^2c^2\theta^2}{4\lambda\eta}(\int_0^t g(t-s)\|\nabla u(s)\|_A ds)^2 \\ &\leq\; 2\eta E(t) + \tfrac{\eta_1^2c^2\theta^2}{2\lambda\eta}(g\ o\ \nabla u)_A(t) + \tfrac{\eta_1^2c^2\theta^2}{2\lambda\eta}\|\nabla u(t)\|_A^2. \end{aligned} \tag{4.36}$$

Estimate for $F_{11} := -2(1+\theta)\int_\Omega c(x)K(u_t(t))h^T\nabla u(t)dx$. From Cauchy-Schwartz inequality also using Mean value theorem and having in mind **(H2)**, we get

$$\begin{aligned}
\mid F_{11}\mid &\leq l^{-\frac12}\tfrac{2^{\frac32}(1+\theta)}{\lambda^{\frac12}}R(x_0)\tfrac{l^{\frac12}}{2^{\frac12}}\|\nabla u(t)\|_A\|c\|_\infty^{\frac12}(\textstyle\int_\Omega c(x)K(u_t(t))^2dx)^{\frac12}\\
&\leq l^{-\frac12}\tfrac{2^{\frac32}(1+\theta)}{\lambda^{\frac12}}R(x_0)\tfrac{l^{\frac12}}{2^{\frac12}}\|\nabla u(t)\|_A\|C\|_\infty^{\frac12}\|K'\|_\infty^{\frac12}(\textstyle\int_\Omega c(x)u_t(t)K(u_t(t))dx)^{\frac12} \qquad (4.37)\\
&\leq l^{-\frac12}\tfrac{2^{\frac12}(1+\theta)}{\lambda^{\frac12}}C_KR(x_0)\eta E'(t)+l^{-\frac12}\tfrac{2^{\frac12}(1+\theta)}{\eta\lambda^{\frac12}}C_KR(x_0)\textstyle\int_\Omega c(x)u_t(t)K(u_t(t))dx,
\end{aligned}$$

where $\|K'\|_\infty^{\frac12} = \max_{t\in\mathbb{R}}\mid K'(t)\mid$ and $C_K = \|C\|_\infty^{\frac12}\tau^{\frac12}$, τ is the constant defined in **(H2)**.

Estimate for $F_{12} := -\theta\int_\Omega c(x)K(u_t(x,t))u(x,t)dx$. Analogously like (4.37) we have

$$\mid F_{12}\mid\leq \frac{l^{-\frac12}\theta C_Kc}{(2\lambda)^{\frac12}}\eta E(t)+\frac{l^{-\frac12}\theta C_Kc}{4\eta(2\lambda)^{\frac12}}\int_\Omega c(x)u_t(t)K(u_t(t))dx. \qquad (4.38)$$

Estimate for $F_{13} := 2(1+\theta)\int_0^t g(t-s)(\int_\Omega c(x)K(u_t(t))h^T\nabla u(s)dx)ds$. Analogously like (4.37) we have

$$\begin{aligned}
\mid F_{13}\mid &\leq \tfrac{2(1+\theta)R(x_0)}{\lambda^{\frac12}(4\eta)}\left((\textstyle\int_0^t g(t-s)\|\nabla u(s)\|_Ads)^2+\eta\int_\Omega c(x)^2K(u_t(t))^2dx\right)\\
&\leq \tfrac{(1+\theta)R(x_0)}{\lambda^{\frac12}(2\eta)}(g\ o\ \nabla u)_A(t)+\tfrac{(1+\theta)R(x_0)}{\lambda^{\frac12}(2\eta)}\|\nabla u(t)\|_A^2 \qquad (4.39)\\
&\quad+\tfrac{2(1+\theta)R(x_0)}{\lambda^{\frac12}}C_K\eta\textstyle\int_\Omega c(x)u_t(t)K(u_t(t))dx.
\end{aligned}$$

Estimate for $F_{14} := \theta\int_0^t g(t-s)(\int_\Omega c(x)K(u_t(t))u(s)dx)ds$.Analogously like (4.37) and (4.39) we have

$$\begin{aligned}
\mid F_{14}\mid &\leq \tfrac{c\theta}{\eta\lambda^{\frac12}}(g\ o\ \nabla u)_A(t)+\tfrac{c\theta}{\eta\lambda^{\frac12}}\|\nabla u(t)\|_A^2\\
&\quad+\tfrac{c\theta C_K^2}{\lambda^{\frac12}}\eta\textstyle\int_\Omega c(x)u_t(t)K(u_t(t))dx.
\end{aligned} \qquad (4.40)$$

Estimate for $F_{15} := 2n\int_{Q_1}u_t(t)^2dx+\frac1\lambda\max_{x\in\Omega}\mid\nabla\psi\mid_A\mid m(x)\mid_2\int_{Q_1/Q_0}u_t(t)^2dx$.

$$\mid F_{15}\mid\leq(2n+\frac1\lambda\max_{\forall x\in\Omega}\mid\nabla\psi\mid_A\mid m\mid_2)\int_\Omega u_t(t)^2dx. \qquad (4.41)$$

Then estimate for $\int_\Omega u_t(t)^2dx$ using **(H2)** and **(H1)** we have :

$$\int_\Omega u_t(t)^2dx\leq\frac{1}{C_0c_2}\left(\int_{\Omega_1}c(x)K(u_t(t))u_t(t)dx\right).$$

Using Cauchy-Schwartz inequality, we get:

$$\mid F_{15}\mid\leq(2n+\frac1\Lambda\max_{\forall x\in\Omega}\mid\nabla\psi\mid_A\mid m(x)\mid_2)(\frac{1}{C_0c_2})\int_\Omega c(x)K(u_t(t))u_t(t)dx. \qquad (4.42)$$

Estimate for $F_{16} := \sum_{k=1}^n\int_\Omega\nabla z(t)^T\frac{\partial}{\partial x_k}(A)\nabla z(t)h_kdx$. We have

$$\begin{aligned}
X^T\tfrac{\partial}{\partial x_k}(A)X &= \textstyle\sum_{i=1}^n\tfrac{\partial}{\partial x_k}(a_{ii})x_i^2+\sum_{1<i<j\leq n}\tfrac{\partial}{\partial x_k}(a_{ij})2x_ix_j\\
&\leq \textstyle\sum_{i=1}^n\tfrac{\partial}{\partial x_k}(a_{ii})x_{ii}^2+\sum_{1<i<j\leq n}\tfrac{\partial}{\partial x_k}(a_{ij})(x_i^2+x_j^2)\\
&\leq max_{1\leq i\leq n}\|\tfrac{\partial}{\partial x_k}(a_{ii})\|_{+\infty}X^TX \qquad (4.43)\\
&\quad+max_{i\neq j}\|\tfrac{\partial}{\partial x_k}(a_{ij})\|_{+\infty}((n-1)X^TX+(n-1)X^TX)\\
&\leq \tfrac{2(n-1)}{\lambda}\|\tfrac{\partial}{\partial x_k}A\|_\infty X^TAX.
\end{aligned}$$

$$\begin{aligned} \mid F_{16} \mid &\leq \frac{R(x_0)}{\lambda} \mid \|\frac{\partial}{\partial x} A\|_\infty \mid_2 \|\nabla z(t)\|_A^2 \\ &\leq \frac{R(x_0)}{\lambda} \mid \|\frac{\partial}{\partial x} A\|_\infty \mid_2 (4(g\ o\ \nabla u)_A(t) + 6\|\nabla u(t)\|_A^2), \end{aligned} \tag{4.44}$$

where $\mid \|\frac{\partial}{\partial x} A\|_\infty \mid_2 = (\sum_{i=1}^n \|\frac{\partial}{\partial x_i} A\|_\infty^2)^{\frac{1}{2}}$ and $\|\frac{\partial}{\partial x_i} A\|_\infty = \max_{\forall x\in\Omega} \max_{1\leq k,j\leq n} \mid \frac{\partial}{\partial x_i} a_{k,j}(x) \mid$. Combining (4.30)-(4.44) we obtain

$$\begin{aligned} \rho^{'}(t) &\leq -\left[k_1 - \eta M\right] E(t) - \left[2^{-1}k_1 - 4\eta(1+C_A)\right] \int_0^t g(s)ds \|\nabla u(t)\|_A^2 + M_1(\eta)(g\ o\ \nabla u)_A(t) \\ &\quad + M_2(\eta)\|\nabla u(t)\|_A^2 + M_3(\eta) \int_\Omega c(x) u_t(t) K(u_t(t)) dx, \end{aligned} \tag{4.45}$$

where

$$\begin{aligned} M &= 22\eta + \lambda^{-\frac{1}{2}} 4g(0) + l^{-\frac{1}{2}} \frac{2^{\frac{1}{2}}(1+\theta)}{\lambda^{\frac{1}{2}}} C_K R(x_0) + \frac{l^{-\frac{1}{2}}\theta C_K c}{(2\lambda)^{\frac{1}{2}}}\eta, \\ M_1(\eta) &= 2^{-1}k_1 + 4 + 4C_A + (\lambda\eta)^{-1} 2\epsilon_1^2 R(x_0)^2 + \frac{(\eta_1 c\theta)^2}{2\lambda\eta} + \frac{(1+\theta)R(x_0)}{\lambda^{\frac{1}{2}}(2\eta)} + (\theta c)(\eta\lambda^{\frac{1}{2}})^{-1} \\ &\quad + 4\frac{R(x_0)}{\lambda} \mid \|\frac{\partial}{\partial x} A\|_\infty \mid_2, \\ M_2(\eta) &= \eta^{-1} + 4 + C_A(5 + (2\eta)^{-1}) + (2\eta\lambda^{\frac{1}{2}})^{-1} g(0) R(x_0)^2 + (2\lambda)^{-1}\eta_1^2 R(x_0)^2 + \frac{c^2\theta}{4\lambda\eta} + \frac{\eta_1^2 c^2\theta^2}{2\lambda\eta} \\ &\quad + \frac{(1+\theta)R(x_0)}{\lambda^{\frac{1}{2}}(2\eta)} + \frac{c\theta}{\eta\lambda^{\frac{1}{2}}} + 6\frac{R(x_0)}{\lambda} \mid \|\frac{\partial}{\partial x} A\|_\infty \mid_2, \\ M_3(\eta) &= \frac{c\theta C_K^2}{\lambda^{\frac{1}{2}}}\eta + \frac{2(1+\theta)R(x_0)}{\lambda^{\frac{1}{2}}} C_K \eta + \frac{l^{-\frac{1}{2}}\theta C_K c}{4\eta(2\lambda)^{\frac{1}{2}}} + l^{-\frac{1}{2}} \frac{2^{\frac{1}{2}}(1+\theta)}{\eta\lambda^{\frac{1}{2}}} C_K R(x_0) \\ &\quad + (2n + \frac{1}{\Lambda} \max_{\forall x\in\Omega} \mid \nabla\psi \mid_A \mid m(x) \mid_2)(\frac{1}{C_0 c_2}). \end{aligned}$$

Choosing $\eta > 0$ sufficiently small such that
$k_2 = k_1 - \eta M > 0$ and $2^{-1}k_1 - 4\eta(1+C_A) > 0$.
From (2.4),(4.1) ,(4.45) and considering the assumption **(H2)**, we obtain

$$\begin{aligned} E_\epsilon^{'}(t) &= E^{'}(t) + \epsilon\rho^{'}(t) \\ &\leq -\epsilon k_2 E(t) - \left[2^{-1}\eta_2 - \epsilon M_1(\eta)\right](g\ o\ \nabla u)_A(t) - \left[2^{-1}g(0) - \epsilon M_2(\eta)\right]\|\nabla u(t)\|_A^2 \\ &\quad -(1-\epsilon M_3(\eta)) \int_\Omega c(x) u_t(t) K(u_t(t)) dx. \end{aligned} \tag{4.46}$$

Choosing $\epsilon > 0$ small enough such that
$2^{-1}\eta_2 - \epsilon M_1(\eta) > 0$, $2^{-1}g(0) - \epsilon M_2(\eta) > 0$ and $1 - \epsilon M_3(\eta) > 0$
and from the propositions 1 we deduce

$$E_\epsilon^{'}(t) \leq -\epsilon k_2(\epsilon\delta + 1)^{-1} E_\epsilon(t). \tag{4.47}$$

Hence, we deduce the exponential rate of decay. □

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