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Soft Derivative

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ABSTRACT: This study presents a comprehensive investigation of soft and upper (lower) soft derivatives within the framework of Molodtsov's soft set theory, extending it through the introduction of novel and complementary notions: left and right soft derivatives. It rigorously develops the fundamental properties of these concepts, including algebraic and order-theoretic rules, and establishes their relationships with boundedness and soft continuity types. The paper offers insightful geometric interpretations that significantly enhance conceptual understanding. Furthermore, it introduces absolute ε-extrema, investigates their fundamental properties, and characterizes the local (τ,ε)-extrema defined by Molodtsov. This study presents analogs of Rolle's Theorem for upper and lower soft derivatives and interprets them geometrically with supporting visual illustrations. Moreover, it establishes and geometrically interprets analogs of the Mean Value Theorem for upper and lower soft derivatives. By systematically investigating fundamental concepts in soft analysis and presenting detailed results, the present paper constructs a comprehensive foundation that strengthens the mathematical structure of soft analysis and paves the way for advanced developments, such as soft integrals, soft directional derivatives, and soft gradients.

Key Words: Soft sets, soft analysis, soft derivative.

Contents

1. Introduction

Soft set theory, since its inception by Molodtsov [6], has rapidly evolved into a powerful framework for modeling uncertainty across diverse domains, such as game theory, operations research, and soft analysis [1,2,3,4,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26]. The ability of the framework in addressing real-world problems has spurred extensive theoretical advancements, particularly through Molodtsov's book [9], which characterizes foundational concepts. Despite these strides, the following critical barrier hinders broader adoption: Many of the studies in the Molodtsov school remain confined to Russian-language sources, limiting global accessibility. Although recent studies – e.g., Sapan et al.'s exploration of the soft limit and soft continuity [27] – have begun addressing these limitations, it has been observed that the soft derivative, central to soft analysis and first introduced alongside upper and lower soft derivatives, has been discussed in fragments, thus lacking a holistic systematic development. Moreover, the existing literature overlooks left and right soft derivatives, the relationship between the soft derivative and both boundedness and soft continuity types, and geometric interpretations. Therefore, this study rigorously bridges the aforementioned gaps and enhances global usability, synthesizing scattered Russian-language sources and providing detailed proofs.

This paper addresses these gaps through three primary objectives:

i. To redefine soft and upper (lower) soft derivatives within a single consistent framework and introduce left and right soft derivatives.

ii. To establish implications linking the aforementioned soft derivative types with boundedness and the corresponding forms of soft continuity.

iii. To extend the results in classical analysis to soft contexts, including Rolle's Theorem and the Mean Value Theorem, and to enrich them with illustrative examples and geometric interpretations.

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This study makes notable conceptual and technical contributions to the relevant literature as follows:

i. It is the first systematic treatment of left and right soft derivatives and the soft derivative types' interplay with boundedness and soft continuity types. This allows for investigating the closed-interval properties for left and right soft derivatives and, as with upper and lower soft derivatives, proving that the nonemptiness of the intersection of the left and right soft derivatives implies that of the soft derivative and vice versa.

ii. It extends algebraic and order-theoretic rules, such as sum, difference, and scalar multiplication, to arbitrary scalars and functions by eliminating restrictive assumptions.

iii. It introduces absolute ε-extrema, relates them to local (τ, ε)-extrema, and proposes some implications and equivalences concerning local (τ, ε)-extrema.

iv. It provides geometric interpretations for all soft derivative types by supporting them with visualizations and offers geometric expositions of Rolle's Theorem and the Mean Value Theorem within the framework of soft set theory, complete with illustrated figures for Rolle's Theorem.

The remainder of the paper is organized as follows: Section 2 presents the essential definitions and properties of soft set theory that are required in the next section. Section 3 redefines soft and upper (lower) soft derivatives, proposes left and right soft derivatives, and explores the implications of the soft derivative associated with boundedness and the types of soft continuity. Moreover, it provides closed-interval properties and the necessary and sufficient conditions for the nonemptiness of the soft derivative by upper, lower, left, and right soft derivatives. The section also investigates algebraic and order-theoretic rules and derives geometric interpretations for the aforementioned soft derivative types. Afterward, it introduces absolute ε-extrema and associates them with local (τ, ε)-extrema. Accordingly, this section geometrically interprets the analog of Rolle's Theorem and establishes the analog of the Mean Value Theorem. The final section concludes the paper by discussing the obtained findings and outlining potential directions for future research in soft analysis.

2. Preliminaries

This section presents some of the basic definitions and properties to be employed in the next section. Across this paper, the notations $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^-$, $\mathbb{R}^+$, and $\mathbb{R}^{\geq 0}$ represent the sets of all integers, positive integers, real numbers, negative real numbers, positive real numbers, and nonnegative real numbers, respectively.

Definition 2.1 *[6,9] Let U be a universal set, E be a parameter set, and $f : E \to P(U)$ be a function. Then, (f, E) (briefly f) is called a soft set parameterized via E over U (briefly a soft set over U).*

Example 2.1 *Let $f : \mathbb{Z} \to P(\mathbb{R})$ be a function defined by $f(x) = [x, x+4]$. Then, f is a soft set over $\mathbb{R}$.*

Definition 2.2 *[6,9] Let M be a set called a model set, U be a universal set, E be a parameter set, and $f : M \times E \to P(U)$ be a function. Then, f is called a soft mapping parameterized via $M \times E$ over U (briefly a soft mapping over U).*

Throughout this study as in [9,26], let $\tau, \lambda, \kappa : \mathbb{R} \to P(\mathbb{R})$, $\varepsilon, \alpha, \beta : \mathbb{R} \to \mathbb{R}^{\geq 0}$, and $\delta : \mathbb{R} \to \mathbb{R}^+$ be seven functions such that $P(\mathbb{R})$ is the set of all classic subsets of $\mathbb{R}$, i.e., the power set of $\mathbb{R}$, and $\tau(a)$, $\lambda(a)$, and $\kappa(a)$ are sets of points close to the point a, but not equal to a. In addition, $\tilde{\tau}(a) = \tau(a) \cup \{a\}$, for all $a \in \mathbb{R}$.

Moreover, the set of all points belonging to $\tau(a)$ and are greater than a is defined by $\tau^+(a) := \tau(a) \cap (a, \infty)$, and the set of all points belonging to $\tau(a)$ and are less than a is defined by $\tau^-(a) := \tau(a) \cap (-\infty, a)$. Besides, if $\tau^-(a) = \emptyset$, for all $a \in \mathbb{R}$, then τ is called a right mapping, and if $\tau^+(a) = \emptyset$, for all $a \in \mathbb{R}$, then τ is called a left mapping. Furthermore, $\tau_\delta(a)$ is defined by $\tau_\delta(a) := [a - \delta(a), a) \cup (a, a + \delta(a)]$. Therefore, $\tau_\delta^+(a) = (a, a + \delta(a)]$ and $\tau_\delta^-(a) = [a - \delta(a), a)$. Here, it can be observed that $\tau(a) = \tau^+(a) \cup \tau^-(a)$ and $\tau_\delta(a) = \tau_\delta^+(a) \cup \tau_\delta^-(a)$. Additionally, $\tilde{\tau}^+(a) = \tau^+(a) \cup \{a\}$, $\tilde{\tau}^-(a) = \tau^-(a) \cup \{a\}$, $\tilde{\tau}_\delta(a) = \tau_\delta(a) \cup \{a\}$, $\tilde{\tau}_\delta^+(a) = \tau_\delta^+(a) \cup \{a\}$, and $\tilde{\tau}_\delta^-(a) = \tau_\delta^-(a) \cup \{a\}$, for all $a \in \mathbb{R}$. In addition, $\tau_{D(f)}(a) := \tau(a) \cap \mathrm{Dom}(f)$, $\tau_{D(f)}^-(a) := \tau^-(a) \cap \mathrm{Dom}(f)$, $\tau_{D(f)}^+(a) := \tau^+(a) \cap \mathrm{Dom}(f)$, $\tilde{\tau}_{D(f)}(a) := \tilde{\tau}(a) \cap \mathrm{Dom}(f)$, $\tilde{\tau}_{D(f)}^-(a) := \tilde{\tau}^-(a) \cap \mathrm{Dom}(f)$, and $\tilde{\tau}_{D(f)}^+(a) := \tilde{\tau}^+(a) \cap \mathrm{Dom}(f)$, for all $a \in \mathbb{R}$. Here, $\mathrm{Dom}(f)$ stands for the domain set of f.

Definition 2.3 *[6,9,26,27] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$.*

i. *If* $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$*, then* f *is said to be* (τ, ε)*-soft continuous at the point* a*.*

ii. *If* $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow f(x) \leq f(a) + \varepsilon(a)$*, then* f *is said to be upper* (τ, ε)*-soft continuous at the point* a*.*

iii. *If* $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow f(a) - \varepsilon(a) \leq f(x)$*, then* f *is said to be lower* (τ, ε)*-soft continuous at the point* a*.*

iv. *If* $x \in \tilde{\tau}^{-}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$*, then* f *is said to be left* (τ, ε)*-soft continuous at the point* a*.*

v. *If* $x \in \tilde{\tau}^{+}_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon(a)$*, then* f *is said to be right* (τ, ε)*-soft continuous at the point* a*.*

3. Soft Derivative

This section presents soft derivative provided in [5,6,9,26] with several theoretical contributions. Throughout this study, let $\Phi(A, B) := \{f \mid f : A \to B \text{ is a function}\}$.

Definition 3.1 *[5,6,9,26] Let* $A \subseteq \mathbb{R}$*,* $f : A \to \mathbb{R}$ *be a function,* $a \in A$*, and* $L \in \mathbb{R}$*. Then, the real number* L *is called an upper* (τ, ε)*-soft derivative of* f *at the point* a *if* $x \in \tau(a) \Rightarrow (x \in A \wedge f(x) \leq f(a) + L(x - a) + \varepsilon(a))$*.*

Definition 3.2 *[5,6,9,26] Let* $A \subseteq \mathbb{R}$*,* $f : A \to \mathbb{R}$ *be a function,* $a \in A$*, and* $L \in \mathbb{R}$*. Then, the real number* L *is called a lower* (τ, ε)*-soft derivative of* f *at the point* a *if* $x \in \tau(a) \Rightarrow (x \in A \wedge f(x) \geq f(a) + L(x - a) - \varepsilon(a))$*.*

Definition 3.3 *[5,6,9,26] Let* $A \subseteq \mathbb{R}$*,* $f : A \to \mathbb{R}$ *be a function,* $a \in A$*, and* $L \in \mathbb{R}$*. Then, the real number* L *is called a* (τ, ε)*-soft derivative of* f *at the point* a *if* $x \in \tau(a) \Rightarrow (x \in A \wedge |f(x) - f(a) - L(x - a)| \leq \varepsilon(a))$*.*

Note 1 *Molodtsov [19] defines* (τ, ε)*-soft derivative of order* n *as follows:*

"Let $A \subseteq \mathbb{R}$*,* $f : A \to \mathbb{R}$ *be a function,* $a \in A$*, and* $(L_1, L_2, \cdots, L_n) \in \mathbb{R}^n := \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$ *such that* $n \in \mathbb{Z}^+$*. Then, the ordered n-tuple* $(L_1, L_2, \cdots, L_n)$ *is called a* (τ, ε)*-soft derivative of order* n *of* f *at the point* a *if*

$$\sup_{x \in \tau_{D(f)}(a)} \left\{ \left| f(x) - f(a) - \sum_{i=1}^{n} L_i (x - a)^i \right| \right\} \leq \varepsilon(a)$$

The set of all (τ, ε)*-soft derivatives of order* n *of* f *at the point* a *is denoted by* $\mathfrak{D}_n(f, a, \tau, \varepsilon)$*. If* $\mathfrak{D}_n(f, a, \tau, \varepsilon) = \emptyset$*, then the* (τ, ε)*-soft derivative of order* n *of* f *at the point* a *does not exist." Hence, for* $n = 1$*, the hypothesis of this concept becomes*

$$\sup_{x \in \tau_{D(f)}(a)} \left\{ \left| f(x) - f(a) - L(x - a) \right| \right\} \leq \varepsilon(a)$$

Moreover, the statement $\sup_{x \in \tau_{D(f)}(a)} \left\{ \left| f(x) - f(a) - L(x - a) \right| \right\} \leq \varepsilon(a)$ *is equivalent to the statement* $x \in \tau_{D(f)}(a) \Rightarrow |f(x) - f(a) - L(x - a)| \leq \varepsilon(a)$*. Therefore, this study uses Definitions 3.1-3.3 as in Definitions 3.4-3.6, respectively. Furthermore, it readjusts the related properties provided in [5,6,9,26] according to Definitions 3.4-3.6.*

Definition 3.4 *Let* $A \subseteq \mathbb{R}$*,* $f : A \to \mathbb{R}$ *be a function,* $a \in A$*,* $\tau_{D(f)}(a) \neq \emptyset$*, and* $L \in \mathbb{R}$*. Then, the real number* L *is called an upper* (τ, ε)*-soft derivative of* f *at the point* a *if* $x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq f(a) + L(x - a) + \varepsilon(a)$*. The set of all upper* (τ, ε)*-soft derivatives of* f *at the point* a *is denoted by* $\overline{D}(f, a, \tau, \varepsilon)$*. If* $\overline{D}(f, a, \tau, \varepsilon) = \emptyset$*, then the upper* (τ, ε)*-soft derivative of* f *at the point* a *does not exist.*

Definition 3.5 *Let* $A \subseteq \mathbb{R}$*,* $f : A \to \mathbb{R}$ *be a function,* $a \in A$*,* $\tau_{D(f)}(a) \neq \emptyset$*, and* $L \in \mathbb{R}$*. Then, the real number* L *is called a lower* (τ, ε)*-soft derivative of* f *at the point* a *if* $x \in \tau_{D(f)}(a) \Rightarrow f(x) \geq f(a) + L(x - a) - \varepsilon(a)$*. The set of all lower* (τ, ε)*-soft derivatives of* f *at the point* a *is denoted by* $\underline{D}(f, a, \tau, \varepsilon)$*. If* $\underline{D}(f, a, \tau, \varepsilon) = \emptyset$*, then the lower* (τ, ε)*-soft derivative of* f *at the point* a *does not exist.*

Definition 3.6 *Let* $A \subseteq \mathbb{R}$*,* $f : A \to \mathbb{R}$ *be a function,* $a \in A$*,* $\tau_{D(f)}(a) \neq \emptyset$*, and* $L \in \mathbb{R}$*. Then, the real number* L *is called a* (τ, ε)*-soft derivative of* f *at the point* a *if* $x \in \tau_{D(f)}(a) \Rightarrow \left| f(x) - f(a) - L(x - a) \right| \leq \varepsilon(a)$*. The set of all* (τ, ε)*-soft derivatives of* f *at the point* a *is denoted by* $D(f, a, \tau, \varepsilon)$*. If* $D(f, a, \tau, \varepsilon) = \emptyset$*, then the* (τ, ε)*-soft derivative of* f *at the point* a *does not exist.*

In addition, left (τ, ε)- and right (τ, ε)-soft derivatives are defined as follows:

Definition 3.7 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau^-_{D(f)}(a) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a left (τ, ε)-soft derivative of f at the point a if $x \in \tau^-_{D(f)}(a) \Rightarrow \left|f(x) - f(a) - L(x-a)\right| \leq \varepsilon(a)$. The set of all left (τ, ε)-soft derivatives of f at the point a is denoted by $D(f, a, \tau^-, \varepsilon)$. If $D(f, a, \tau^-, \varepsilon) = \emptyset$, then the left (τ, ε)-soft derivative of f at the point a does not exist.*

Definition 3.8 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau^+_{D(f)}(a) \neq \emptyset$, and $L \in \mathbb{R}$. Then, the real number L is called a right (τ, ε)-soft derivative of f at the point a if $x \in \tau^+_{D(f)}(a) \Rightarrow \left|f(x) - f(a) - L(x-a)\right| \leq \varepsilon(a)$. The set of all right (τ, ε)-soft derivatives of f at the point a is denoted by $D(f, a, \tau^+, \varepsilon)$. If $D(f, a, \tau^+, \varepsilon) = \emptyset$, then the right (τ, ε)-soft derivative of f at the point a does not exist.*

Note 2 *[5] For the upper and lower soft derivatives, the values of $\pm\infty$ are allowed.*

Example 3.1 *Let $f : [0,2] \to \mathbb{R}$ be a function defined by $f(x) = x^2$, $a = 1$, $\delta(1) = 2$, and $\varepsilon(1) = 2$. Then, $\tau^+_\delta(1) \cap [0,2] = (1,2]$, $\tau^-_\delta(1) \cap [0,2] = [0,1)$, and $\tau_\delta(1) \cap [0,2] = [0,2] \setminus \{1\}$. Hence,*

$$\begin{aligned}
\forall x \in [0,2] \setminus \{1\},\ |f(x) - f(a) - L(x-a)| \leq 2 &\Leftrightarrow \forall x \in [0,2] \setminus \{1\},\ |x^2 - 1 - L(x-1)| \leq 2 \\
&\Leftrightarrow \forall x \in [0,2] \setminus \{1\},\ -x^2 + 1 - 2 \leq -L(x-1) \leq -x^2 + 1 + 2 \\
&\Leftrightarrow \forall x \in [0,2] \setminus \{1\},\ \tfrac{x^2-1}{x-1} - \tfrac{2}{|x-1|} \leq L \leq \tfrac{x^2-1}{x-1} + \tfrac{2}{|x-1|} \\
&\Leftrightarrow L \in [1,3]
\end{aligned}$$

Therefore, $D(f, 1, \tau_\delta, \varepsilon) = [1,3]$. Similarly, $\overline{D}(f, 1, \tau_\delta, \varepsilon) = [1, \infty]$, $\underline{D}(f, 1, \tau_\delta, \varepsilon) = [-\infty, 3]$, $D\left(f, 1, \tau^-_\delta, \varepsilon\right) = [-1, 3]$, and $D\left(f, 1, \tau^+_\delta, \varepsilon\right) = [1,5]$.

Note 3 *[9,26] Each of the concepts of upper and lower (τ, ε)-soft derivatives, left (τ, ε)- and right (τ, ε)-soft derivatives, and (τ, ε)-soft derivative is a soft mapping parameterized via $\Phi(A, \mathbb{R}) \times A \times \Phi(\mathbb{R}, P(\mathbb{R})) \times \Phi\left(\mathbb{R}, \mathbb{R}^{\geq 0}\right)$ over $\mathbb{R}$ such that $\emptyset \neq A \subseteq \mathbb{R}$.*

Theorem 3.1 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a)$ be bounded above. If there exists the upper (τ, ε)-soft derivative of f at the point a, then there exists a function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that f is upper (τ, ε^*)-soft continuous at the point a.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a)$ be bounded above, and there exist the upper (τ, ε)-soft derivative of f at the point a. Then, $\tau_{D(f)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq f(a) + L(x-a) + \varepsilon(a)$$

Thus, for any function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that $\varepsilon^*(a) = \left| \sup\limits_{x \in \tau_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a)$, since

$$x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq f(a) + L(x-a) + \varepsilon(a) \leq f(a) + \varepsilon^*(a)$$

and

$$x = a \Rightarrow f(x) \leq f(a) + \varepsilon^*(a)$$

then

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow f(x) \leq f(a) + \varepsilon^*(a)$$

Consequently, f is upper (τ, ε^*)-soft continuous at the point a. □

Theorem 3.2 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a)$ be bounded below. If there exists the lower (τ, ε)-soft derivative of f at the point a, then there exists a function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that f is lower (τ, ε^*)-soft continuous at the point a.*

The proof is similar to the proof of Theorem 3.1 by any function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that

$$\varepsilon^*(a) = \left| \inf_{x \in \tau_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a)$$

Theorem 3.3 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^-_{D(f)}(a)$ be bounded. If there exists the left (τ, ε)-soft derivative of f at the point a, then there exists a function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that f is left (τ, ε^*)-soft continuous at the point a.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau^-_{D(f)}(a)$ be bounded, and there exist the left (τ, ε)-soft derivative of f at the point a. Then, $\tau^-_{D(f)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$x \in \tau^-_{D(f)}(a) \Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a)$$

Thus, for any function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that

$$\varepsilon^*(a) = \max \left\{ \left| \inf_{x \in \tau^-_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a), \left| \sup_{x \in \tau^-_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a) \right\}$$

since

$$\begin{aligned} x \in \tau^-_{D(f)}(a) &\Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a) \\ &\Rightarrow \inf_{x \in \tau^-_{D(f)}(a)} \{L(x-a)\} - \varepsilon(a) \leq f(x) - f(a) \leq \sup_{x \in \tau^-_{D(f)}(a)} \{L(x-a)\} + \varepsilon(a) \\ &\Rightarrow -\left| \inf_{x \in \tau^-_{D(f)}(a)} \{L(x-a)\} \right| - \varepsilon(a) \leq f(x) - f(a) \leq \left| \sup_{x \in \tau^-_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a) \\ &\Rightarrow |f(x) - f(a)| \leq \varepsilon^*(a) \end{aligned}$$

and

$$x = a \Rightarrow |f(x) - f(a)| \leq \varepsilon^*(a)$$

then

$$x \in \tilde{\tau}^-_{D(f)}(a) \Rightarrow |f(x) - f(a)| \leq \varepsilon^*(a)$$

Consequently, f is left (τ, ε^*)-soft continuous at the point a. □

Theorem 3.4 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^+_{D(f)}(a)$ be bounded. If there exists the right (τ, ε)-soft derivative of f at the point a, then there exists a function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that f is right (τ, ε^*)-soft continuous at the point a.*

The proof is similar to the proof of Theorem 3.3 by any function $\varepsilon^* : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that

$$\varepsilon^*(a) = \max \left\{ \left| \inf_{x \in \tau^+_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a), \left| \sup_{x \in \tau^+_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a) \right\}$$

Theorem 3.5 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If f is upper (τ, ε)-soft continuous at the point a, then there exists the upper (τ, ε)-soft derivative of f at the point a.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a) \neq \emptyset$, and f be upper (τ, ε)-soft continuous at the point a. Then,

$$x \in \tilde{\tau}_{D(f)}(a) \Rightarrow f(x) \leq f(a) + \varepsilon(a)$$

Hence,

$$\begin{aligned} x \in \tau_{D(f)}(a) &\Rightarrow x \in \tilde{\tau}_{D(f)}(a) \\ &\Rightarrow f(x) \leq f(a) + 0(x-a) + \varepsilon(a) \end{aligned}$$

Thus, $0 \in \overline{D}(f, a, \tau, \varepsilon)$. Consequently, there exists the upper (τ, ε)-soft derivative of f at the point a. □

Theorem 3.6 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If f is lower (τ, ε)-soft continuous at the point a, then there exists the lower (τ, ε)-soft derivative of f at the point a.*

Theorem 3.7 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^-_{D(f)}(a) \neq \emptyset$. If f is left (τ, ε)-soft continuous at the point a, then there exists the left (τ, ε)-soft derivative of f at the point a.*

Theorem 3.8 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^+_{D(f)}(a) \neq \emptyset$. If f is right (τ, ε)-soft continuous at the point a, then there exists the right (τ, ε)-soft derivative of f at the point a.*

The proofs of Theorems 3.6-3.8 are similar to the proof of Theorem 3.5.

Theorem 3.9 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a)$ be bounded above. If there exists the upper (τ, ε)-soft derivative of f at the point a, then f is bounded above on $\tau_{D(f)}(a)$.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a)$ be bounded above, and there exist the upper (τ, ε)-soft derivative of f at the point a. Then, $\tau_{D(f)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq f(a) + L(x - a) + \varepsilon(a)$$

Since $\tau_{D(f)}(a)$ is bounded above, then $\sup\limits_{x \in \tau_{D(f)}(a)} \{L(x - a)\} \in \mathbb{R}$. Therefore,

$$\sup_{x \in \tau_{D(f)}(a)} \{f(x)\} \leq \sup_{x \in \tau_{D(f)}(a)} \{f(a) + L(x - a) + \varepsilon(a)\} = f(a) + \sup_{x \in \tau_{D(f)}(a)} \{L(x - a)\} + \varepsilon(a) \in \mathbb{R}$$

Consequently, f is bounded above on $\tau_{D(f)}(a)$. □

Theorem 3.10 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a)$ be bounded below. If there exists the lower (τ, ε)-soft derivative of f at the point a, then f is bounded below on $\tau_{D(f)}(a)$.*

Theorem 3.11 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^-_{D(f)}(a)$ be bounded. If there exists the left (τ, ε)-soft derivative of f at the point a, then f is bounded on $\tau^-_{D(f)}(a)$.*

Theorem 3.12 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^+_{D(f)}(a)$ be bounded. If there exists the right (τ, ε)-soft derivative of f at the point a, then f is bounded on $\tau^+_{D(f)}(a)$.*

The proofs of Theorems 3.10-3.12 are similar to the proof of Theorem 3.9.

Theorem 3.13 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If f is bounded above on $\tau_{D(f)}(a)$, then there exists a function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that $\overline{D}(f, a, \tau, \varepsilon) \neq \emptyset$.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a) \neq \emptyset$, and f be bounded above on $\tau_{D(f)}(a)$. Then, there exists an $M \in \mathbb{R}$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq M$$

Thus,

$$x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq f(a) + 0(x - a) + M - f(a) \leq f(a) + 0(x - a) + |M - f(a)|$$

Hence, $0 \in \overline{D}(f, a, \tau, \varepsilon)$ for the function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by $\varepsilon(x) = |M - f(a)|$. Consequently, $\overline{D}(f, a, \tau, \varepsilon) \neq \emptyset$. □

Theorem 3.14 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If f is bounded below on $\tau_{D(f)}(a)$, then there exists a function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that $\underline{D}(f, a, \tau, \varepsilon) \neq \emptyset$.*

The proof is similar to the proof of Theorem 3.13.

Theorem 3.15 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^-_{D(f)}(a) \neq \emptyset$. If f is bounded on $\tau^-_{D(f)}(a)$, then there exists a function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that $D(f, a, \tau^-, \varepsilon) \neq \emptyset$.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau^-_{D(f)}(a) \neq \emptyset$, and f be bounded on $\tau^-_{D(f)}(a)$. Assume that $|f(x)| \leq M$, for all $x \in \tau^-_{D(f)}(a)$. Then, the proof of the theorem is similar to the proof of Theorem 3.13 by the function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by $\varepsilon(x) = \max\{|M + f(a)|, |M - f(a)|\}$. □

Theorem 3.16 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau^+_{D(f)}(a) \neq \emptyset$. If f is bounded on $\tau^+_{D(f)}(a)$, then there exists a function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that $D(f, a, \tau^+, \varepsilon) \neq \emptyset$.*

The proof is similar to the proof of Theorem 3.15.

Theorem 3.17 *[9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. If there exists the (τ, ε)-soft derivative of f at the point a, then*

$$D(f, a, \tau, \varepsilon) = \left[\sup_{x \in \tau_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} - \frac{\varepsilon(a)}{|x - a|}\right\}, \inf_{x \in \tau_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} + \frac{\varepsilon(a)}{|x - a|}\right\}\right]$$

Theorem 3.18 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. If there exists the left (τ, ε)-soft derivative of f at the point a, then*

$$D(f, a, \tau^-, \varepsilon) = \left[\sup_{x \in \tau^-_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} - \frac{\varepsilon(a)}{|x - a|}\right\}, \inf_{x \in \tau^-_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} + \frac{\varepsilon(a)}{|x - a|}\right\}\right]$$

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and there exist the left (τ, ε)-soft derivative of f at the point a. Then, $\tau^-_{D(f)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that, for all $x \in \tau^-_{D(f)}(a)$,

$$\begin{aligned}
\left|f(x) - f(a) - L(x - a)\right| \leq \varepsilon(a) &\Rightarrow -(f(x) - f(a)) - \varepsilon(a) \leq -L(x - a) \leq -(f(x) - f(a)) + \varepsilon(a) \\
&\Rightarrow \tfrac{-(f(x)-f(a))-\varepsilon(a)}{-(x-a)} \leq L \leq \tfrac{-(f(x)-f(a))+\varepsilon(a)}{-(x-a)} \\
&\Rightarrow \tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{x-a} \leq L \leq \tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{x-a} \\
&\Rightarrow \tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{|x-a|} \leq L \leq \tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{|x-a|}
\end{aligned}$$

Hence,

$$\sup_{x \in \tau^-_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{|x-a|}\right\} \leq L \quad \text{and} \quad L \leq \inf_{x \in \tau^-_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{|x-a|}\right\}$$

Consequently,

$$D(f, a, \tau^-, \varepsilon) = \left[\sup_{x \in \tau^-_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} - \frac{\varepsilon(a)}{|x - a|}\right\}, \inf_{x \in \tau^-_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} + \frac{\varepsilon(a)}{|x - a|}\right\}\right]$$

□

Theorem 3.19 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. If there exists the right (τ, ε)-soft derivative of f at the point a, then*

$$D(f, a, \tau^+, \varepsilon) = \left[\sup_{x \in \tau^+_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} - \frac{\varepsilon(a)}{|x - a|}\right\}, \inf_{x \in \tau^+_{D(f)}(a)} \left\{\frac{f(x) - f(a)}{x - a} + \frac{\varepsilon(a)}{|x - a|}\right\}\right]$$

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and there exist the right (τ, ε)-soft derivative of f at the point a. Then, $\tau^+_{D(f)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that, for all $x \in \tau^+_{D(f)}(a)$,

$$\begin{aligned}
|f(x) - f(a) - L(x - a)| \leq \varepsilon(a) &\Rightarrow -(f(x) - f(a)) - \varepsilon(a) \leq -L(x - a) \leq -(f(x) - f(a)) + \varepsilon(a) \\
&\Rightarrow \tfrac{-(f(x)-f(a))+\varepsilon(a)}{-(x-a)} \leq L \leq \tfrac{-(f(x)-f(a))-\varepsilon(a)}{-(x-a)} \\
&\Rightarrow \tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{x-a} \leq L \leq \tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{x-a} \\
&\Rightarrow \tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{|x-a|} \leq L \leq \tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{|x-a|}
\end{aligned}$$

Hence,

$$\sup_{x\in\tau^+_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}-\frac{\varepsilon(a)}{|x-a|}\right\}\leq L \quad\text{and}\quad L\leq \inf_{x\in\tau^+_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}+\frac{\varepsilon(a)}{|x-a|}\right\}$$

Consequently,

$$D(f,a,\tau^+,\varepsilon)=\left[\sup_{x\in\tau^+_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}-\frac{\varepsilon(a)}{|x-a|}\right\},\inf_{x\in\tau^+_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}+\frac{\varepsilon(a)}{|x-a|}\right\}\right]$$

□

Theorem 3.20 *[9,26] Let $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$ be a function, and $a\in A$. If there exists the upper (τ,ε)-soft derivative of f at the point a, then*

$$\overline{D}(f,a,\tau,\varepsilon)=\left[\sup_{x\in\tau^+_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}-\frac{\varepsilon(a)}{|x-a|}\right\},\inf_{x\in\tau^-_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}+\frac{\varepsilon(a)}{|x-a|}\right\}\right]$$

Theorem 3.21 *[9,26] Let $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$ be a function, and $a\in A$. If there exists the lower (τ,ε)-soft derivative of f at the point a, then*

$$\underline{D}(f,a,\tau,\varepsilon)=\left[\sup_{x\in\tau^-_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}-\frac{\varepsilon(a)}{|x-a|}\right\},\inf_{x\in\tau^+_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}+\frac{\varepsilon(a)}{|x-a|}\right\}\right]$$

Example 3.2 *Let $f:[0,2]\to\mathbb{R}$ be a function defined by $f(x)=x^2$, $a=0$, $\delta(0)=2$, $\delta(1)=3$, $\varepsilon_1(1)=\varepsilon_1(0)=1$, and $\varepsilon_2(1)=\varepsilon_2(0)=\frac{1}{2}$. Then, $\tau^+_\delta(0)\cap[0,2]=(0,2]\cap[0,2]=(0,2]$, $\tau^-_\delta(0)\cap[0,2]=[-2,0)\cap[0,2]=\emptyset$, and $\tau_\delta(0)\cap[0,2]=(0,2]$. Hence,*

$$\overline{D}(f,0,\tau_\delta,\varepsilon_1)=\left[\sup_{x\in(0,2]}\left\{x-\frac{1}{x}\right\},\inf_{x\in\emptyset}\left\{x+\frac{1}{|x|}\right\}\right]=\left[\frac{3}{2},\infty\right]$$

$$\underline{D}(f,0,\tau_\delta,\varepsilon_1)=\left[\sup_{x\in\emptyset}\left\{x-\frac{1}{|x|}\right\},\inf_{x\in(0,2]}\left\{x+\frac{1}{x}\right\}\right]=[-\infty,2]$$

and

$$D(f,0,\tau_\delta,\varepsilon_1)=\left[\sup_{x\in(0,2]}\left\{x-\frac{1}{x}\right\},\inf_{x\in(0,2]}\left\{x+\frac{1}{x}\right\}\right]=\left[\frac{3}{2},2\right]$$

It can be observed that $D(f,0,\tau_\delta,\varepsilon_1)=\overline{D}(f,0,\tau_\delta,\varepsilon_1)\cap\underline{D}(f,0,\tau_\delta,\varepsilon_1)$. Moreover,

$$\overline{D}(f,0,\tau_\delta,\varepsilon_2)=\left[\sup_{x\in(0,2]}\left\{x-\frac{\frac{1}{2}}{x}\right\},\inf_{x\in\emptyset}\left\{x+\frac{\frac{1}{2}}{|x|}\right\}\right]=\left[\frac{7}{4},\infty\right]$$

$$\underline{D}(f,0,\tau_\delta,\varepsilon_2)=\left[\sup_{x\in\emptyset}\left\{x-\frac{\frac{1}{2}}{|x|}\right\},\inf_{x\in(0,2]}\left\{x+\frac{\frac{1}{2}}{x}\right\}\right]=\left[-\infty,\sqrt{2}\right]$$

and

$$D(f,0,\tau_\delta,\varepsilon_2)=\left[\sup_{x\in(0,2]}\left\{x-\frac{\frac{1}{2}}{x}\right\},\inf_{x\in(0,2]}\left\{x+\frac{\frac{1}{2}}{x}\right\}\right]=\left[\frac{7}{4},\sqrt{2}\right]=\emptyset$$

It is clear that $\overline{D}(f,0,\tau_\delta,\varepsilon_2)\cap\underline{D}(f,0,\tau_\delta,\varepsilon_2)=\emptyset$. Besides, for $\tau_\delta(1)=[-2,1)\cup(1,4]$,

$$D(f,1,\tau^-_\delta,\varepsilon_1)=\left[\sup_{x\in[0,1)}\left\{x+1-\frac{1}{|x-1|}\right\},\inf_{x\in[0,1)}\left\{x+1+\frac{1}{|x-1|}\right\}\right]=[0,2]$$

$$D(f,1,\tau^+_\delta,\varepsilon_1)=\left[\sup_{x\in(1,2]}\left\{x+1-\frac{1}{|x-1|}\right\},\inf_{x\in(1,2]}\left\{x+1+\frac{1}{|x-1|}\right\}\right]=[2,4]$$

and

$$D(f,1,\tau_\delta,\varepsilon_1)=\left[\sup_{x\in[0,2]\setminus\{1\}}\left\{x+1-\frac{1}{|x-1|}\right\},\inf_{x\in[0,2]\setminus\{1\}}\left\{x+1+\frac{1}{|x-1|}\right\}\right]=[2,2]$$

It can be observed that $D(f,1,\tau_\delta,\varepsilon_1)=D\left(f,1,\tau_\delta^-,\varepsilon_1\right)\cap D\left(f,1,\tau_\delta^+,\varepsilon_1\right)$*. Furthermore,*

$$D(f,1,\tau_\delta^-,\varepsilon_2)=\left[\sup_{x\in[0,1)}\left\{x+1-\frac{\frac{1}{2}}{|x-1|}\right\},\inf_{x\in[0,1)}\left\{x+1+\frac{\frac{1}{2}}{|x-1|}\right\}\right]=\left[2-\sqrt{2},\frac{3}{2}\right]$$

$$D(f,1,\tau_\delta^+,\varepsilon_2)=\left[\sup_{x\in(1,2]}\left\{x+1-\frac{\frac{1}{2}}{|x-1|}\right\},\inf_{x\in(1,2]}\left\{x+1+\frac{\frac{1}{2}}{|x-1|}\right\}\right]=\left[\frac{5}{2},2+\sqrt{2}\right]$$

and

$$D(f,1,\tau_\delta,\varepsilon_2)=\left[\sup_{x\in[0,2]\setminus\{1\}}\left\{x+1-\frac{\frac{1}{2}}{|x-1|}\right\},\inf_{x\in[0,2]\setminus\{1\}}\left\{x+1+\frac{\frac{1}{2}}{|x-1|}\right\}\right]=\left[\frac{5}{2},\frac{3}{2}\right]=\emptyset$$

It is clear that $D\left(f,1,\tau_\delta^-,\varepsilon_2\right)\cap D\left(f,1,\tau_\delta^+,\varepsilon_2\right)=\emptyset$.

Note 4 *Example 3.2 indicates that the existence of upper and lower soft derivatives or left and right soft derivatives does not require the existence of soft derivatives. The following theorems prove the situation of the existence.*

Theorem 3.22 *Let* $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$ *be a function, and* $a\in A$*. Then,* $\overline{D}(f,a,\tau,\varepsilon)\cap\underline{D}(f,a,\tau,\varepsilon)\neq\emptyset$ *if and only if there exists the* (τ,ε)*-soft derivative of* f *at the point* a*.*

Proof: Let $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$ be a function, and $a\in A$.
($\Rightarrow$): Let $\overline{D}(f,a,\tau,\varepsilon)\cap\underline{D}(f,a,\tau,\varepsilon)\neq\emptyset$. Then, $\tau_{D(f)}(a)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that

$$x\in\tau_{D(f)}(a)\Rightarrow f(x)\leq f(a)+L(x-a)+\varepsilon(a)$$

and

$$x\in\tau_{D(f)}(a)\Rightarrow f(x)\geq f(a)+L(x-a)-\varepsilon(a)$$

Thus,

$$x\in\tau_{D(f)}(a)\Rightarrow f(a)+L(x-a)-\varepsilon(a)\leq f(x)\leq f(a)+L(x-a)+\varepsilon(a)$$

Hence,

$$x\in\tau_{D(f)}(a)\Rightarrow|f(x)-f(a)-L(x-a)|\leq\varepsilon(a)$$

Consequently, $L\in D(f,a,\tau,\varepsilon)$ and thus there exists the (τ,ε)-soft derivative of f at the point a.
($\Leftarrow$): Let there exist the (τ,ε)-soft derivative of f at the point a. Then, $\tau_{D(f)}(a)\neq\emptyset$ and there exists an $L\in\mathbb{R}$ such that

$$x\in\tau_{D(f)}(a)\Rightarrow\left|f(x)-f(a)-L(x-a)\right|\leq\varepsilon(a)$$

Therefore,

$$x\in\tau_{D(f)}(a)\Rightarrow f(x)\leq f(a)+L(x-a)+\varepsilon(a)$$

and

$$x\in\tau_{D(f)}(a)\Rightarrow f(x)\geq f(a)+L(x-a)-\varepsilon(a)$$

Consequently, $L\in\overline{D}(f,a,\tau,\varepsilon)\cap\underline{D}(f,a,\tau,\varepsilon)$ and thus $\overline{D}(f,a,\tau,\varepsilon)\cap\underline{D}(f,a,\tau,\varepsilon)\neq\emptyset$. □

Theorem 3.23 *[6,9,26] Let* $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$ *be a function, and* $a\in A$*. Then,*

$$\overline{D}(f,a,\tau,\varepsilon)\cap\underline{D}(f,a,\tau,\varepsilon)=D(f,a,\tau,\varepsilon)$$

Theorem 3.24 *Let* $A\subseteq\mathbb{R}$, $f:A\to\mathbb{R}$ *be a function,* $a\in A$, $\tau_{D(f)}^-(a)\neq\emptyset$*, and* $\tau_{D(f)}^+(a)\neq\emptyset$*. Then,* $D(f,a,\tau^-,\varepsilon)\cap D(f,a,\tau^+,\varepsilon)\neq\emptyset$ *if and only if there exists the* (τ,ε)*-soft derivative of* f *at the point* a*. Moreover,*

$$D(f,a,\tau^-,\varepsilon)\cap D(f,a,\tau^+,\varepsilon)=D(f,a,\tau,\varepsilon)$$

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau^{-}_{D(f)}(a) \neq \emptyset$, and $\tau^{+}_{D(f)}(a) \neq \emptyset$.
($\Rightarrow$): Let $D(f, a, \tau^{-}, \varepsilon) \cap D(f, a, \tau^{+}, \varepsilon) \neq \emptyset$. Then, there exists an $L \in \mathbb{R}$ such that

$$x \in \tau^{-}_{D(f)}(a) \Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a)$$

and

$$x \in \tau^{+}_{D(f)}(a) \Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a)$$

Thus,

$$x \in \tau_{D(f)}(a) \Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a)$$

Hence, there exists the (τ, ε)-soft derivative of f at the point a.
($\Leftarrow$): Let there exist the (τ, ε)-soft derivative of f at the point a. Then, $\tau_{D(f)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a)$$

Since $\tau^{-}_{D(f)}(a) \neq \emptyset$ and $\tau^{+}_{D(f)}(a) \neq \emptyset$,

$$\begin{aligned} x \in \tau^{-}_{D(f)}(a) &\Rightarrow x \in \tau_{D(f)}(a) \\ &\Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a) \end{aligned}$$

and

$$\begin{aligned} x \in \tau^{+}_{D(f)}(a) &\Rightarrow x \in \tau_{D(f)}(a) \\ &\Rightarrow \left| f(x) - f(a) - L(x-a) \right| \leq \varepsilon(a) \end{aligned}$$

Therefore, $L \in D(f, a, \tau^{-}, \varepsilon) \cap D(f, a, \tau^{+}, \varepsilon)$ and thus $D(f, a, \tau^{-}, \varepsilon) \cap D(f, a, \tau^{+}, \varepsilon) \neq \emptyset$.
Moreover, let $S = D(f, a, \tau^{-}, \varepsilon) \cap D(f, a, \tau^{+}, \varepsilon)$. Then,

$$S = \left[\max\left\{ \sup_{x \in \tau^{+}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} - \frac{\varepsilon(a)}{|x-a|} \right\}, \sup_{x \in \tau^{-}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} - \frac{\varepsilon(a)}{|x-a|} \right\} \right\}, \min\left\{ \inf_{x \in \tau^{+}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} + \frac{\varepsilon(a)}{|x-a|} \right\}, \inf_{x \in \tau^{-}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} + \frac{\varepsilon(a)}{|x-a|} \right\} \right\} \right]$$

Since

$$\max\left\{ \sup_{x \in \tau^{-}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} - \frac{\varepsilon(a)}{|x-a|} \right\}, \sup_{x \in \tau^{+}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} - \frac{\varepsilon(a)}{|x-a|} \right\} \right\} = \sup_{x \in \tau_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} - \frac{\varepsilon(a)}{|x-a|} \right\}$$

and

$$\min\left\{ \inf_{x \in \tau^{-}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} + \frac{\varepsilon(a)}{|x-a|} \right\}, \inf_{x \in \tau^{+}_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} + \frac{\varepsilon(a)}{|x-a|} \right\} \right\} = \inf_{x \in \tau_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} + \frac{\varepsilon(a)}{|x-a|} \right\}$$

then

$$S = \left[\sup_{x \in \tau_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} - \frac{\varepsilon(a)}{|x-a|} \right\}, \inf_{x \in \tau_{D(f)}(a)} \left\{ \frac{f(x)-f(a)}{x-a} + \frac{\varepsilon(a)}{|x-a|} \right\} \right]$$

Consequently,

$$D(f, a, \tau^{-}, \varepsilon) \cap D(f, a, \tau^{+}, \varepsilon) = D(f, a, \tau, \varepsilon)$$

□

Corollary 3.1 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a)$ be bounded. If there exists the (τ, ε)-soft derivative of f at the point a, then there exists a function $\varepsilon^{*} : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that f is (τ, ε^{*})-soft continuous at the point a.*

The proof is similar to the proof of Theorem 3.3 by any function $\varepsilon^{*} : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that

$$\varepsilon^{*}(a) = \max\left\{ \left| \sup_{x \in \tau_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a), \left| \inf_{x \in \tau_{D(f)}(a)} \{L(x-a)\} \right| + \varepsilon(a) \right\}$$

Corollary 3.2 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If f is (τ, ε)-soft continuous at the point a, then there exists the (τ, ε)-soft derivative of f at the point a.*

The proof is similar to the proof of Theorem 3.5.

Corollary 3.3 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a) \neq \emptyset$, and $\tau_{D(f)}(a)$ be bounded. If there exists the (τ, ε)-soft derivative of f at the point a, then f is bounded on $\tau_{D(f)}(a)$.*

The proof can be observed from Theorems 3.11, 3.12, and 3.24.

Corollary 3.4 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If f is bounded on $\tau_{D(f)}(a)$, then there exists a function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that $D(f, a, \tau, \varepsilon) \neq \emptyset$.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a) \neq \emptyset$, and f be bounded on $\tau_{D(f)}(a)$. Assume that there exist $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$, for all $x \in \tau_{D(f)}(a)$. Then, the proof can be observed from the proofs of Theorems 3.13, 3.14, and 3.22 by a function $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that

$$\varepsilon(a) = \max\left\{\left|m - f(a)\right|, \left|M - f(a)\right|\right\}$$

□

Example 3.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x + 3$, $a = 2$, $\delta(2) = 3$, and $\varepsilon(2) = 4$. Then,*

$$\begin{aligned}
\forall x \in \tau_\delta(2) \cap \mathbb{R},\ \left|f(x) - f(a) - L(x - a)\right| \leq \varepsilon &\Leftrightarrow \forall x \in \tau_\delta(2) \cap \mathbb{R},\ \left|x + 3 - 5 - L(x - 2)\right| \leq 4 \\
&\Leftrightarrow \forall x \in \tau_\delta(2) \cap \mathbb{R},\ -2 - 2L \leq x(1 - L) \leq 6 - 2L \\
&\Leftrightarrow L \in \left[-\tfrac{1}{3}, \tfrac{7}{3}\right]
\end{aligned}$$

Moreover, from Theorem 3.17,

$$D(f, 2, \tau_\delta, \varepsilon) = \left[\sup_{x \in \tau_\delta(2) \cap \mathbb{R}} \left\{1 - \frac{4}{|x - 2|}\right\}, \inf_{x \in \tau_\delta(2) \cap \mathbb{R}} \left\{1 + \frac{4}{|x - 2|}\right\}\right] = \left[-\frac{1}{3}, \frac{7}{3}\right]$$

Example 3.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} \varepsilon(0) - \frac{\varepsilon(0)x}{\delta(0)}, & x > 0 \\ 0, & x = 0 \\ \varepsilon(0) + \frac{2\varepsilon(0)x}{\delta(0)}, & x < 0 \end{cases}$$

such that $\varepsilon(0) > 0$. For $a = 0$,

$$\overline{D}(f, 0, \tau_\delta, \varepsilon) = \left[\sup_{x \in \tau_\delta^+(0) \cap \mathbb{R}} \left\{\frac{f(x)}{x} - \frac{\varepsilon(0)}{|x|}\right\}, \inf_{x \in \tau_\delta^-(0) \cap \mathbb{R}} \left\{\frac{f(x)}{x} + \frac{\varepsilon(0)}{|x|}\right\}\right] = \left[-\frac{\varepsilon(0)}{\delta(0)}, \frac{2\varepsilon(0)}{\delta(0)}\right]$$

$$\underline{D}(f, 0, \tau_\delta, \varepsilon) = \left[\sup_{x \in \tau_\delta^-(0) \cap \mathbb{R}} \left\{\frac{f(x)}{x} - \frac{\varepsilon(0)}{|x|}\right\}, \inf_{x \in \tau_\delta^+(0) \cap \mathbb{R}} \left\{\frac{f(x)}{x} + \frac{\varepsilon(0)}{|x|}\right\}\right] = \left[0, \frac{\varepsilon(0)}{\delta(0)}\right]$$

and

$$D(f, 0, \tau_\delta, \varepsilon) = \left[\sup_{x \in \tau_\delta(0) \cap \mathbb{R}} \left\{\frac{f(x)}{x} - \frac{\varepsilon(0)}{|x|}\right\}, \inf_{x \in \tau_\delta(0) \cap \mathbb{R}} \left\{\frac{f(x)}{x} + \frac{\varepsilon(0)}{|x|}\right\}\right] = \left[0, \frac{\varepsilon(0)}{\delta(0)}\right]$$

Thus, $\overline{D}(f, 0, \tau_\delta, \varepsilon) \cap \underline{D}(f, 0, \tau_\delta, \varepsilon) = \left[-\frac{\varepsilon(0)}{\delta(0)}, \frac{2\varepsilon(0)}{\delta(0)}\right] \cap \left[0, \frac{\varepsilon(0)}{\delta(0)}\right] = \left[0, \frac{\varepsilon(0)}{\delta(0)}\right] = D(f, 0, \tau_\delta, \varepsilon)$.

Theorem 3.25 *[9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\beta(a) \leq \alpha(a)$. If there exists the (τ, β)-soft derivative of f at the point a, then there exists the (τ, α)-soft derivative of f at the point a. Moreover, $D(f, a, \tau, \beta) \subseteq D(f, a, \tau, \alpha)$.*

Theorem 3.26 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\beta(a) \leq \alpha(a)$. If there exists the upper (τ,β)-soft derivative, then there exists the upper (τ,α)-soft derivative of f at the point a. Moreover, $\overline{D}(f,a,\tau,\beta) \subseteq \overline{D}(f,a,\tau,\alpha)$.*

Theorem 3.27 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\beta(a) \leq \alpha(a)$. If there exists the lower (τ,β)-soft derivative, then there exists the lower (τ,α)-soft derivative of f at the point a. Moreover, $\underline{D}(f,a,\tau,\beta) \subseteq \underline{D}(f,a,\tau,\alpha)$.*

Theorem 3.28 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\beta(a) \leq \alpha(a)$. If there exists the left (τ,β)-soft derivative of f at the point a, then there exists the left (τ,α)-soft derivative of f at the point a. Moreover, $D(f,a,\tau^-,\beta) \subseteq D(f,a,\tau^-,\alpha)$.*

Theorem 3.29 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\beta(a) \leq \alpha(a)$. If there exists the right (τ,β)-soft derivative of f at the point a, then there exists the right (τ,α)-soft derivative of f at the point a. Moreover, $D(f,a,\tau^+,\beta) \subseteq D(f,a,\tau^+,\alpha)$.*

Theorem 3.30 *[9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\emptyset \neq \lambda_{D(f)}(a) \subseteq \tau_{D(f)}(a)$. If there exists the (τ,ε)-soft derivative of f at the point a, then there exists the (λ,ε)-soft derivative of f at the point a. Moreover, $D(f,a,\tau,\varepsilon) \subseteq D(f,a,\lambda,\varepsilon)$.*

Theorem 3.31 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\emptyset \neq \lambda^-_{D(f)}(a) \subseteq \tau^-_{D(f)}(a)$, and $\emptyset \neq \lambda^+_{D(f)}(a) \subseteq \tau^+_{D(f)}(a)$. If there exists the upper (τ,ε)-soft derivative, then there exists the upper (λ,ε)-soft derivative of f at the point a. Moreover, $\overline{D}(f,a,\tau,\varepsilon) \subseteq \overline{D}(f,a,\lambda,\varepsilon)$.*

Theorem 3.32 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\emptyset \neq \lambda^-_{D(f)}(a) \subseteq \tau^-_{D(f)}(a)$, and $\emptyset \neq \lambda^+_{D(f)}(a) \subseteq \tau^+_{D(f)}(a)$. If there exists the lower (τ,ε)-soft derivative, then there exists the lower (λ,ε)-soft derivative of f at the point a. Moreover, $\underline{D}(f,a,\tau,\varepsilon) \subseteq \underline{D}(f,a,\lambda,\varepsilon)$.*

Theorem 3.33 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\emptyset \neq \lambda^-_{D(f)}(a) \subseteq \tau^-_{D(f)}(a)$. If there exists the left (τ,ε)-soft derivative of f at the point a, then there exists the left (λ,ε)-soft derivative of f at the point a. Moreover, $D(f,a,\tau^-,\varepsilon) \subseteq D(f,a,\lambda^-,\varepsilon)$.*

Theorem 3.34 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\emptyset \neq \lambda^+_{D(f)}(a) \subseteq \tau^+_{D(f)}(a)$. If there exists the right (τ,ε)-soft derivative of f at the point a, then there exists the right (λ,ε)-soft derivative of f at the point a. Moreover, $D(f,a,\tau^+,\varepsilon) \subseteq D(f,a,\lambda^+,\varepsilon)$.*

Theorem 3.35 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R} \setminus \{0\}$. Then, there exists the (τ,ε)-soft derivative of f at the point a if and only if there exists the $(\tau,|t|\varepsilon)$-soft derivative of tf at the point a. Moreover, $D(tf,a,\tau,|t|\varepsilon) = tD(f,a,\tau,\varepsilon)$.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R} \setminus \{0\}$.
$(\Rightarrow)$: Let there exist the (τ,ε)-soft derivative of f at the point a. Then, $\tau_{D(f)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow \left|f(x) - f(a) - L(x-a)\right| \leq \varepsilon(a)$$

Since

$$\begin{aligned} x \in \tau_{D(tf)}(a) &\Rightarrow x \in \tau_{D(f)}(a) \\ &\Rightarrow \left|f(x) - f(a) - L(x-a)\right| \leq \varepsilon(a) \\ &\Rightarrow \left|(tf)(x) - (tf)(a) - tL(x-a)\right| \leq |t|\varepsilon(a) \end{aligned}$$

then $tL \in D(tf,a,\tau,|t|\varepsilon)$. Thus, there exists the $(\tau,|t|\varepsilon)$-soft derivative of tf at the point a.
$(\Leftarrow)$: Let there exist the $(|t|\tau,\varepsilon)$-soft derivative of tf at the point a. Then, $\tau_{D(tf)}(a) \neq \emptyset$ and there exists an $L \in \mathbb{R}$ such that

$$x \in \tau_{D(tf)}(a) \Rightarrow \left|(tf)(x) - (tf)(a) - L(x-a)\right| \leq |t|\varepsilon(a)$$

Since

$$\begin{aligned} x \in \tau_{D(f)}(a) &\Rightarrow x \in \tau_{D(tf)}(a) \\ &\Rightarrow \left|(tf)(x) - (tf)(a) - L(x-a)\right| \leq |t|\varepsilon(a) \\ &\Rightarrow \left|f(x) - f(a) - \tfrac{L}{t}(x-a)\right| \leq \varepsilon(a) \end{aligned}$$

then $\frac{L}{t} \in D(f, a, \tau, \varepsilon)$. Thus, there exists the (τ, ε)-soft derivative of f at the point a.
Moreover, for $t > 0$,

$$\begin{aligned} D(tf, a, \tau, |t|\varepsilon) &= \left[\sup_{x\in\tau_{D(tf)}(a)} \left\{\tfrac{(tf)(x)-(tf)(a)}{x-a} - \tfrac{|t|\varepsilon(a)}{|x-a|}\right\}, \inf_{x\in\tau_{D(tf)}(a)} \left\{\tfrac{(tf)(x)-(tf)(a)}{x-a} + \tfrac{|t|\varepsilon(a)}{|x-a|}\right\}\right] \\ &= t\left[\sup_{x\in\tau_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{|x-a|}\right\}, \inf_{x\in\tau_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{|x-a|}\right\}\right] \\ &= tD(f, a, \tau, \varepsilon) \end{aligned}$$

and for $t < 0$,

$$\begin{aligned} D(tf, a, \tau, |t|\varepsilon) &= \left[\sup_{x\in\tau_{D(tf)}(a)} \left\{\tfrac{(tf)(x)-(tf)(a)}{x-a} - \tfrac{|t|\varepsilon(a)}{|x-a|}\right\}, \inf_{x\in\tau_{D(tf)}(a)} \left\{\tfrac{(tf)(x)-(tf)(a)}{x-a} + \tfrac{|t|\varepsilon(a)}{|x-a|}\right\}\right] \\ &= \left[t\inf_{x\in\tau_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{|x-a|}\right\}, t\sup_{x\in\tau_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{|x-a|}\right\}\right] \\ &= t\left[\sup_{x\in\tau_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} - \tfrac{\varepsilon(a)}{|x-a|}\right\}, \inf_{x\in\tau_{D(f)}(a)} \left\{\tfrac{f(x)-f(a)}{x-a} + \tfrac{\varepsilon(a)}{|x-a|}\right\}\right] \\ &= tD(f, a, \tau, \varepsilon) \end{aligned}$$

Consequently, $D(tf, a, \tau, |t|\varepsilon) = tD(f, a, \tau, \varepsilon)$. □

Corollary 3.5 *[6,9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. If there exists the (τ, ε)-soft derivative of f at the point a, then there exists the (τ, ε)-soft derivative of $-f$ at the point a. Moreover, $D(-f, a, \tau, \varepsilon) = -D(f, a, \tau, \varepsilon)$.*

Corollary 3.6 *[6,9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R}^+$. If there exists the (τ, ε)-soft derivative of f at the point a, then there exists the $(\tau, t\varepsilon)$-soft derivative of tf at the point a. Moreover, $D(tf, a, \tau, t\varepsilon) = tD(f, a, \tau, \varepsilon)$.*

Theorem 3.36 *[5] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R}^+$. Then, there exists the upper (τ, ε)-soft derivative of f at the point a if and only if there exists the upper $(\tau, t\varepsilon)$-soft derivative of tf at the point a. Moreover, $\overline{D}(tf, a, \tau, t\varepsilon) = t\overline{D}(f, a, \tau, \varepsilon)$.*

Theorem 3.37 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R}^-$. Then, there exists the lower (τ, ε)-soft derivative of f at the point a if and only if there exists the upper $(\tau, |t|\varepsilon)$-soft derivative of tf at the point a. Moreover, $t\underline{D}(f, a, \tau, \varepsilon) = \overline{D}(tf, a, \tau, |t|\varepsilon)$.*

Theorem 3.38 *[5] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R}^+$. Then, there exists the lower (τ, ε)-soft derivative of f at the point a if and only if there exists the lower $(\tau, t\varepsilon)$-soft derivative of tf at the point a. Moreover, $\underline{D}(tf, a, \tau, t\varepsilon) = t\underline{D}(f, a, \tau, \varepsilon)$.*

Theorem 3.39 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R}^-$. Then, there exists the lower (τ, ε)-soft derivative of f at the point a if and only if there exists the upper $(\tau, |t|\varepsilon)$-soft derivative of tf at the point a. Moreover, $t\overline{D}(f, a, \tau, \varepsilon) = \underline{D}(tf, a, \tau, |t|\varepsilon)$.*

Theorem 3.40 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R} \setminus \{0\}$. Then, there exists the left (τ, ε)-soft derivative of f at the point a if and only if there exists the left $(\tau, |t|\varepsilon)$-soft derivative of tf at the point a. Moreover, $D(tf, a, \tau^-, |t|\varepsilon) = tD(f, a, \tau^-, \varepsilon)$.*

Theorem 3.41 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $t \in \mathbb{R} \setminus \{0\}$. Then, there exists the right (τ, ε)-soft derivative of f at the point a if and only if there exists the right $(\tau, |t|\varepsilon)$-soft derivative of tf at the point a. Moreover, $D(tf, a, \tau^+, |t|\varepsilon) = tD(f, a, \tau^+, \varepsilon)$.*

The proofs of Theorems 3.36-3.41 are similar to the proof of Theorem 3.35.

Example 3.5 *Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be three functions defined by $f(x) = x + 2$, $g(x) = 3x + 6$, and $h(x) = -2x - 4$. Then, for $a = 1$, $\delta_1(1) = 2$, $\delta_2(1) = 4$, $\varepsilon_1(1) = 3$, and $\varepsilon_2(1) = 4$,*

$$D(g, 1, \tau_{\delta_1}, 3\varepsilon_1) = \left[-\frac{3}{2}, \frac{15}{2}\right] = 3\left[-\frac{1}{2}, \frac{5}{2}\right] = 3D(f, 1, \tau_{\delta_1}, \varepsilon_1)$$

$$D(h, 1, \tau_{\delta_1}, 2\varepsilon_1) = [-5, 1] = -2\left[-\frac{1}{2}, \frac{5}{2}\right] = -2D(f, 1, \tau_{\delta_1}, \varepsilon_1)$$

$$D(f, 1, \tau_{\delta_1}, \varepsilon_1) \subseteq D(f, 1, \tau_{\delta_1}, \varepsilon_2)$$

and

$$D(f, 1, \tau_{\delta_2}, \varepsilon_1) \subseteq D(f, 1, \tau_{\delta_1}, \varepsilon_1)$$

Here, $D(f, 1, \tau_{\delta_2}, \varepsilon_1) = \left[\frac{1}{4}, \frac{7}{4}\right]$ and $D(f, 1, \tau_{\delta_1}, \varepsilon_2) = [-1, 3]$.

Theorem 3.42 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, and $a \in A$. If there exist the (τ, α)-soft derivative of f and the (λ, β)-soft derivative of g at the point a, then there exists the (κ, ε)-soft derivative of $f + g$ at the point a such that $\emptyset \neq \kappa_{D(f+g)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(g)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover,*

$$D(f, a, \tau, \alpha) + D(g, a, \lambda, \beta) \subseteq D(f + g, a, \kappa, \varepsilon)$$

Proof: Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and there exist the (τ, α)-soft derivative of f and the (λ, β)-soft derivative of g at the point a. Then, there exist $L_1, L_2 \in \mathbb{R}$ such that

$$x \in \tau_{D(f)}(a) \Rightarrow \left|f(x) - f(a) - L_1(x - a)\right| \leq \alpha(a)$$

and

$$x \in \lambda_{D(g)}(a) \Rightarrow \left|g(x) - g(a) - L_2(x - a)\right| \leq \beta(a)$$

Therefore, since

$$\left|(f + g)(x) - (f + g)(a) - (L_1 + L_2)(x - a)\right| \leq \left|f(x) - f(a) - L_1(x - a)\right| + \left|g(x) - g(a) - L_2(x - a)\right|$$

then

$$\begin{aligned} x \in \kappa_{D(f+g)}(a) \;&\Rightarrow\; x \in \tau_{D(f)}(a) \cap \lambda_{D(g)}(a) \\ &\Rightarrow\; \left|(f + g)(x) - (f + g)(a) - (L_1 + L_2)(x - a)\right| \leq \alpha(a) + \beta(a) \leq \varepsilon(a) \end{aligned}$$

Thus, $L_1 + L_2 \in D(f + g, a, \kappa,\ \varepsilon)$. Hence, there exists the (κ, ε)-soft derivative of $f + g$ at the point a. Moreover, for $S = D(f, a, \tau, \alpha) + D(g, a, \lambda, \beta)$, from Theorem 3.17 and properties of infimum and supremum, since

$$\frac{(f + g)(x) - (f + g)(a)}{x - a} - \frac{\varepsilon(a)}{|x - a|} \leq \frac{(f + g)(x) - (f + g)(a)}{x - a} - \frac{(\alpha + \beta)(a)}{|x - a|}, \quad \text{for all } x \in \kappa_{D(f+g)}(a)$$

and

$$\frac{(f + g)(x) - (f + g)(a)}{x - a} + \frac{(\alpha + \beta)(a)}{|x - a|} \leq \frac{(f + g)(x) - (f + g)(a)}{x - a} + \frac{\varepsilon(a)}{|x - a|}, \quad \text{for all } x \in \kappa_{D(f+g)}(a)$$

then

$$\begin{aligned}
S &= \left[\sup_{x\in\tau_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}-\frac{\alpha(a)}{|x-a|}\right\}, \inf_{x\in\tau_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}+\frac{\alpha(a)}{|x-a|}\right\}\right]+\left[\sup_{x\in\lambda_{D(g)}(a)}\left\{\frac{g(x)-g(a)}{x-a}-\frac{\beta(a)}{|x-a|}\right\}, \inf_{x\in\lambda_{D(g)}(a)}\left\{\frac{g(x)-g(a)}{x-a}+\frac{\beta(a)}{|x-a|}\right\}\right]\\
&= \left[\sup_{x\in\tau_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}-\frac{\alpha(a)}{|x-a|}\right\}+\sup_{x\in\lambda_{D(g)}(a)}\left\{\frac{g(x)-g(a)}{x-a}-\frac{\beta(a)}{|x-a|}\right\}, \inf_{x\in\tau_{D(f)}(a)}\left\{\frac{f(x)-f(a)}{x-a}+\frac{\alpha(a)}{|x-a|}\right\}+\inf_{x\in\lambda_{D(g)}(a)}\left\{\frac{g(x)-g(a)}{x-a}+\frac{\beta(a)}{|x-a|}\right\}\right]\\
&\subseteq \left[\sup_{x\in\kappa_{D(f+g)}(a)}\left\{\frac{(f+g)(x)-(f+g)(a)}{x-a}-\frac{(\alpha+\beta)(a)}{|x-a|}\right\}, \inf_{x\in\kappa_{D(f+g)}(a)}\left\{\frac{(f+g)(x)-(f+g)(a)}{x-a}+\frac{(\alpha+\beta)(a)}{|x-a|}\right\}\right]\\
&\subseteq \left[\sup_{x\in\kappa_{D(f+g)}(a)}\left\{\frac{(f+g)(x)-(f+g)(a)}{x-a}-\frac{\varepsilon(a)}{|x-a|}\right\}, \inf_{x\in\kappa_{D(f+g)}(a)}\left\{\frac{(f+g)(x)-(f+g)(a)}{x-a}+\frac{\varepsilon(a)}{|x-a|}\right\}\right]\\
&= D(f+g,a,\kappa,\varepsilon)
\end{aligned}$$

□

Corollary 3.7 *[6,9,26] Let $A\subseteq\mathbb{R}$, $f,g:A\to\mathbb{R}$ be two functions, and $a\in A$. If there exist the (τ,α)-soft derivative of f and the (τ,β)-soft derivative of g at the point a, then there exists the $(\tau,\alpha+\beta)$-soft derivative of $f+g$ at the point a. Moreover, $D(f,a,\tau,\alpha)+D(g,a,\tau,\beta)\subseteq D(f+g,a,\tau,\alpha+\beta)$.*

Theorem 3.43 *Let $A\subseteq\mathbb{R}$, $f,g:A\to\mathbb{R}$ be two functions, and $a\in A$. If there exist the upper (τ,α)-soft derivative of f and the upper (λ,β)-soft derivative of g at the point a, then there exists the upper (κ,ε)-soft derivative of $f+g$ at the point a such that $\emptyset\neq\kappa_{D(f+g)}(a)\subseteq\tau_{D(f)}(a)\cap\lambda_{D(g)}(a)$ and $\alpha(a)+\beta(a)\leq\varepsilon(a)$. Moreover, $\overline{D}(f,a,\tau,\alpha)+\overline{D}(g,a,\lambda,\beta)\subseteq\overline{D}(f+g,a,\kappa,\varepsilon)$.*

Theorem 3.44 *Let $A\subseteq\mathbb{R}$, $f,g:A\to\mathbb{R}$ be two functions, and $a\in A$. If there exist the lower (τ,α)-soft derivative of f and the lower (λ,β)-soft derivative of g at the point a, then there exists the lower (κ,ε)-soft derivative of $f+g$ at the point a such that $\emptyset\neq\kappa_{D(f+g)}(a)\subseteq\tau_{D(f)}(a)\cap\lambda_{D(g)}(a)$ and $\alpha(a)+\beta(a)\leq\varepsilon(a)$. Moreover, $\underline{D}(f,a,\tau,\alpha)+\underline{D}(g,a,\lambda,\beta)\subseteq\underline{D}(f+g,a,\kappa,\varepsilon)$.*

Theorem 3.45 *Let $A\subseteq\mathbb{R}$, $f,g:A\to\mathbb{R}$ be two functions, and $a\in A$. If there exist the left (τ,α)-soft derivative of f and the left (λ,β)-soft derivative of g at the point a, then there exists the left (κ,ε)-soft derivative of $f+g$ at the point a such that $\emptyset\neq\kappa^-_{D(f+g)}(a)\subseteq\tau^-_{D(f)}(a)\cap\lambda^-_{D(g)}(a)$ and $\alpha(a)+\beta(a)\leq\varepsilon(a)$. Moreover, $D(f,a,\tau^-,\alpha)+D(g,a,\lambda^-,\beta)\subseteq D(f+g,a,\kappa^-,\varepsilon)$.*

Theorem 3.46 *Let $A\subseteq\mathbb{R}$, $f,g:A\to\mathbb{R}$ be two functions, and $a\in A$. If there exist the right (τ,α)-soft derivative of f and the right (λ,β)-soft derivative of g at the point a, then there exists the right (κ,ε)-soft derivative of $f+g$ at the point a such that $\emptyset\neq\kappa^+_{D(f+g)}(a)\subseteq\tau^+_{D(f)}(a)\cap\lambda^+_{D(g)}(a)$ and $\alpha(a)+\beta(a)\leq\varepsilon(a)$. Moreover, $D(f,a,\tau^+,\alpha)+D(g,a,\lambda^+,\beta)\subseteq D(f+g,a,\kappa^+,\varepsilon)$.*

The proofs of Theorems 3.43-3.46 are similar to the proof of Theorem 3.42.

Corollary 3.8 *Let $A\subseteq\mathbb{R}$, $f,g:A\to\mathbb{R}$ be two functions, and $a\in A$. If there exist the (τ,α)-soft derivative of f and the (λ,β)-soft derivative of g at the point a, then there exists the (κ,ε)-soft derivative of $f-g$ at the point a such that $\emptyset\neq\kappa_{D(f-g)}(a)\subseteq\tau_{D(f)}(a)\cap\lambda_{D(g)}(a)$ and $\alpha(a)+\beta(a)\leq\varepsilon(a)$. Moreover,*

$$D(f,a,\tau,\alpha)-D(g,a,\lambda,\beta)\subseteq D(f-g,a,\kappa,\varepsilon)$$

Proof: Let $A\subseteq\mathbb{R}$, $f,g:A\to\mathbb{R}$ be two functions, $a\in A$, and there exist the (τ,α)-soft derivative of f and the (λ,β)-soft derivative of g at the point a. Since there exists the (λ,β)-soft derivative of g at the point a, then there exists the (λ,β)-soft derivative of $-g$ at the point a and $-D(g,a,\lambda,\beta)=D(-g,a,\lambda,\beta)$ from Corollary 3.5. Therefore, from Theorem 3.42, there exists the (κ,ε)-soft derivative of $f-g=f+(-g)$ at the point a and

$$D(f,a,\tau,\alpha)-D(g,a,\lambda,\beta)=D(f,a,\tau,\alpha)+D(-g,a,\lambda,\beta)\subseteq D(f+(-g),a,\kappa,\varepsilon)=D(f-g,a,\kappa,\varepsilon)$$

□

Corollary 3.9 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, and $a \in A$. If there exist the upper (τ, α)-soft derivative of f and the upper (λ, β)-soft derivative of g at the point a, then there exists the upper (κ, ε)-soft derivative of $f - g$ at the point a such that $\emptyset \neq \kappa_{D(f-g)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(g)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\overline{D}(f, a, \tau, \alpha) - \overline{D}(g, a, \lambda, \beta) \subseteq \overline{D}(f - g, a, \kappa, \varepsilon)$.*

Corollary 3.10 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, and $a \in A$. If there exist the lower (τ, α)-soft derivative of f and the lower (λ, β)-soft derivative of g at the point a, then there exists the lower (κ, ε)-soft derivative of $f - g$ at the point a such that $\emptyset \neq \kappa_{D(f-g)}(a) \subseteq \tau_{D(f)}(a) \cap \lambda_{D(g)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $\underline{D}(f, a, \tau, \alpha) - \underline{D}(g, a, \lambda, \beta) \subseteq \underline{D}(f - g, a, \kappa, \varepsilon)$.*

Corollary 3.11 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, and $a \in A$. If there exist the left (τ, α)-soft derivative of f and the left (λ, β)-soft derivative of g at the point a, then there exists the left (κ, ε)-soft derivative of $f - g$ at the point a such that $\emptyset \neq \kappa^-_{D(f-g)}(a) \subseteq \tau^-_{D(f)}(a) \cap \lambda^-_{D(g)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $D(f, a, \tau^-, \alpha) - D(g, a, \lambda^-, \beta) \subseteq D(f - g, a, \kappa^-, \varepsilon)$.*

Corollary 3.12 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, and $a \in A$. If there exist the right (τ, α)-soft derivative of f and the right (λ, β)-soft derivative of g at the point a, then there exists the right (κ, ε)-soft derivative of $f - g$ at the point a such that $\emptyset \neq \kappa^+_{D(f-g)}(a) \subseteq \tau^+_{D(f)}(a) \cap \lambda^+_{D(g)}(a)$ and $\alpha(a) + \beta(a) \leq \varepsilon(a)$. Moreover, $D(f, a, \tau^+, \alpha) - D(g, a, \lambda^+, \beta) \subseteq D(f - g, a, \kappa^+, \varepsilon)$.*

The proofs of Corollaries 3.9-3.12 are similar to the proof of Corollary 3.8.

Example 3.6 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be the absolute value function and signum function defined by $f(x) = |x|$ and $g(x) = \operatorname{sgn}(x)$, respectively, $\delta(0) \leq 2$, $\varepsilon_1(0) = 2$, and $\varepsilon_2(0) = 3$. Therefore, $(f + g)(x) = |x| + \operatorname{sgn}(x)$ and $(f - g)(x) = |x| - \operatorname{sgn}(x)$, for all $x \in \mathbb{R}$. Then, for $a = 0$,*

$$D(f, 0, \tau_\delta, \varepsilon_1) + D(g, 0, \tau_\delta, \varepsilon_2) = \left[1 - \tfrac{2}{\delta(0)}, -1 + \tfrac{2}{\delta(0)}\right] + \left[-\tfrac{2}{\delta(0)}, \tfrac{4}{\delta(0)}\right] \subseteq \left[1 - \tfrac{4}{\delta(0)}, -1 + \tfrac{6}{\delta(0)}\right] = D(f + g, 0, \tau_\delta, \varepsilon_1 + \varepsilon_2)$$

and

$$D(f, 0, \tau_\delta, \varepsilon_1) - D(g, 0, \tau_\delta, \varepsilon_2) = \left[1 - \tfrac{2}{\delta(0)}, -1 + \tfrac{2}{\delta(0)}\right] - \left[-\tfrac{2}{\delta(0)}, \tfrac{4}{\delta(0)}\right] \subseteq \left[1 - \tfrac{6}{\delta(0)}, -1 + \tfrac{4}{\delta(0)}\right] = D(f - g, 0, \tau_\delta, \varepsilon_1 + \varepsilon_2)$$

Example 3.7 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by $f(x) = x^2$ and $g(x) = x^2 + 1$, respectively, $a = 1$, $\alpha(1) = 2$, and $\beta(1) = 4$. Then, for $\tau_{\delta_1}(1) = [0, 2] \setminus \{1\}$ and $\lambda_{\delta_2}(1) = [-1, 3] \setminus \{1\}$ such that $\delta_1(1) = 1$ and $\delta_2(1) = 2$, $D(f, 1, \tau_{\delta_1}, \alpha) = [1, 3]$ and $D(g, 1, \lambda_{\delta_2}, \beta) = [2, 2]$. Moreover, for $(f + g)(x) = 2x^2 + 1$, $(f - g)(x) = -1$, $\kappa_{\delta_3}(1) = \left[\frac{1}{2}, \frac{3}{2}\right]$ such that $\delta_3(1) = \frac{1}{2}$, and $\varepsilon(1) = 6$, $D(f + g, 1, \kappa_{\delta_3}, \varepsilon) = [-7, 15]$ and $D(f - g, 1, \kappa_{\delta_3}, \varepsilon) = [-12, 12]$. Therefore,*

$$D(f, 1, \tau_{\delta_1}, \alpha) + D(g, 1, \lambda_{\delta_2}, \beta) = [3, 5] \subseteq D(f + g, 1, \kappa_{\delta_3}, \varepsilon)$$

and

$$D(f, 1, \tau_{\delta_1}, \alpha) - D(g, 1, \lambda_{\delta_2}, \beta) = [-1, 1] \subseteq D(f - g, 1, \kappa_{\delta_3}, \varepsilon)$$

Theorem 3.47 *[9,26] Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and $t \in \mathbb{R}$. If $g(x) = f(x) + tx$, for all $x \in \tau_{D(f)}(a)$, and there exists the (τ, ε)-soft derivative of f at the point a, then there exists the (τ, ε)-soft derivative of g at the point a. Moreover, $D(g, a, \tau, \varepsilon) = D(f, a, \tau, \varepsilon) + t$.*

Theorem 3.48 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and $t \in \mathbb{R}$. If $g(x) = f(x) + tx$, for all $x \in \tau_{D(f)}(a)$, and there exists the upper (τ, ε)-soft derivative of f at the point a, then there exists the upper (τ, ε)-soft derivative of g at the point a. Moreover, $\overline{D}(g, a, \tau, \varepsilon) = \overline{D}(f, a, \tau, \varepsilon) + t$.*

Theorem 3.49 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and $t \in \mathbb{R}$. If $g(x) = f(x) + tx$, for all $x \in \tau_{D(f)}(a)$, and there exists the lower (τ, ε)-soft derivative of f at the point a, then there exists the lower (τ, ε)-soft derivative of g at the point a. Moreover, $\underline{D}(g, a, \tau, \varepsilon) = \underline{D}(f, a, \tau, \varepsilon) + t$.*

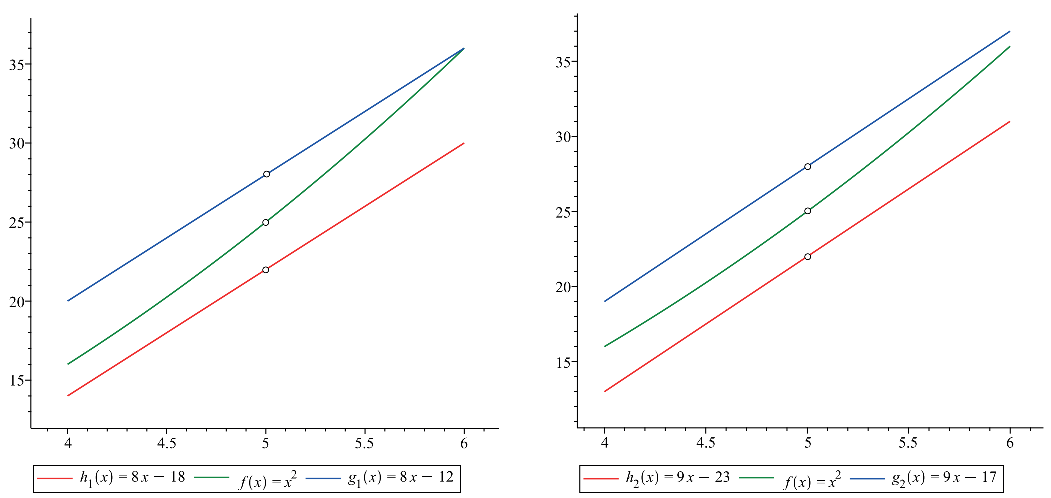
Theorem 3.50 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and $t \in \mathbb{R}$. If $g(x) = f(x) + tx$, for all $x \in \tau^-_{D(f)}(a)$, and there exists the left (τ, ε)-soft derivative of f at the point a, then there exists the left (τ, ε)-soft derivative of g at the point a. Moreover, $D(g, a, \tau^-, \varepsilon) = D(f, a, \tau^-, \varepsilon) + t$.*

Theorem 3.51 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and $t \in \mathbb{R}$. If $g(x) = f(x) + tx$, for all $x \in \tau^+_{D(f)}(a)$, and there exists the right (τ, ε)-soft derivative of f at the point a, then there exists the right (τ, ε)-soft derivative of g at the point a. Moreover, $D(g, a, \tau^+, \varepsilon) = D(f, a, \tau^+, \varepsilon) + t$.*

Remark 3.1 *The geometric interpretation of the soft derivative of a function f at a point a is the tangent of the slope angle of the bandwidth $2\varepsilon(a)$ bounded by two linear functions $f(a)+L(x-a)+\varepsilon(a)$ and $f(a)+L(x-a)-\varepsilon(a)$ containing the entire graph of f on the set $\tau_{D(f)}(a)$. For example, let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x^2$, $a = 5$, $\delta(5) = 1$, and $\varepsilon(5) = 3$. Then, $D(f, 5, \tau_\delta, \varepsilon) = [8, 12]$. Consider the following linear functions and ordered pairs:*

for $L = 8 \in [8, 12]$,	$h_1(x) = f(a) + L(x-a) - \varepsilon(a) = 8x - 18$	$A_1 = (x, h_1(x))$
	$g_1(x) = f(a) + L(x-a) + \varepsilon(a) = 8x - 12$	$B_1 = (x, g_1(x))$
for $L = 9 \in [8, 12]$,	$h_2(x) = f(a) + L(x-a) - \varepsilon(a) = 9x - 23$	$A_2 = (x, h_2(x))$
	$g_2(x) = f(a) + L(x-a) + \varepsilon(a) = 9x - 17$	$B_2 = (x, g_2(x))$
for $L = 10 \in [8, 12]$,	$h_3(x) = f(a) + L(x-a) - \varepsilon(a) = 10x - 28$	$A_3 = (x, h_3(x))$
	$g_3(x) = f(a) + L(x-a) + \varepsilon(a) = 10x - 22$	$B_3 = (x, g_3(x))$
for $L = 11 \in [8, 12]$,	$h_4(x) = f(a) + L(x-a) - \varepsilon(a) = 11x - 33$	$A_4 = (x, h_4(x))$
	$g_4(x) = f(a) + L(x-a) + \varepsilon(a) = 11x - 27$	$B_4 = (x, g_4(x))$
for $L = 12 \in [8, 12]$,	$h_5(x) = f(a) + L(x-a) - \varepsilon(a) = 12x - 38$	$A_5 = (x, h_5(x))$
	$g_5(x) = f(a) + L(x-a) + \varepsilon(a) = 12x - 32$	$B_5 = (x, g_5(x))$

Thus, it can be observed that for all $i \in I_5 = \{1, 2, 3, 4, 5\}$ and for all $x \in \tau_\delta(5) \cap \mathbb{R}$, $h_i(x) \leq f(x) \leq g_i(x)$ and the Euclidean distance of the ordered pairs $A_i = (x, g_i(x))$ and $B_i = (x, h_i(x))$ is $2\varepsilon(a)$, where $|A_iB_i| = \sqrt{(x-x)^2 + (g_i(x) - h_i(x))^2} = 6 = 2\varepsilon(a)$. Figures 1-3 demonstrate the graphs of the functions h_i, f, and g_i, for all $i \in I_5$, on the set $\tau_\delta(5) \cap \mathbb{R}$, separately and together, respectively. Besides, for all $L \in D(f, 5, \tau_\delta, \varepsilon) = [8, 12]$, the pairs of all linear functions h and g form two bundles of lines (see Figure 4).



(a) (b)

Figure 1: Graphs of (a) h_1, f, and g_1 and (b) h_2, f, and g_2 on the set $\tau_\delta(5) \cap \mathbb{R}$.

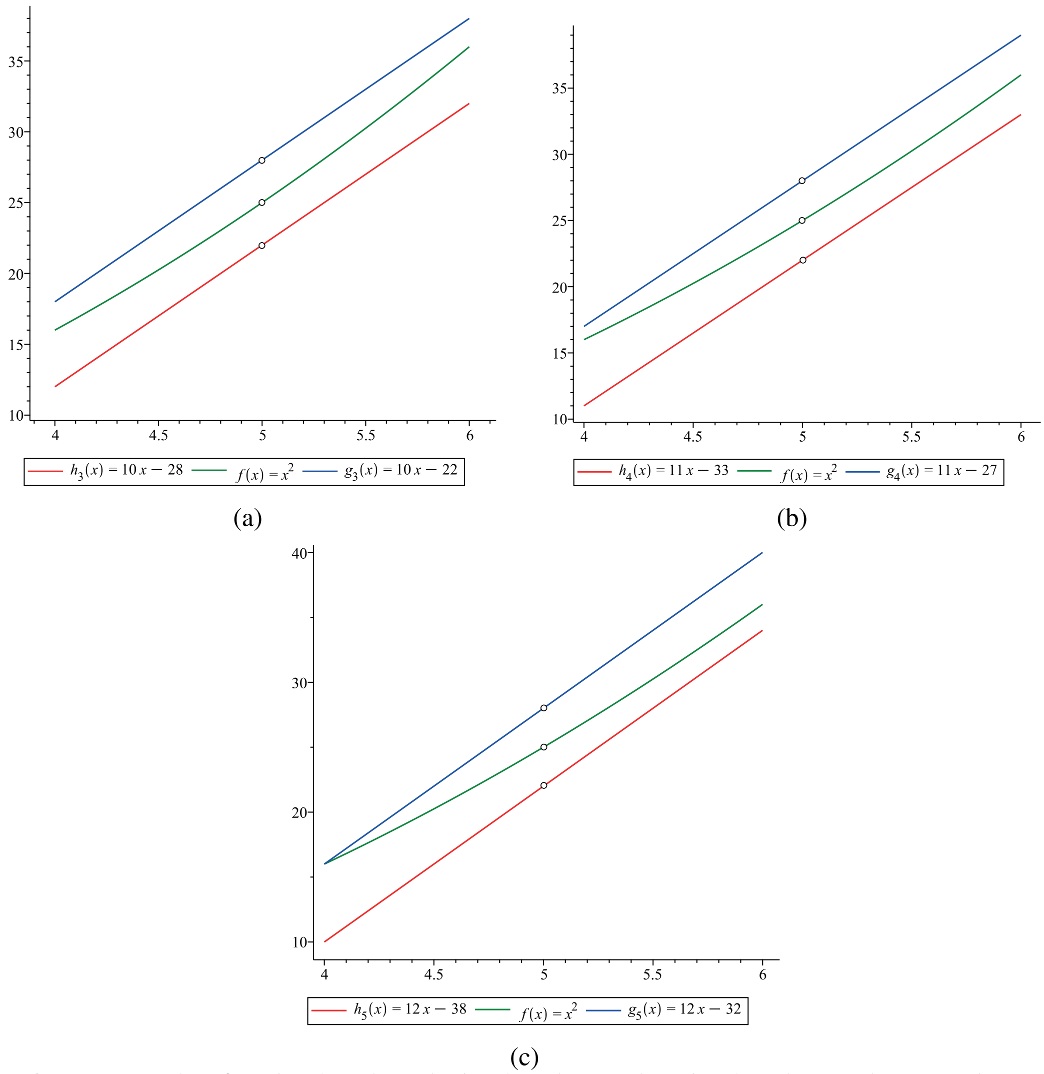


Figure 2: Graphs of (a) h_3, f, and g_3, (b) h_4, f, and g_4, and (c) h_5, f, and g_5 on the set $\tau_\delta(5) \cap \mathbb{R}$.

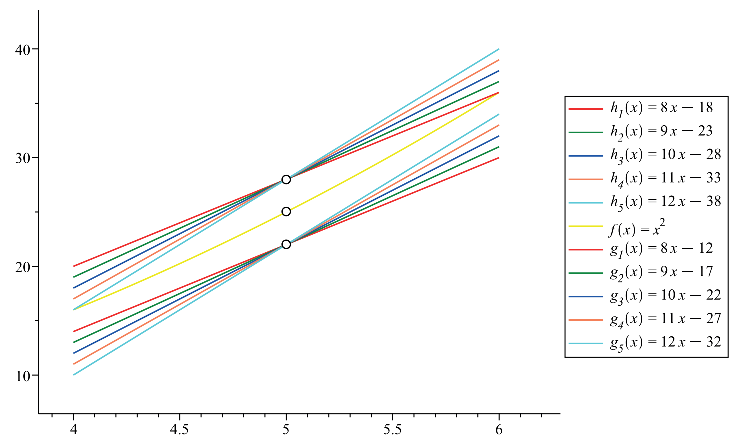


Figure 3: Graphs of the functions h_i, f, and g_i, for all $i \in I_5$, on the set $\tau_\delta(5) \cap \mathbb{R}$.

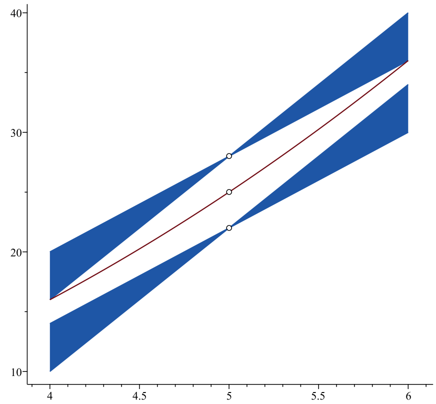


Figure 4: Bundles of lines formed by the pairs of all linear functions, for all $L \in [8, 12]$.

Further, $D\left(f, 5, \tau_\delta^-, \varepsilon\right) = [6, 12]$. *Consider the following linear functions and ordered pairs:*

for $L = 6 \in [6, 12]$,	$m_1(x) = f(a) + L(x-a) - \varepsilon(a) = 6x - 8$ $n_1(x) = f(a) + L(x-a) + \varepsilon(a) = 6x - 2$	$C_1 = (x, m_1(x))$ $D_1 = (x, n_1(x))$
for $L = 7 \in [6, 12]$,	$m_2(x) = f(a) + L(x-a) - \varepsilon(a) = 7x - 13$ $n_2(x) = f(a) + L(x-a) + \varepsilon(a) = 7x - 7$	$C_2 = (x, m_2(x))$ $D_2 = (x, n_2(x))$
for $L = 8 \in [6, 12]$,	$m_3(x) = f(a) + L(x-a) - \varepsilon(a) = 8x - 18$ $n_3(x) = f(a) + L(x-a) + \varepsilon(a) = 8x - 12$	$C_3 = (x, h_1(x))$ $D_3 = (x, g_1(x))$
for $L = 9 \in [6, 12]$,	$m_4(x) = f(a) + L(x-a) - \varepsilon(a) = 9x - 23$ $n_4(x) = f(a) + L(x-a) + \varepsilon(a) = 9x - 17$	$C_4 = (x, h_2(x))$ $D_4 = (x, g_2(x))$
for $L = 10 \in [6, 12]$,	$m_5(x) = f(a) + L(x-a) - \varepsilon(a) = 10x - 28$ $n_5(x) = f(a) + L(x-a) + \varepsilon(a) = 10x - 22$	$C_5 = (x, h_3(x))$ $D_5 = (x, g_3(x))$
for $L = 11 \in [6, 12]$,	$m_6(x) = f(a) + L(x-a) - \varepsilon(a) = 11x - 33$ $n_6(x) = f(a) + L(x-a) + \varepsilon(a) = 11x - 27$	$C_6 = (x, h_4(x))$ $D_6 = (x, g_4(x))$
for $L = 12 \in [6, 12]$,	$m_7(x) = f(a) + L(x-a) - \varepsilon(a) = 12x - 38$ $n_7(x) = f(a) + L(x-a) + \varepsilon(a) = 12x - 32$	$C_7 = (x, h_5(x))$ $D_7 = (x, g_5(x))$

Hence, it can be observed that for all $i \in I_7 = \{1, 2, 3, 4, 5, 6, 7\}$ *and for all* $x \in \tau_\delta^-(5) \cap \mathbb{R}$, $m_i(x) \leq f(x) \leq n_i(x)$ *and the Euclidean distance of the ordered pairs* $C_i = (x, n_i(x))$ *and* $D_i = (x, m_i(x))$ *is* $2\varepsilon(a)$*, where* $|C_iD_i| = \sqrt{(x-x)^2 + (n_i(x) - m_i(x))^2} = 6 = 2\varepsilon(a)$. *Figure 5 indicates the graphs of* m_i*,* f*, and* n_i *on the set* $\tau_\delta^-(5) \cap \mathbb{R}$*, for all* $i \in I_7$.

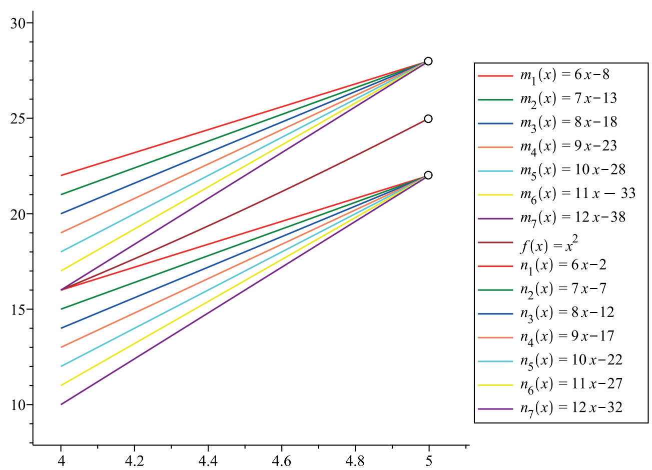


Figure 5: Graphs of the functions m_i, f, and n_i, for all $i \in I_7$, on the set $\tau_\delta^-(5) \cap \mathbb{R}$.

In addition, for all $L \in D\left(f, 5, \tau_\delta^-, \varepsilon\right) = [6, 12]$*, the pairs of all linear functions m and n form two bundles of lines (see Figure 6).*

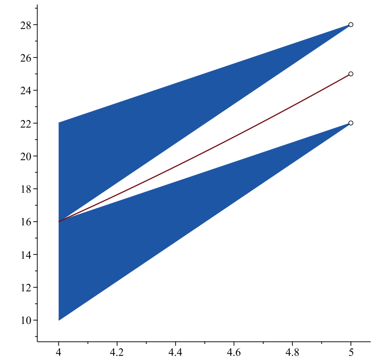


Figure 6: Bundles of lines formed by the pairs of all linear functions, for all $L \in [6, 12]$.

Furthermore, $D\left(f, 5, \tau_\delta^+, \varepsilon\right) = [8, 14]$*. Consider the following linear functions and ordered pairs:*

for $L = 8 \in [8, 14]$,	$p_1(x) = f(a) + L(x-a) - \varepsilon(a) = 8x - 18$ $r_1(x) = f(a) + L(x-a) + \varepsilon(a) = 8x - 12$	$E_1 = (x, h_1(x))$ $F_1 = (x, g_1(x))$
for $L = 9 \in [8, 14]$,	$p_2(x) = f(a) + L(x-a) - \varepsilon(a) = 9x - 23$ $r_2(x) = f(a) + L(x-a) + \varepsilon(a) = 9x - 17$	$E_2 = (x, h_2(x))$ $F_2 = (x, g_2(x))$
for $L = 10 \in [8, 14]$,	$p_3(x) = f(a) + L(x-a) - \varepsilon(a) = 10x - 28$ $r_3(x) = f(a) + L(x-a) + \varepsilon(a) = 10x - 22$	$E_3 = (x, h_3(x))$ $F_3 = (x, g_3(x))$
for $L = 11 \in [8, 14]$,	$p_4(x) = f(a) + L(x-a) - \varepsilon(a) = 11x - 33$ $r_4(x) = f(a) + L(x-a) + \varepsilon(a) = 11x - 27$	$E_4 = (x, h_4(x))$ $F_4 = (x, g_4(x))$
for $L = 12 \in [8, 14]$,	$p_5(x) = f(a) + L(x-a) - \varepsilon(a) = 12x - 38$ $r_5(x) = f(a) + L(x-a) + \varepsilon(a) = 12x - 32$	$E_5 = (x, h_5(x))$ $F_5 = (x, g_5(x))$
for $L = 13 \in [8, 14]$,	$p_6(x) = f(a) + L(x-a) - \varepsilon(a) = 13x - 43$ $r_6(x) = f(a) + L(x-a) + \varepsilon(a) = 13x - 37$	$E_6 = (x, h_5(x))$ $F_6 = (x, g_5(x))$
for $L = 14 \in [8, 14]$,	$p_7(x) = f(a) + L(x-a) - \varepsilon(a) = 14x - 48$ $r_7(x) = f(a) + L(x-a) + \varepsilon(a) = 14x - 42$	$E_7 = (x, h_5(x))$ $F_7 = (x, g_5(x))$

Thereby, it can be observed that for all $i \in I_7$ *and for all* $x \in \tau_\delta^+(5) \cap \mathbb{R}$, $p_i(x) \leq f(x) \leq r_i(x)$ *and the Euclidean distance of the ordered pairs* $E_i = (x, r_i(x))$ *and* $F_i = (x, p_i(x))$ *is* $2\varepsilon(a)$*, where* $|E_iF_i| = \sqrt{(x-x)^2 + (r_i(x) - p_i(x))^2} = 6 = 2\varepsilon(a)$*. Figure 7 shows the graphs of* p_i*,* f*, and* r_i *on the set* $\tau_\delta^+(5) \cap \mathbb{R}$*, for all* $i \in I_7$*. Additionally, for all* $L \in D\left(f, 5, \tau_\delta^+, \varepsilon\right) = [8, 14]$*, the pairs of all linear functions p and r form two bundles of lines (see Figure 8).*

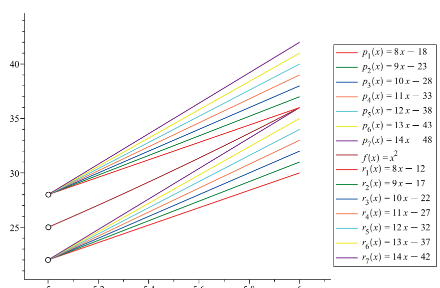


Figure 7: Graphs of the functions p_i, f, and r_i, for all $i \in I_7$, on the set $\tau_\delta^+(5) \cap \mathbb{R}$.

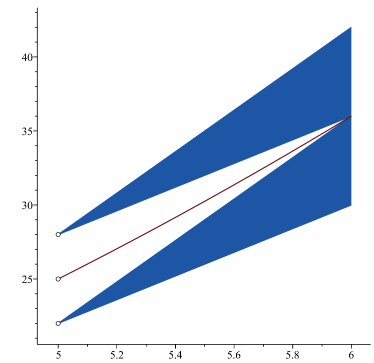
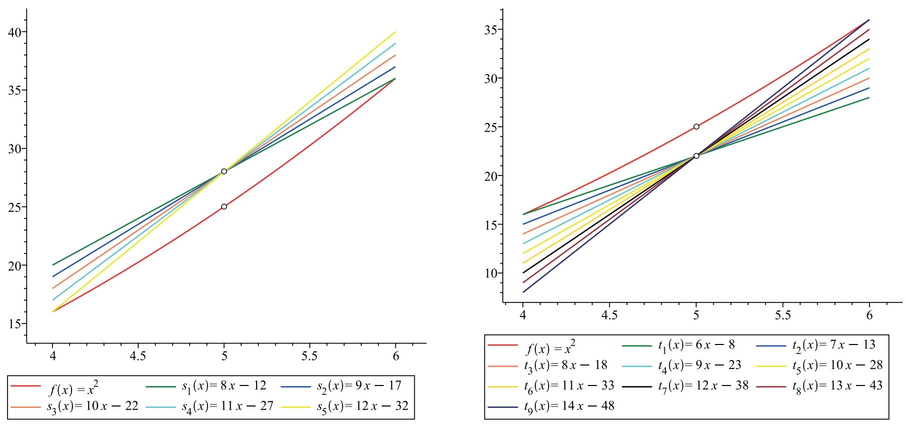


Figure 8: Bundles of lines formed by the pairs of all linear functions, for all $L \in [8, 14]$.

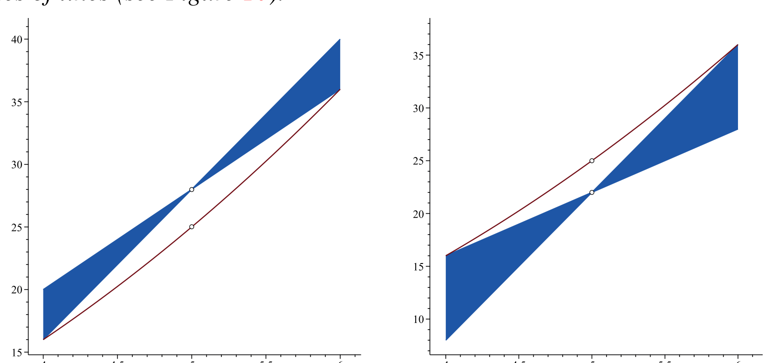
Moreover, $\overline{D}(f, 5, \tau_\delta, \varepsilon) = [8, 12]$ *and* $\underline{D}(f, 5, \tau_\delta, \varepsilon) = [6, 14]$*. For* $8, 9, 10, 11, 12 \in [8, 12]$*, let* $s_1(x) = 8x - 12$*,* $s_2(x) = 9x - 17$*,* $s_3(x) = 10x - 22$*,* $s_4(x) = 11x - 27$*, and* $s_5(x) = 12x - 32$*, and for* $6, 7, 8, 9, 10, 11, 12, 13, 14 \in [6, 14]$*, let* $t_1(x) = 6x - 8$*,* $t_2(x) = 7x - 13$*,* $t_3(x) = 8x - 18$*,* $t_4(x) = 9x - 23$*,* $t_5(x) = 10x - 28$*,* $t_6(x) = 11x - 33$*,* $t_7(x) = 12x - 38$*,* $t_8(x) = 13x - 43$*, and* $t_9(x) = 14x - 48$*. Figure 9 demonstrates the graphs of* s_i*,* f*, and* t_j *on the set* $\tau_\delta(5) \cap \mathbb{R}$*, for all* $i \in I_5$ *and* $j \in I_9 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$*.*



(a) (b)

Figure 9: Graphs of (a) s_i and f, for all $i \in I_5$, and (b) f and t_j, for all $j \in I_9$, on the set $\tau_\delta(5) \cap \mathbb{R}$.

Besides, for all $L \in \overline{D}(f, 5, \tau_\delta, \varepsilon) = [8, 12]$ *and* $L \in \underline{D}(f, 5, \tau_\delta, \varepsilon) = [6, 14]$*, the pairs of all linear functions s and t form two bundles of lines (see Figure 10).*



(a) (b)

Figure 10: Bundles of lines formed by the pairs of all linear functions, (a) for all $L \in [8, 12]$ and (b) for all $L \in [6, 14]$.

Definition 3.9 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. Then, the point a is called an absolute ε-maximum (or a global ε-maximum) point of f, and $f(a)$ is called absolute ε-maximum (or global ε-maximum) value if $f(x) \leq f(a) + \varepsilon(a)$, for all $x \in A$.*

Definition 3.10 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. Then, the point a is called an absolute ε-minimum (or a global ε-minimum) point of f, and $f(a)$ is called absolute ε-minimum (or global ε-minimum) value if $f(x) \geq f(a) - \varepsilon(a)$, for all $x \in A$.*

Note 5 *Absolute ε-maxima and absolute ε-minima are also called absolute ε-extrema. Moreover, absolute ε-maximum values and absolute ε-minimum values are also called absolute ε-extremum values.*

Definition 3.11 *[9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. Then, the point a is called a local (τ, ε)-maximum point of f, and $f(a)$ is called local (τ, ε)-maximum value if $f(x) \leq f(a) + \varepsilon(a)$, for all $x \in \tau_{D(f)}(a)$.*

Definition 3.12 *[9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. Then, the point a is called a local (τ, ε)-minimum point of f, and $f(a)$ is called local (τ, ε)-minimum value if $f(x) \geq f(a) - \varepsilon(a)$, for all $x \in \tau_{D(f)}(a)$.*

Note 6 *Local (τ, ε)-maxima and local (τ, ε)-minima are also called local (τ, ε)-extrema. Moreover, local (τ, ε)-maximum values and local (τ, ε)-minimum values are also called local (τ, ε)-extremum values.*

Theorem 3.52 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If the point a is an absolute ε-maximum (absolute ε-minimum) point of f, then the point a is a local (τ, ε)-maximum (local (τ, ε)-minimum) point of f.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a) \neq \emptyset$, and the point a be an absolute ε-maximum (absolute ε-minimum) point of f. Therefore, $f(x) \leq f(a) + \varepsilon(a)$ $(f(x) \geq f(a) - \varepsilon(a))$, for all $x \in A$. Then, $f(x) \leq f(a) + \varepsilon(a)$ $(f(x) \geq f(a) - \varepsilon(a))$, for all $x \in \tau_{D(f)}(a) \subseteq A$. Hence, the point a is a local (τ, ε)-maximum (local (τ, ε)-minimum) point of f. □

Theorem 3.53 *[9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If the point a is a local (τ, ε)-maximum point of f, then $0 \in \overline{D}(f, a, \tau, \varepsilon)$.*

Theorem 3.54 *[9,26] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If the point a is a local (τ, ε)-minimum point of f, then $0 \in \underline{D}(f, a, \tau, \varepsilon)$.*

Corollary 3.13 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. If the point a is an absolute ε-maximum (absolute ε-minimum) point of f, then $0 \in \overline{D}(f, a, \tau, \varepsilon)$ $(0 \in \underline{D}(f, a, \tau, \varepsilon))$.*

The proof can be observed from Theorems 3.52-3.54.

Theorem 3.55 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. If $0 \in \overline{D}(f, a, \tau, \varepsilon)$, then a is a local (τ, ε)-maximum point of f.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $0 \in \overline{D}(f, a, \tau, \varepsilon)$. Then, $\tau_{D(f)}(a) \neq \emptyset$ and $x \in \tau_{D(f)}(a) \Rightarrow f(x) \leq f(a) + 0(x - a) + \varepsilon(a)$. Thus, $f(x) \leq f(a) + \varepsilon(a)$, for all $x \in \tau_{D(f)}(a)$. That is, a is a local (τ, ε)-maximum point of f. □

Theorem 3.56 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, and $a \in A$. If $0 \in \underline{D}(f, a, \tau, \varepsilon)$, then a is a local (τ, ε)-minimum point of f.*

The proof is similar to the proof of Theorem 3.55.

Corollary 3.14 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. Then, a is a local (τ, ε)-maximum point (local (τ, ε)-minimum point) of f if and only if $0 \in \overline{D}(f, a, \tau, \varepsilon)$ $(0 \in \underline{D}(f, a, \tau, \varepsilon))$.*

Theorem 3.57 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a) \neq \emptyset$, and f be (τ, ε)-soft continuous at the point a. Then, a is a local (τ, ε)-maximum point and a local (τ, ε)-minimum point of f.*

Proof: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, $\tau_{D(f)}(a) \neq \emptyset$, and f be (τ, ε)-soft continuous at the point a. Then, $x \in \tilde{\tau}_{D(f)}(a) \Rightarrow \left|f(x) - f(a)\right| \leq \varepsilon(a)$. Then, for all $x \in \tau_{D(f)}(a) \subseteq \tilde{\tau}_{D(f)}(a)$,

$$\begin{aligned}\left|f(x) - f(a)\right| \leq \varepsilon(a) &\Rightarrow -\varepsilon(a) \leq f(x) - f(a) \leq \varepsilon(a)\\ &\Rightarrow f(a) - \varepsilon(a) \leq f(x) \wedge f(x) \leq f(a) + \varepsilon(a)\end{aligned}$$

Hence, a is a local (τ, ε)-maximum point and a local (τ, ε)-minimum point of f. □

Theorem 3.58 *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in A$, and $\tau_{D(f)}(a) \neq \emptyset$. Then, $0 \in D(f, a, \tau, \varepsilon)$ if and only if a is a local (τ, ε)-maximum point and a local (τ, ε)-minimum point of f.*

The proof can be observed from Theorem 3.22 and Corollary 3.14.

Example 3.8 *[9,26] Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} \varepsilon(0) - \frac{\varepsilon(0)x}{\delta(0)}, & x > 0 \\ 0, & x = 0 \\ \varepsilon(0) + \frac{3\varepsilon(0)x}{\delta(0)}, & x < 0 \end{cases}$ such that $\varepsilon(0) > 0$. Since, for all $x \in \tau_\delta(0) \cap \mathbb{R}$,*

$$x \in (0, \delta(0)] \Rightarrow f(x) = \varepsilon(0) - \frac{\varepsilon(0)x}{\delta(0)} \leq \varepsilon(0) - \frac{\varepsilon(0)x}{\delta(0)} + \frac{\varepsilon(0)x}{\delta(0)} = f(0) + \varepsilon(0)$$

and

$$x \in [-\delta(0), 0) \Rightarrow f(x) = \varepsilon(0) + \frac{3\varepsilon(0)x}{\delta(0)} \leq \varepsilon(0) + \frac{3\varepsilon(0)x}{\delta(0)} - \frac{3\varepsilon(0)x}{\delta(0)} = f(0) + \varepsilon(0)$$

then $f(x) \leq f(0) + \varepsilon$, for all $x \in \tau_\delta(0) \cap \mathbb{R}$. Hence, $a = 0$ is a local (τ, ε)-maximum point of f. However, for $x = -\delta(0)$, since $f(0) - \varepsilon(0) = -\varepsilon(0) \nleq \varepsilon(0) + \frac{3\varepsilon(0)x}{\delta(0)}$, then 0 is not a local (τ, ε)-minimum point of f. Thus, $0 \notin D(f, 0, \tau_\delta, \varepsilon) = \left[\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta}\right]$.

The following two theorems can be considered analogs of Rolle's theorem for the upper soft derivative and the lower soft derivative, respectively.

Theorem 3.59 *[9,26] Let $f : [a, b] \to \mathbb{R}$ be a function, f be bounded above on the set $[a, b]$, $\tau_{D(f)}(c) \neq \emptyset$, for all $c \in [a, b]$, and $\inf_{x \in [a,b]} \{\varepsilon(x)\} > 0$. Then, there exists a local (τ, ε)-maximum point $c \in [a, b]$ of f such that $0 \in \overline{D}(f, c, \tau, \varepsilon)$.*

Theorem 3.60 *[9,26] Let $f : [a, b] \to \mathbb{R}$ be a function, f be bounded below on the set $[a, b]$, $\tau_{D(f)}(c) \neq \emptyset$, for all $c \in [a, b]$, and $\inf_{x \in [a,b]} \{\varepsilon(x)\} > 0$. Then, there exists a local (τ, ε)-minimum point $c \in [a, b]$ of f such that $0 \in \underline{D}(f, c, \tau, \varepsilon)$.*

Note 7 *If a function $f : [a, b] \to \mathbb{R}$ provides the hypotheses of Theorem 3.59, then there exists a linear function $y = f(c) + \varepsilon(c)$ whose slope is 0, which bounds the function f from above on $\tau_{D(f)}(c)$ such that $c \in [a, b]$ is a local (τ, ε)-maximum point of f. Similarly, if a function $f : [a, b] \to \mathbb{R}$ provides the hypotheses of Theorem 3.60, then there exists a linear function $y = f(c) - \varepsilon(c)$ whose slope is 0, which bounds the function f from below on $\tau_{D(f)}(c)$ such that $c \in [a, b]$ is a local (τ, ε)-minimum point of f.*

Example 3.9 *Let $f : [-1, 1] \to \mathbb{R}$, $\tau : \mathbb{R} \to P(\mathbb{R})$, and $\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ be three functions defined by $f(x) = x^2$,*

$$\tau(x) = \begin{cases} \left(x, x + \frac{1}{2}\right], & x \leq -1 \\ \left[x - \frac{x+1}{4}, x\right) \cup \left(x, x + \frac{1-x}{4}\right], & -1 < x < 1 \\ \left[x - \frac{1}{2}, x\right), & x \geq 1 \end{cases}$$

and $\varepsilon(x) = \frac{\cos(x)+2}{3}$*, respectively. It can be observed that* f *is bounded above and below on the set* $[-1, 1]$*. Moreover,* $\inf\limits_{x\in[-1,1]} \{\varepsilon(x)\} = \frac{1}{3} > 0$*. Thus, the hypotheses of Theorems 3.59 and 3.60 are valid. Therefore, there exists a local* (τ, ε)*-maximum point* $c_1 \in [-1, 1]$ *of* f *such that* $0 \in \overline{D}(f, c_1, \tau, \varepsilon)$ *and a local* (τ, ε)*-minimum point* $c_2 \in [-1, 1]$ *of* f *such that* $0 \in \underline{D}(f, c_2, \tau, \varepsilon)$*. Since*

$$f(x) \le f\left(-\frac{1}{2}\right) + \varepsilon\left(-\frac{1}{2}\right), \quad \text{for all } x \in \tau_{D(f)}\left(-\frac{1}{2}\right) = \left[-\frac{5}{8}, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{8}\right]$$

$$f(x) \le f(0) + \varepsilon(0), \quad \text{for all } x \in \tau_{D(f)}(0) = \left[-\frac{1}{4}, 0\right) \cup \left(0, \frac{1}{4}\right]$$

and

$$f(x) \le f\left(\frac{1}{2}\right) + \varepsilon\left(\frac{1}{2}\right), \quad \text{for all } x \in \tau_{D(f)}\left(\frac{1}{2}\right) = \left[\frac{1}{8}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{5}{8}\right]$$

then $c = -\frac{1}{2}$, $c = 0$, *and* $c = \frac{1}{2}$ *are local* (τ, ε)*-maximum points of* f *(see Figure 11). Similarly, because*

$$f(x) \ge f\left(-\frac{1}{2}\right) - \varepsilon\left(-\frac{1}{2}\right), \quad \text{for all } x \in \tau_{D(f)}\left(-\frac{1}{2}\right) = \left[-\frac{5}{8}, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{8}\right]$$

$$f(x) \ge f(0) - \varepsilon(0), \quad \text{for all } x \in \tau_{D(f)}(0) = \left[-\frac{1}{4}, 0\right) \cup \left(0, \frac{1}{4}\right]$$

and

$$f(x) \ge f\left(\frac{1}{2}\right) - \varepsilon\left(\frac{1}{2}\right), \quad \text{for all } x \in \tau_{D(f)}\left(\frac{1}{2}\right) = \left[\frac{1}{8}, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{5}{8}\right]$$

then $c = -\frac{1}{2}$, $c = 0$, *and* $c = \frac{1}{2}$ *are local* (τ, ε)*-minimum points of* f *(see Figure 11). Besides, as, for all* $c \in [-1, 1]$,

$$\begin{aligned}\overline{D}(f, c, \tau, \varepsilon) &= \left[\sup_{x\in\tau^{+}_{D(f)}(c)} \left\{\frac{x^2-c^2}{x-c} - \frac{\frac{\cos(c)+2}{3}}{|x-c|}\right\}, \inf_{x\in\tau^{-}_{D(f)}(c)} \left\{\frac{x^2-c^2}{x-c} + \frac{\frac{\cos(c)+2}{3}}{|x-c|}\right\}\right] \\ &= \begin{cases} \left[\frac{1}{6}(-17 - 4\cos(1)), \infty\right], & c = -1 \\ \left[-\infty, \frac{1}{6}(17 + 4\cos(1))\right], & c = 1 \\ \left[\frac{21c^2-18c+16\cos(c)+29}{12c-12}, \frac{21c^2+18c+16\cos(c)+29}{12c+12}\right], & c \in (-1, 1) \end{cases}\end{aligned}$$

then $0 \in \overline{D}(f, c, \tau, \varepsilon)$*, for all* $c \in (-1, 1)$*. Hence,* $0 \in \overline{D}\left(f, -\frac{1}{2}, \tau, \varepsilon\right)$, $0 \in \overline{D}(f, 0, \tau, \varepsilon)$*, and* $0 \in \overline{D}\left(f, \frac{1}{2}, \tau, \varepsilon\right)$*. In addition, since, for all* $c \in [-1, 1]$,

$$\underline{D}(f, c, \tau, \varepsilon) = \begin{cases} \left[-\infty, \frac{1}{6}(4\cos(1) - 1)\right], & c = -1 \\ \left[\frac{1}{6}(1 - 4\cos(1)), \infty\right], & c = 1 \\ \left[\frac{21c^2+18c-16\cos(c)-35}{12c+12}, \frac{21c^2-18c-16\cos(c)-35}{12c-12}\right], & c \in (-1, 1) \end{cases}$$

then $0 \in \underline{D}(f, c, \tau, \varepsilon)$*, for all* $c \in (-1, 1)$*. Thereby,* $0 \in \underline{D}\left(f, -\frac{1}{2}, \tau, \varepsilon\right)$, $0 \in \underline{D}(f, 0, \tau, \varepsilon)$*, and* $0 \in \underline{D}\left(f, \frac{1}{2}, \tau, \varepsilon\right)$*.*

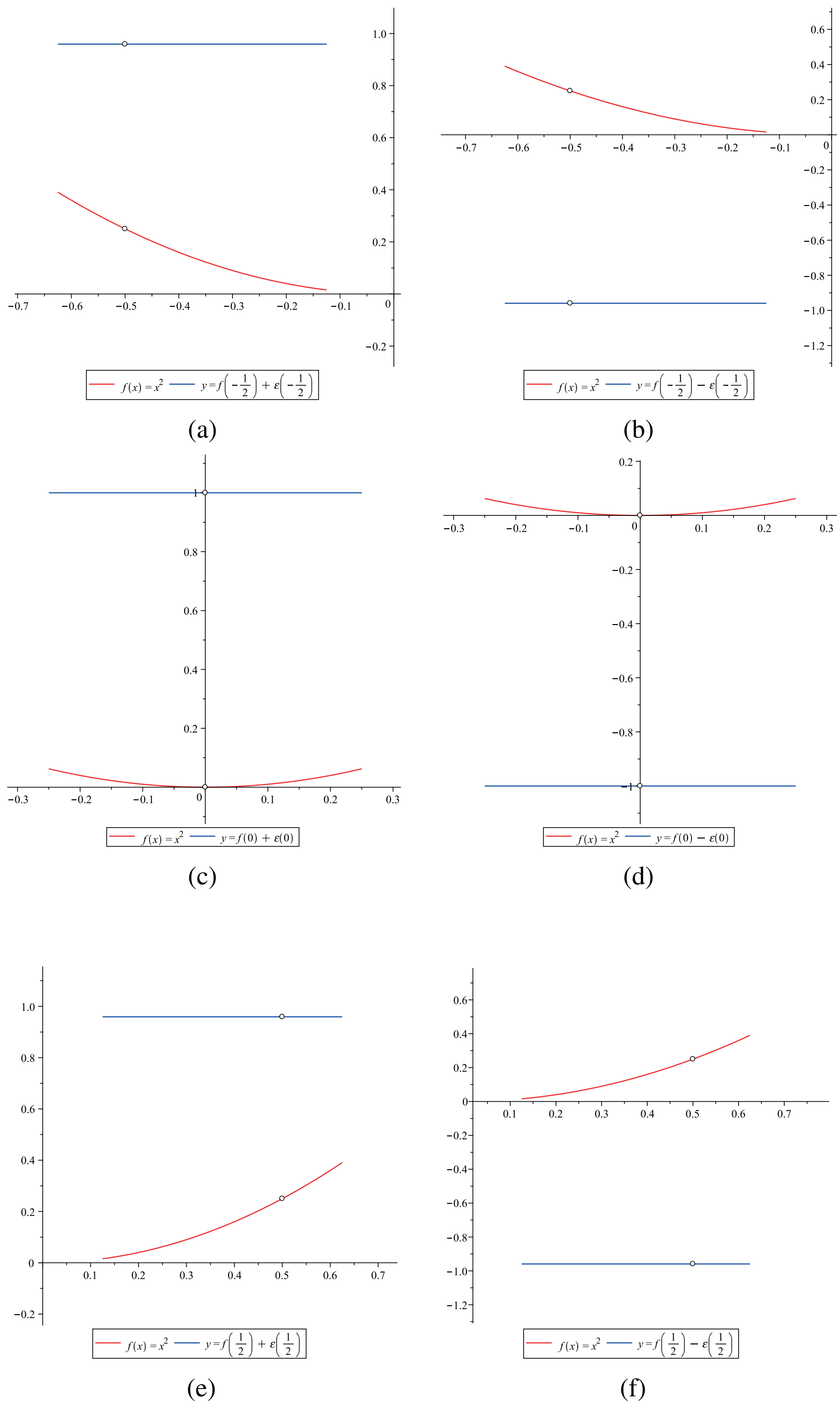


Figure 11: Graphs of (a) $f(x) = x^2$ and $y = f\left(-\frac{1}{2}\right) + \varepsilon\left(-\frac{1}{2}\right)$ on the set $\tau_{D(f)}\left(-\frac{1}{2}\right)$, (b) $f(x) = x^2$ and $y = f\left(-\frac{1}{2}\right) - \varepsilon\left(-\frac{1}{2}\right)$ on the set $\tau_{D(f)}\left(-\frac{1}{2}\right)$, (c) $f(x) = x^2$ and $y = f(0) + \varepsilon(0)$ on the set $\tau_{D(f)}(0)$, (d) $f(x) = x^2$ and $y = f(0) - \varepsilon(0)$ on the set $\tau_{D(f)}(0)$ (e) $f(x) = x^2$ and $y = f\left(\frac{1}{2}\right) + \varepsilon\left(\frac{1}{2}\right)$ on the set $\tau_{D(f)}\left(\frac{1}{2}\right)$, and (f) $f(x) = x^2$ and $y = f\left(\frac{1}{2}\right) - \varepsilon\left(\frac{1}{2}\right)$ on the set $\tau_{D(f)}\left(\frac{1}{2}\right)$.

The following two theorems can be considered analogs of the Mean Value Theorem for the upper and lower soft derivatives, respectively.

Theorem 3.61 *Let $f : [a,b] \to \mathbb{R}$ be a function, f be bounded above on the set $[a,b]$, $\tau_{D(f)}(c) \neq \emptyset$, for all $c \in [a,b]$, and $\inf_{x\in[a,b]}\{\varepsilon(x)\} > 0$. Then, there exists a local (τ,ε)-maximum point $m \in [a,b]$ of f such that $\frac{f(b)-f(a)}{b-a} \in \overline{D}(f,m,\tau,\varepsilon)$.*

Proof: Let $f : [a,b] \to \mathbb{R}$ be a function, f be bounded above on the set $[a,b]$, $\tau_{D(f)}(c) \neq \emptyset$, for all $c \in [a,b]$, and $\inf_{x\in[a,b]}\{\varepsilon(x)\} > 0$. Since f is bounded above on the set $[a,b]$, then $\sup_{x\in[a,b]}\{f(x)\} \in \mathbb{R}$. From properties of supremum, there exists at least one $c \in [a,b]$ such that $\sup_{x\in[a,b]}\{f(x)\} - \inf_{x\in[a,b]}\{\varepsilon(x)\} < f(c)$. Let $g : [a,b] \to \mathbb{R}$ be a function defined by $g(x) = f(x)-f(c)-\frac{f(b)-f(a)}{b-a}(x-c)$. Since f is bounded above on the set $[a,b]$, g is bounded above on the set $[a,b]$. Then, g holds the conditions of Theorem 3.59. Therefore, there exists at least one local (τ,ε)-maximum point $m \in [a,b]$ of g such that $0 \in \overline{D}(g,m,\tau,\varepsilon)$. Hence, for all $x \in \tau_{D(g)}(m) = \tau_{D(f)}(m)$,

$$\begin{aligned} g(x) \leq g(m) + 0\,(x-m) + \varepsilon(m) &\Rightarrow g(x) \leq g(m) + \varepsilon(m) \\ &\Rightarrow f(x) \leq f(m) + \frac{f(b)-f(a)}{b-a}(x-m) + \varepsilon(m) \\ &\Rightarrow \frac{f(b)-f(a)}{b-a} \in \overline{D}(f,m,\tau,\varepsilon) \end{aligned}$$

□

Theorem 3.62 *Let $f : [a,b] \to \mathbb{R}$ be a function, f be bounded below on the set $[a,b]$, $\tau_{D(f)}(c) \neq \emptyset$, for all $c \in [a,b]$, and $\inf_{x\in[a,b]}\{\varepsilon(x)\} > 0$. Then, there exists a local (τ,ε)-minimum point $m \in [a,b]$ of f such that $\frac{f(b)-f(a)}{b-a} \in \underline{D}(f,m,\tau,\varepsilon)$.*

The proof is similar to the proof of Theorem 3.61 by properties of infimum and Theorem 3.60.

Note 8 *If a function $f : [a,b] \to \mathbb{R}$ provides the hypotheses of Theorem 3.61, then there exists a linear function whose slope is $\frac{f(b)-f(a)}{b-a}$, which bounds the function f from above on $\tau_{D(f)}(c)$ such that $c \in [a,b]$ is a local (τ,ε)-maximum point of f. Similarly, if a function $f : [a,b] \to \mathbb{R}$ provides the hypotheses of Theorem 3.62, then there exists a linear function whose slope is $\frac{f(b)-f(a)}{b-a}$, which bounds the function f from below on $\tau_{D(f)}(c)$ such that $c \in [a,b]$ is a local (τ,ε)-minimum point of f.*

Example 3.10 *The function f in Example 3.9 provides the conditions in Theorems 3.61 and 3.62. Moreover, for $[a,b] = [-1,1]$, $\frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(-1)}{1-(-1)} = 0$. From Example 3.9, there exists a local (τ,ε)-maximum point $m_1 \in [-1,1]$ of f, e.g., $-\frac{1}{2}, 0$, and $\frac{1}{2}$, such that $\frac{f(b)-f(a)}{b-a} = 0 \in \overline{D}(f,m_1,\tau,\varepsilon)$ and a local (τ,ε)-minimum point $m_2 \in [-1,1]$ of f, e.g., $-\frac{1}{2}, 0$, and $\frac{1}{2}$, such that $\frac{f(b)-f(a)}{b-a} = 0 \in \underline{D}(f,m_2,\tau,\varepsilon)$.*

Theorem 3.63 *Let $A \subseteq \mathbb{R}$, $f,g : A \to \mathbb{R}$ be two functions, $a \in A$, and there exist the (τ,ε)-soft derivatives of f and g at the point a. If $D(f,a,\tau,\varepsilon) = D(g,a,\tau,\varepsilon)$, then $0 \in D(f-g,a,\tau,2\varepsilon)$.*

Proof: Let $A \subseteq \mathbb{R}$, $f,g : A \to \mathbb{R}$ be two functions, $a \in A$, there exist the (τ,ε)-soft derivatives of f and g at the point a, and $D(f,a,\tau,\varepsilon) = D(g,a,\tau,\varepsilon)$. Then, $\tau_{D(f)}(a) = \tau_{D(g)}(a) \neq \emptyset$, and for an $L \in D(f,a,\tau,\varepsilon) = D(g,a,\tau,\varepsilon)$,

$$x \in \tau_{D(f)}(a) \Rightarrow \left|f(x) - f(a) - L(x-a)\right| \leq \varepsilon(a)$$

and

$$x \in \tau_{D(g)}(a) \Rightarrow \left|g(x) - g(a) - L(x-a)\right| \leq \varepsilon(a)$$

Hence,

$$\begin{aligned} x \in \tau_{D(f-g)}(a) &\Rightarrow \left|f(x) - f(a) - L(x-a)\right| \leq \varepsilon(a) \wedge \left|g(x) - g(a) - L(x-a)\right| \leq \varepsilon(a) \\ &\Rightarrow \left|(f-g)(x) - (f-g)(a) - 0(x-a)\right| \leq 2\varepsilon(a) \\ &\Rightarrow 0 \in D(f-g,a,\tau,2\varepsilon) \end{aligned}$$

□

Theorem 3.64 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and there exist the left (τ, ε)-soft derivatives of f and g at the point a. If $D(f, a, \tau^-, \varepsilon) = D(g, a, \tau^-, \varepsilon)$, then $0 \in D(f - g, a, \tau^-, 2\varepsilon)$.*

Theorem 3.65 *Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$ be two functions, $a \in A$, and there exist the right (τ, ε)-soft derivatives of f and g at the point a. If $D(f, a, \tau^+, \varepsilon) = D(g, a, \tau^+, \varepsilon)$, then $0 \in D(f - g, a, \tau^+, 2\varepsilon)$.*

4. Conclusion

This study intends to contribute to the field of soft analysis by redefining concepts of soft and upper (lower) soft derivatives and proposing left and right soft derivatives. It establishes essential equivalence conditions for the nonemptiness of the soft derivative at a point, demonstrating that such a presence is contingent upon the intersection, being nonempty, of upper and lower soft derivatives, as well as that of left and right soft derivatives. This foundational understanding is crucial for further explorations in soft analysis. Moreover, this investigation clarifies the relationships between soft derivatives and such fundamental properties as boundedness and soft continuity types. Further, this study contends that soft continuity guarantees the existence of local (τ, ε)-extrema and provides definitions of absolute ε-extrema and characterizations for local (τ, ε)-extrema.

In addition to theoretical contributions, this study derives geometric interpretations of soft derivatives and analogs of Rolle's Theorem and the Mean Value Theorem and substantiates these results through graphical representations. These interpretations offer a deeper insight into the mechanism of soft derivatives and their applications, which are essential for both theoretical and practical progress in soft analysis. This study also includes illustrative examples of the concepts concerned, which strengthen the discussed theoretical framework. These examples serve to clarify the definitions and properties of soft derivatives, making them more accessible and practically relevant for future researchers. This research lays a robust groundwork for future investigations into such subject matters as soft integrals, directional soft derivative, and soft gradients, and into their applications in optimization and decision making. By expanding the core principles of soft analysis, this study opens new pathways for interdisciplinary research and practical applications.

Conflict of Interest

All the authors declare no conflict of interest.

Data Availability Statement

Data sharing does not apply to this article as no new data were created or analyzed in this study.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

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