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Computing Dominating Number and Dominant Metric Dimension for Zero Divisor Graphs of Order at most 10 of Small Finite Commutative Rings

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Contents

1. Abstract

In this article we compute the dominating number (DN) and dominant metric dimension (Ddim) of zero divisor graphs of some small finite commutative rings with order not exceeding 14. Consider a commutative ring denoted as and let represent its zero-divisor graph (ZD-graph). The vertices of these graphs correspond to the non-zero divisors (ZD) within the commutative ring (CR), where an edge connects two distinct vertices if their product in the ring results in zero. This paper focuses on studying the domination number and dominant metric dimension for zero divisor graphs of orders 3, 4, 5, 6, 7, 8, 9, and 10 within a small finite commutative ring with a unity. Employing a combination of computational methods and mathematical techniques, our research sheds light on the structural nuances of these small commutative rings, enhancing our comprehension of their algebraic behavior and paving the way for potential applications in algebraic theory and related fields.

keywords: algebraic structures, zero divisor graphs, dominant metric dimensions, equivalence classes, metric-dimension, compressed zero-divisor graph.

2. Introduction

Beck [1] proposed the connection between graph theory and algebra by introducing a ZD-graph of a CR R. The primary focus OF author's [1] was on the coloring of nodes in a graph, specifically on the ring elements that corresponded to these nodes. Note that, a zero vertex is linked to all other vertices in this case. $Z^o(R)$ denotes this type of ZD-graph in literature. In [2], Anderson and Livingston conducted a study on a ZD-graph in which each node represents a nonzero ZD. According to them an undirected graph obtained by considering x and y as nodes joined by an edge iff $xy = 0$ is called a ZD-graph of R. The study of Anderson and Livingston emphasizes the case of finite rings, as finite graphs can be obtained when R is finite. Their task was to determine whether a graph is complete for a given ring or a star for a given ring. Here $Z(R)$ will denote this type of ZD-graph of R. This ZD-graph definition differs slightly from Beck's ZD-graph definition for R. Remember, zero is not considered as a vertex of ZD-graph here, hence $Z(R) \subseteq Z^0(R)$. Anderson and Livingston [2] found the correlation between properties of ring R and the graph properties of associated $Z(R)$; Moreover, this study yields essential outcomes concerning $Z(R)$. Redmond [3] expanded the ZD-graph idea from unital
commutative rings (CR) to noncommutative rings. Different methods were presented by him to characterize the ZD-graph related to a noncommutative ring, encompassing both undirected & directed graphs.

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Redmond [4] extended this work using a ZD-graph for a commutative ring and transformed it into an ideal-based ZD graph. The aim was to generalize the method by substituting elements with zero products with elements whose product belongs to a particular ideal I of ring R.

An ideal based ZD-graph denoted by $\Gamma_I(R)$ can be obtained by considering two nonzero ZD, x and y of R as nodes and there is an edge between them iff $xy \in I$ such that $\{xy \in I$, for some, $y \in R - I\}$. Numerous authors have defined various types of graphs, including total graphs, unit graphs, ideal-based ZD-graph, Jacobson graphs, ZD-graph of equivalence classes, and more. These works can be found in sources such as [5-9]. Readers may refer to [10, 11] for a deeper understanding of graph theory, and for some fundamental definitions of ring theory, [12, 13] can be consulted. Reader may also study some distinguishing parameters for different graphs in [24-25].

The graph associated with R provides a remarkable demonstration of the properties of $L(R)$, the lattice of ideals of R. This graph enables us to visualize and analyze the algebraic properties of rings through graph theory techniques. In [5], the properties of $Z(R)$ were analyzed by the authors and found to be interesting. To study $Z(R)$, we stick to the method presented by Anderson and Livingston in [5], where non-zero ZD are vertex set for $Z(R)$. Unless otherwise stated, we consider R is a finite unital CR in this paper. $L(R)$ denotes set of non-zero ZD, as discussed earlier. Consider the ring R contains only single maximal ideal then it is called local.

Let $x \in R$, then annihilator of x is denoted and defined as $\operatorname{ann}(x) = \{y \in R : xy = 0\}$. An element $r \in R$, is a nilpotent element if $r^m = 0$, for some positive integer m. Reduced ring is defined as a ring containing no nilpotent elements (i.e., non-zero elements). Z_n is called ring of integers modulo n and defined as $Z_n = \{0, 1, 2, \ldots, n-1\}$, while ring of gaussian integers modulo n is denoted and defined as $Z_n[i] = \{x + iy : x, y \in Z_n$ and $i^2 = -1\}$ under the complex multiplication and addition, and F denotes finite field. Osba et al. introduced the graph for $Z_n[i]$ in [14], $Z(Z_n[i])$ denotes the ZD-graph of the Gaussian integer ring $Z_n[i]$.

For a graph G, a subset $S \subseteq V(G)$ is said to be a dominating set (DS) of G if for $\forall x \in V(G) \mid S$, $\exists$ at least one vertex $u \in S$ such that x is adjacent to u. The minimum cardinality among DS of G is referred as dominating number (DN) of G and denoted by $\gamma(G)$ [26]. Let $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ is an ordered set. If every pair of vertices $u, v \in V(G)$ have distinct representation with respect to W, that is, $r(u \mid W) \neq r(v \mid W)$, where $r(u \mid W) = (d(u, w_1), d(u, w_2), \ldots, d(u, w_k))$ then W is called a resolving set (RS) of G. The RS of G with minimum cardinality is called metric dimension (MD) of G. We denote MD as $\dim(G)$. Brigham, et al. [27] studied MD and DS collectively and named it as resolving domination number, denoted by $\gamma_r(G)$. The concluded that $\max\{\dim(G), \gamma(G)\} \leq \gamma_r(G) \leq \dim(G) + \gamma(G)$. Later, Henning and Oellarmann [28] studied the metric locating DN of graph G, denoted by $\gamma_M(G)$ and found that $\gamma(G) \leq \gamma_M(G) \leq n - 1$. Then, Gonzalez, et al. [29] studied different lower and upper bounds, i.e., $\max\{\dim(G), \gamma(G)\} \leq \gamma_M(G) \leq \dim(G) + \gamma(G)$. A dominant RS of G is an ordered set $W \subseteq V(G)$, such that W is a RS and a DS of G. The minimum cardinality of dominant RS is called a dominant basis of G, while the cardinality of dominant basis is recognized as a dominant MD of G and denoted by $\operatorname{Ddim}(G)$.

The contribution of this article lies in its comprehensive exploration of the dominating number and dominant metric dimension of zero divisor graphs associated with finite commutative rings of order not exceeding 14. By employing a combination of computational methods and mathematical techniques, the research offers valuable insights into the underlying algebraic structures of these small rings. The determination of the dominating number provides a crucial understanding of the minimal set of vertices necessary to control the entire zero divisor graph, thereby revealing the specific elements that exert significant influence on the ring's properties. Concurrently, the introduction of the novel concept of dominant metric dimension adds a new dimension to the analysis by quantifying the graph's ability to identify elements based on distance considerations.

This study contributes to the field of algebraic structures by enhancing our comprehension of the structural nuances inherent in small commutative rings. The insights gained from investigating the dominating number and dominant metric dimension provide researchers with a powerful tool for unraveling the algebraic intricacies of these rings. Such knowledge not only deepens our understanding of fundamental algebraic concepts but also opens avenues for potential applications in algebraic theory and related fields. Researchers in the domain of algebraic structures can leverage the findings of this article to refine existing theories, develop new methodologies, and advance the broader understanding of finite commutative rings. Ultimately, this research significantly enriches the theoretical framework in algebra and contributes to the ongoing dialogue within the academic community on the structural analysis of algebraic systems.

The novelty of finding the Ddim of graphs lies in the fact that it is provides a more complete understanding of the graph's structure and algebraic properties, which can be useful in many applications such as network design, social networking, and communication systems.

3. Main results

Formally, Graph is an ordered pair $G=(V,E)$; here V and E stands for vertices or nodes and edge set, respectively. A graph's order and size are defined as cardinality of nodes set and edges set, respectively. The open neighborhood of a node v is written as $N(v)$, and defined as $\{v \in V(G) : vu \in E(G)\}$, while closed neighborhood of a node u is written as $N[u]$, and defined as $\{u\} \cup N(u)$. The distance $d\left(u', v'\right)$ between two nodes u' and v' is defined as length of shortest path between them, while $d\left(w, e'\right)=\min\left\{d\left(w, u'\right), d\left(w, v'\right)\right\}$ defines the distance between a node w and edge $e'=u'v'$. The length of the longest path is the diameter of the graph which is denoted by $\operatorname{diam}(G)$ Mathematically, $\operatorname{diam}(G)=\sup\{d(r,s) :$ where r and s are distinct vertices in $G\}$. Let H be a subset of vertices along with any subset of edges containing those vertices, forming a subgraph of graph G written as $H \subset G$. The number of edges in the smallest cycle subgraph in graph G is referred to as the girth of the graph, denoted by $\operatorname{gr}(G)$. The maximal complete subgraph of graph G is termed a clique, denoted by K, and $|K|=\omega(G)$ is identified as the clique number.

A graph is said to be regular graph if for every $r \in V, \deg(r)=c$ for a fix, $c \in Z^{+}$. A graph is said to be complete graph if there is a connection between all pairs of vertices. It is represented by k_m, where m is the number of vertices. A graph is considered a complete bipartite graph if it can be divided into two distinct sets of vertices, X and Y, where each vertex in X is connected to every vertex in Y, and it is usually denoted by $k_{m,n}$, where $|X|=m$ and $|Y|=n$. Let G be a connected graph, and if the removal of a vertex results in two or more disconnected components in the graph, the vertex is designated as a cut vertex.

Kelenc et al. [16] studied the edge md (emd) of various graphs, including the path graph P_n, complete graph k_m, and complete bipartite graph $k_{m,n}$. The relationship between the md and the emd allows for the identification of graphs where these two dimensions are equal, as well as for some other graphs G for which $\dim(G)<\dim_E(G)$ or $\dim_E(G)<\dim(G)$. Basically, Kelenc et al. explored the comparison of values $\dim(G)$ and $\dim_E(G)$. Recently, a study on metric parameters for ZD-graphs has been done. Redmond in 2002 [17] studied the ZD-graphs of noncommutative rings, and in 2003 [18] the ideal based ZD-graphs of commutative rings was studied by him. In 2019 [19] the metric dimension of ZD-graphs for ring Z_n was calculated. In 2020 [20] bounds for the emd of ZD-graphs related to rings were studied by Siddiqui et al. Pirzada and Aijaz in 2020 [21] studied ZD-graphs for commutative rings for their metric and upper dimension.

Susilowati et al. [15] determined the Ddim of a specific class of graphs, characterized graphs with particular Ddim, and computed the Ddim of joint and comb products of graphs. In this context, we examine selected findings from references [30-31] as follows:

Remarks

1. For path graph denoted by P_n and cyclic graph $C_n, \gamma\left(P_n\right)=\gamma\left(C_n\right)=\left\lceil\frac{n}{3}\right\rceil$ and $\dim\left(P_n\right)=1$ and $\dim\left(C_n\right)=2$.
2. For a complete graph denoted by $K_n, \gamma\left(K_n\right)=1$ and $\dim\left(K_n\right)=n-1$.
3. For a start graph denoted by $S_n, \gamma\left(S_n\right)=1$ and $\dim\left(S_n\right)=n-2, \forall n \geq 2$.
4. Consider a complete bipartite graph $K_{m,n}, \gamma\left(K_{m,n}\right)=2$ and $\dim\left(K_{m,n}\right)=m+n-2, \forall m,n \geq 2$.

Furthermore, we consider some previous results on dominant metric dimension of G as follows:
Theorem 1 [15]: If C_n is a cyclic graph of order $n \geq 7$, then $\operatorname{Dim}_d\left(C_n\right)=\gamma\left(C_n\right)$.
Theorem 2 [15]: If G is a start graph S_n, having order $n \geq 2$, then $\operatorname{Dim}_d(G)=n-1$.
Theorem 3 [15]: Let $K_{m,n}$ be a complete bipartite graph with $m,n \geq 2$, then $\operatorname{Dim}_d\left(K_{m,n}\right)=\dim\left(K_{m,n}\right)$.

Theorem 4 [15]: If G is a path graph P_n, with $n \geq 4$, then $\mathrm{Dim}_d(P_n) = \gamma(P_n)$.
Theorem 5 [15]: If G is a complete graph K_n, with $n \geq 2$, then $\mathrm{Dim}_d(K_n) = \dim(K_n)$.
Theorem 6 [15]: $\mathrm{Dim}_d(P_n) = 1 \Leftrightarrow G \cong P_n, n = 1, 2$.
Moreover, Ddim for a single vertex graph G is supposed to be zero and for an empty graph it is undefined. Our discussion commences with the subsequent observation. Adirasari, et al. [23] studied the Ddim of corona product graphs.

4. Dominating number and dominant metric dimension of ZD-graphs

Here, we consider simple connected ZD-graphs that possess a countable number of vertices. We analyze some ZD-graphs associated to small finite commutative rings and determine their dominating number and the dominant metric dimension, leading to the subsequent outcomes.

Theorem 3.1: If $\Gamma(R) \cong$ G having 3 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.1.

No. Vertices	R ing (R)	$\boldsymbol{R}$—	Graph $\boldsymbol{G}$	$\boldsymbol{\gamma(G)}$	$\mathrm{Dim}_d(\boldsymbol{G})$
	$\mathbb{Z}_6$	6			
	$\mathbb{Z}_8$	8			
3	$\mathbb{Z}_2[x]/(x^3)$	8	$K_{1,2}$	1	2
	$\mathbb{Z}_4[x]/(2x, x^2-2)$	8			
	$\mathbb{Z}_2[x,y]/(x,y)^2$	8			
	$\mathbb{Z}_4[x]/(2,x)^2$	8			
	$F_4[x]/(x^2)$	16	K_3	1	2
	$\mathbb{Z}_4[x]/(x^2+x+1)$	16			

Table 3.1: ZD-graphs with 3 vertices
Proof. Case (a). The $\Gamma(R) \cong K_{1,2}$ is a special case of bipartite graph and we can consider $K_{1,2}$ as star graph S_n, with $n = 3$. By Remark 3, $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 2$, by Theorem 2.

Case (b). The $\Gamma(R) \cong K_3$ is special case of complete graph K_n, with $n = 3$. Hence by Remark 2 , $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 2$, by using Theorem 5 along with Remark 2. []

Theorem 3.2: If $\Gamma(R) \cong$ G having 4 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.2.

No. Vertices	R ing (R)	$\lvert\boldsymbol{R}\rvert$	Jraph G	$\boldsymbol{\gamma(G)}$	$\mathrm{Dim}_{\boldsymbol{d}}(\boldsymbol{G})$
	$\mathbb{Z}_2 \times F_4$	8	$K_{1,3}$	1	3
4	$\mathbb{Z}_3 \times \mathbb{Z}_3$	9	$K_{2,2}$	2	2
	$\mathbb{Z}_{25}$	25	K_4	1	3
	$\mathbb{Z}_5[x]/(x^2)$	25	K_4		

Table 3.2: ZD -graphs with 4 vertices
Proof. Case (a). When $\Gamma(R) \cong K_{1,3}$, it is a special case of bipartite graph, and we can consider $K_{1,3}$ as star graph S_n, with $n = 4$. By Remark 3, $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 3$, by Theorem 2. On another hand when $\Gamma(R) \cong K_{2,2}$, it is a special case of bipartite graph with $m = n = 2$, By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 2$, by using Theorem 3 along with Remark 4.

Case (b). When $\Gamma(R) \cong K_4$, it is special case of complete graph K_n, with $n = 4$. By Remark 2, $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 3$, by using Theorem 5 along with Remark 2. []

Theorem 3.3: If $\Gamma(R) \cong$ G having 5 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.3.

No. Vertices	R ing (R)	| $\boldsymbol{R}$—	Jraph G	$\boldsymbol{\gamma(G)}$	$\mathrm{Dim}_{\boldsymbol{d}}(\mathbf{G})$
5	$\mathbb{Z}_2 \times \mathbb{Z}_5$	10	$K_{1,4}$	1	4
	$\mathbb{Z}_3 \times F_4$	12	$K_{2,3}$	2	3
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8	Fig. 3.1		
	$\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$	8	Fig. 3.1	2	3

Table 3.3: ZD -graphs with 5 vertices

Proof. Case (a). When $\Gamma(R) \cong K_{1,4}$, it is a special case of bipartite graph, and we can consider $K_{1,4}$ as star graph S_n, with $n = 5$. By Remark 3, $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 4$, by Theorem 2 . On another hand when $\Gamma(R) \cong K_{2,3}$, it is a special case of bipartite graph with $m = 2, \&n = 3$, By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 3$, by using Theorem 3 along with Remark 4 .

Case (b). The ZD -graph for rings $\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[\mathrm{X}]/\left(x^2\right)$ is given in Fig. 3.1. From the figure, the set $\{2, 5\}$ is a dominating set for the graph and hence $\gamma(\Gamma(R)) = 2$. Whereas the set $\{2, 3, 4\}$ is dominant resolving set for the graph, so $\mathrm{Dim}_d(\Gamma(R)) = 3$.]
Figure 3.1

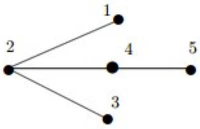

Figure 1: picture 1

Theorem 3.4: If $\Gamma(R) \cong$ G having 6 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.4.

No. Vertices	R ing (R)	$\|R\|$	Jraph G	$\boldsymbol{\gamma(G)}$	$\lim_{\boldsymbol{d}}(G)$
6	$\mathbb{Z}_3 \times \mathbb{Z}_5$	15	$K_{2,4}$		
	$F_4 \times F_4$	16	$K_{3,3}$	2	4
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8	Fig. 3.2	3	3
	$\mathbb{Z}_7[x]/\left(x^2\right)$	49	K_6		
	$\mathbb{Z}_{49}$	49	K_6	1	5

Table 3.4: ZD -graphs with 6 vertices

Proof. Case (a). When $\Gamma(R) \cong K_{2,4}$, it is a special case of bipartite graph with $m = 2\&n = 4$. By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 4$, by using Remark 4 along with Theorem 3. On another hand when $\Gamma(R) \cong K_{3,3}$, it is a special case of bipartite graph with $m = 3\&n = 3$, By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 4$, by using Theorem 3 along with Remark 4.

Case (b). The ZZD-graph for ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is given in Fig. 3.2. From the figure, the set $\{2, 3, 4\}$ is a dominating and resolving set for graph $\Gamma(R)$, hence $\gamma(\Gamma(R)) = \mathrm{Dim}_d(\Gamma(R)) = 3$.

Case (c). When $\Gamma(R) \cong K_6$, it is special case of complete graph K_n, with $n = 6$. By Remark 2, $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 5$, by using Theorem 5 along with Remark 2. []
Figure 3.2

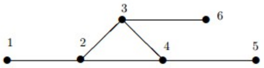

Figure 2: picture 2

Theorem 3.5: If $\Gamma(R) \cong G$ having 7 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.5.

No. Vertices	**R** ing (R)	$\|\boldsymbol{R}\|$	Jraph G	$\boldsymbol{\gamma(G)}$	$\mathrm{Sim}_{\mathrm{d}}$ (G)
	$F_4 \times \mathbb{Z}_5$	20	$K_{3,4}$	2	5
	$\mathbb{Z}_3 \times \mathbb{Z}_4$	12	Fig. 3.3		
	$\mathbb{Z}_3 \times \mathbb{Z}_2[x]/\left(x^2\right)$	12	Fig. 3.3	2	4
	$\mathbb{Z}_{16}$	16	Fig. 3.4		
	$\mathbb{Z}_2[x]/\left(x^4\right)$	16	Fig. 3.4		
	$\mathbb{Z}_4[x]/\left(x^2+2\right)$	16	Fig. 3.4	1	5
	$\mathbb{Z}_4[x]/\left(x^2+2x+2\right)$	16	Fig. 3.4		
	$\mathbb{Z}_4[x]/\left(x^3-2, 2x^2, 2x\right)$	16	Fig. 3.4		
	$\mathbb{Z}_2[x,y]/\left(x^3, xy, y^2\right)$	16	Fig. 3.5		
	$\mathbb{Z}_8[x]/\left(2x, x^2\right)$	16	Fig. 3.5		
	$\mathbb{Z}_4[x]/\left(x^3, 2x^2, 2x\right)$	16	Fig. 3.5	1	5
	$\mathbb{Z}_4[x,y]/\left(x^2\right.$	16	Fig. 3.5		
	$\mathbb{Z}_4[x]/\left(x^2+2x\right)$	16	Fig. 3.6		
	$\mathbb{Z}_8[x]/\left(2x, x^2+4\right)$	16	Fig. 3.6	1	3
	$\mathbb{Z}_2[x,y]/\left(x^2, y^2-xy\right)$	16	Fig. 3.6		
	$\mathbb{Z}_4[x,y]/\left(x^2, y^2-xy, xy\right.$	16	Fig. 3.6		
	$\mathbb{Z}_4[x,y]/\left(x^2, y^2, xy\right.$	16	Fig. 3.7		
	$\mathbb{Z}_2[x,y]/\left(x^2, \hat{y}^2\right)^{-}$	16	Fig. 3.7	1	3
	$\mathbb{Z}_4[x]/\left(x^2\right)$	16	Fig. 3.7		
	$\mathbb{Z}_2[x,y,z]/(x,y,z)^2$	16	K_7		
	$\mathbb{Z}_4[x,y]/\left(x^2, y^2, xy, 2x, 2y\right)$	16	K_7		
	$F_8[x]/\left(x^2\right)$	64	K_7	1	6
	$\mathbb{Z}_4[x]/\left(x^3+x+1\right)$	64	K_7		

Table 3.5: ZD -graphs with 7 vertices

Proof. Case (a). When $\Gamma(R) \cong K_{1,6}$, it is a special case of bipartite graph and we can consider $K_{1,6}$ as star graph S_n with $n=7$. By Remark 3, $\gamma(\Gamma(R))=1$. On another hand, $\mathrm{Dim}_d(\Gamma(R))=6$, by using Remark 3 along with Theorem 2. On another hand, when $\Gamma(R) \cong K_{3,4}$, it is a special case of bipartite graph with $m=3\&n=4$. By Remark 4, $\gamma(\Gamma(R))=2$. On another hand, $\mathrm{Dim}_d(\Gamma(R))=5$, by using Remark 4 along with Theorem 3 .

Case(b). The ZZD-graph for rings $\mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[x]/\left(x^2\right)$ is given in Fig 3.3. From the figure, the set $\{3,6\}$ is a dominating set and resolving set $\{2,4,7\}$ for graph $\Gamma(R)$, hence $\gamma(\Gamma(R))=2$ and $\{1,2,4,7\}$ is $\mathrm{Dim}_d(\Gamma(R))=4$.

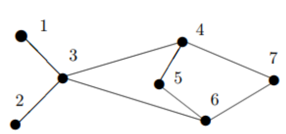

Figure 3: picture 3

Figure 3.3

Case (c). The ZD-graph for ring $\mathbb{Z}_{16}, \mathbb{Z}_2[x]/\left(x^4\right), \mathbb{Z}_4[x]/\left(x^2+2\right), \mathbb{Z}_4[x]/\left(x^2+2x+\quad 2\right)$, $\mathbb{Z}_4[x]/\left(x^3-2, 2x^2, 2x\right)$ is given in Fig. 3.4. From the figure, the set $\{5\}$ is a dominating set and resolving set $\{1,2,3,4,5,7\}$ for graph $\Gamma(R)$, hence $\gamma(\Gamma(R))=1$ and $\mathrm{Dim}_d(\Gamma(R))=5$.

Figure 3.4

Case (d). The ZD-graph for rings: $\mathbb{Z}_2[x,y]/\left(x^3, xy, y^2\right), \mathbb{Z}_8[x]/\left(2x, x^2\right), \mathbb{Z}_4[x]/\left(x^3, 2x^2, 2x\right)$ is given in Fig. 3.5. From the figure, the set $\{5\}$ is a dominating set and resolving set $\{1,2,4,5,6\}$ for graph $\Gamma(R)$,

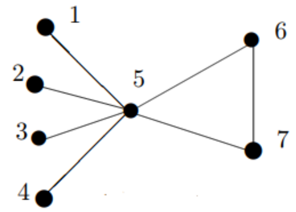

Figure 4: picture 4

hence $\gamma(\Gamma(R)) = 1$ and $\mathrm{Dim}_d(\Gamma(R)) = 5$.

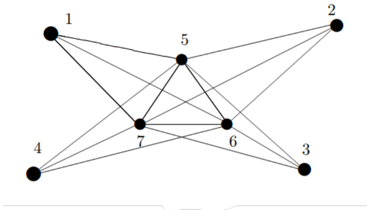

Figure 5: picture 5

Figure 3.5

Case (e). The ZD-graph for rings $\mathbb{Z}_4[x]/\left(x^2+2\right), \mathbb{Z}_4[x]/\left(x^2+2x\right), \mathbb{Z}_8[x]/\left(2x, x^2+\quad 4\right)$, $\mathbb{Z}_2[x,y]/\left(x^2, y^2-xy\right), \mathbb{Z}_4[x]/\left(x^2, y^2-xy, xy-2, 2x, 2y\right)$ is given in Fig. 3.6. From the figure, the set $\{7\}$ is a dominating set and resolving set $\{1,2,5\}$ for graph $\Gamma(R)$, hence $\gamma(\Gamma(R)) = 1$ and $\mathrm{Dim}_d(\Gamma(R)) = 3$.

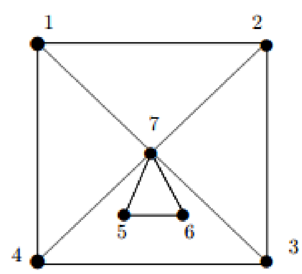

Figure 6: picture 6

Figure 3.6

Case (f). The ZD-graph for rings $\mathbb{Z}_4[x,y]/\left(x^2, y^2, xy-2, 2x, 2y\right), \mathbb{Z}_2[x,y]/\left(x^2, y^2\right), \mathbb{Z}_4[x]/\left(x^2\right)$ is given in Fig. 3.7. From the figure, the set $\{4\}$ is a dominating set and resolving set $\{1,3,5\}$ for graph $\Gamma(R)$, hence $\gamma(\Gamma(R)) = 1$ and$\mathrm{Dim}_d(\Gamma(R)) = 3$.

Figure 3.7

Case (g). When $\Gamma(R) \cong K_7$, it is special case of complete graph K_n, with $n = 7$. By Remark 2, $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 6$, by using Theorem 5 along with Remark 2. []

Theorem 3.6: If $\Gamma(R) \cong$ G having 8 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.6.

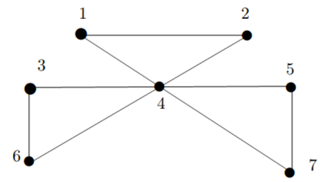

Figure 7: picture 7

No. Vertices	R ing (R)	$\|R\|$	Jraph G	$\boldsymbol{\gamma(G)}$	$\mathbf{im}_{\boldsymbol{d}}\mathbf{(G)}$
	$\mathbb{Z}_2 \times F_8$	16	$K_{1,7}$	1	7
	$\mathbb{Z}_3 \times \mathbb{Z}_7$	21	$K_{2,6}$	2	6
8	$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	$K_{4,4}$	2	4
	$\mathbb{Z}_{27}$	27	Fig. 3.8		
	$\mathbb{Z}_9[x]/\left(3x, x^2-3\right)$	27	Fig. 3.8	1	7

Figure 3.7
Case (g). When $\Gamma(R) \cong K_7$, it is special case of complete graph K_n, with $n = 7$. By Remark 2, $\gamma(\Gamma(R)) = 1$. On another hand, $\operatorname{Dim}_d(\Gamma(R)) = 6$, by using Theorem 5 along with Remark 2. []

Theorem 3.6: If $\Gamma(R) \cong$ G having 8 vertices then $\gamma(G)$ and $\operatorname{Dim}_d(G)$ are given in the Table 3.6.

No. Vertices	R ing (R)	$\|R\|$	Jraph G	$\boldsymbol{\gamma(G)}$	$\mathbf{im}_{\boldsymbol{d}}\mathbf{(G)}$
	$\mathbb{Z}_2 \times F_8$	16	$K_{1,7}$	1	7
	$\mathbb{Z}_3 \times \mathbb{Z}_7$	21	$K_{2,6}$	2	6
8	$\mathbb{Z}_5 \times \mathbb{Z}_5$	25	$K_{4,4}$	2	4
	$\mathbb{Z}_{27}$	27	Fig. 3.8		
	$\mathbb{Z}_9[x]/\left(3x, x^2-3\right)$	27	Fig. 3.8	1	7

able 3.6: ZD-graphs with 8 vertices
Proof. Case (a). When $\Gamma(R) \cong K_{1,7}$, it is a special case of bipartite graph, and we can consider $K_{1,7}$, as a star graph S_n, with $n = 8$. By Remark 3, $\gamma(\Gamma(R)) = 1$. On another hand, $\operatorname{Dim}_d(\Gamma(R)) = 7$, by using Remark 3 along with Theorem 2. On another hand, when $\Gamma(R) \cong K_{2,6}$, it is a special case of bipartite graph with $m = 2\&n = 6$. By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\operatorname{Dim}_d(\Gamma(R)) = 6$, by using Remark 4 along with Theorem 3. On another hand when $\Gamma(R) \cong K_{4,4}$, it is a special case of bipartite graph with $m = 4\&n = 4$, By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\operatorname{Dim}_d(\Gamma(R)) = 4$, by using Theorem 3 along with Remark 4 .

Case (b). The ZD-graph for ring $\mathbb{Z}_{27}, \mathbb{Z}_9[x]/\left(3x, x^2-3\right), \mathbb{Z}_9[x]/\left(3x, x^2-6\right), \mathbb{Z}_3[x]/\left(x^3\right)$ is given in Fig. 3.8. From the figure, the set $\{7\}$ is a dominating set and resolving set $\{1,2,3,4,5,7\}$ for graph $\Gamma(R)$, hence $\gamma(\Gamma(R)) = 1$ and $\operatorname{Dim}_d(\Gamma(R)) = 7$.

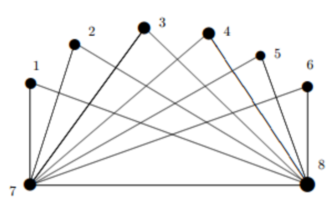

Figure 8: picture 8

Figure 3.8
Case (c). When $\Gamma(R) \cong K_8$, it is special case of complete graph K_n, with $n = 8$. By Remark 2, $\gamma(\Gamma(R)) = 1$. On another hand, $\operatorname{Dim}_d(\Gamma(R)) = 7$, by using Theorem 5 along with Remark 2. []

Theorem 3.7: If $\Gamma(R) \cong$ G having 9 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.7.

No. Vertices	R ing (R)	$\|\boldsymbol{R}\|$	Graph G	$\gamma(\boldsymbol{G})$	) im d (G)
	$\mathbb{Z}_2 \times F_9$	18	$K_{1,8}$	1	8
	$\mathbb{Z}_3 \times F_8$	24	$K_{2,7}$	2	7
9	$F_4 \times \mathbb{Z}_7$	28	$K_{3,6}$	2	7
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	12	Fig. 3.9	3	6
	$\mathbb{Z}_4 \times F_4$	16	Fig. 3.10	2	6
	$\mathbb{Z}_2[x]/\left(x^2\right) \times F_4$	16	Fig. 3.10		

Proof. Case (a). When $\Gamma(R) \cong K_{1,8}$, it is a special case of bipartite graph, and we can consider $K_{1,8}$, as a star graph S_n, with $n=9$. By Remark 3, $\gamma(\Gamma(R))=1$. On another hand, $\mathrm{Dim}_d(\Gamma(R))=8$, by using Remark 3 along with Theorem 2. On another hand, when $\Gamma(R) \cong K_{2,7}$, it is a special case of bipartite graph with $m=2 \& n=7$. By Remark 4, $\gamma(\Gamma(R))=2$. On another hand, $\mathrm{Dim}_d(\Gamma(R))=7$, by using Remark 4 along with Theorem 3. On another hand when $\Gamma(R) \cong K_{3,6}$, it is a special case of bipartite graph with $m=3 \& n=6$, By Remark 4 , $\gamma(\Gamma(R))=2$. On another hand, $\mathrm{Dim}_d(\Gamma(R))=7$, by using Theorem 3 along with Remark 4 .

Case (b).The ZZD-graph for ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ is given in Fig. 3.9. From the figure, the set $\{5,6,8\}$ is a dominating set and resolving set $\{1,2,3,4,7\}$ for graph hence $\gamma(\Gamma(R))=3$ and $\mathrm{Dim}_d(\Gamma(R))=6$.

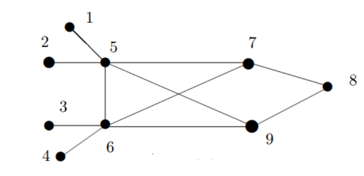

Figure 9: picture 9

Figure 3.9

Case (c). The ZD-graph for ring $\mathbb{Z}_4 \times F_4), \mathbb{Z}_2[x]/\left(x^2\right) \times F_4$ is given in Fig. 3.10. From the figure, the set $\{4,6\}$ is a dominating set and resolving set $\{1,2,3,5,7,9\}$ for graph $\Gamma(R)$, hence $\gamma(\Gamma(R))=2$ and $\mathrm{Dim}_d(\Gamma(R))=6$. []

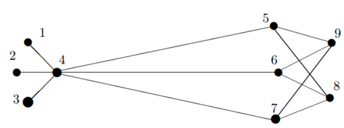

Figure 10: picture 10

Figure 3.10

Theorem 3.8: If $\Gamma(R) \cong$ G having 10 vertices then $\gamma(G)$ and $\mathrm{Dim}_d(G)$ are given in the Table 3.8.

No. Vertices	R ing (R)	$\|\boldsymbol{R}\|$	Graph G	$\gamma(\boldsymbol{G})$	$\mathrm{Dim}_{\boldsymbol{d}}(\boldsymbol{G})$
	$F_4 \times F_8$	32	$K_{3,7}$		
	$\mathbb{Z}_5 \times \mathbb{Z}_7$	35	$K_{4,6}$		
	$\mathbb{Z}_{11}[x]/\left(x^2\right)$	121	K_{10}		

Table 3.8: ZD-graphs with 10 vertices

Proof. Case (a). When $\Gamma(R) \cong K_{2,8}$, it is a special case of bipartite graph with $m=2 \& n=8$. By Remark 4, $\gamma(\Gamma(R))=2$. On another hand, $\mathrm{Dim}_d(\Gamma(R))=8$, by using Remark 4 along with Theorem

3. On another hand when $\Gamma(R) \cong K_{3,7}$, it is a special case of bipartite graph with $m = 3 \& n = 7$, By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 8$, by using Theorem 3 along with Remark 4. On another hand when $\Gamma(R) \cong K_{4,6}$, it is a special case of bipartite graph with $m = 4 \& n = 6$, By Remark 4, $\gamma(\Gamma(R)) = 2$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 8$, by using Theorem 3 along with Remark 4.

Case (c). When $\Gamma(R) \cong K_{10}$, it is special case of complete graph K_n, with $n = 10$. By Remark 2, $\gamma(\Gamma(R)) = 1$. On another hand, $\mathrm{Dim}_d(\Gamma(R)) = 9$, by using Theorem 5 along with Remark 2. []

Declaration

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- Author's contribution: All the authors equally contributed towards this work.

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