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Fuzzy Random Variables and Transforms: A Modern Perspective on Signal Processing

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ABSTRACT: A new idea for the signal, system, and transformation of control and signal processing systems is presented in this work. The fundamental mathematical properties of the $\mathcal{Z}$-transform for discrete time signals (DTS) and the $\mathcal{L}$-transform for continuous time signals (CTS), as well as several unique applications and working techniques, are examined in this article. To define the fuzzy randomised signal and calculate its lifespan, we applied the idea of fuzzy sets. The likelihood ratio, mean residual life, mean residual life order, and other significant reliability operators are also discovered, along with a large number of stochastic orders of the CTS and DTS. Lemmas and dependability theorems were illustrated with the concept of fuzzy randomised signal. Lastly, the significance of determining the lifespan of the signal was elucidated in this essay.

Key Words: $\mathcal{Z}$-transforms, $\mathcal{L}$-transforms, continuous signal, discrete signal, Hazard rate, mean residual life, mean inactivity time.

Contents

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1 Introduction

A signal is any substance that transmits or transfers information. Any signal can be classified as continuous or discrete based on its number of sources, dimensions, and other characteristics. We continuously state analogue signals for all values of the independent variable. An analogue signal is referred to as a continuous-time signal when time is its independent variable. The most commonly employed independent variable in signals is time, denoted by the letter "t". Current researchers deduced the use of signals in current trends by both military and civilian communication applications widely employ frequency hopping signals due to their low likelihood of interception [15,17,23,28,32]. The identification of multi-source signals from the wide-band spectrum without any prior signal information and the identification of each hop from the received signal mixture to the related network stations are two difficult and ongoing problems in the field of signal recognition and processing. Discrete signals are those that have discrete intervals of the independent variable defined in them. A discrete signal is called a discrete time signal when time is its independent variable. Before discussing the signal processing and control system, you must be an expert in it. A system is any apparatus or process that takes an input signal and outputs it. The characteristics of the system dictate the connection between the input and output signals. A system's inputs and outputs are known as signals. These could be any physical quantity, such as voltage, current, pressure, temperature, or even information, that varies over time. The primary points of contact between a system and a signal are the input-output relationship, system features, signal processing, and system representation. Systems can be represented using a variety of models, including as transfer functions, state-space representations, and block diagrams. The structure, connections, and interactions between the input and output signals of the system are all made easier to understand with the aid of these representations. The systems engineering (SE) community has long called for a theory of SE to place it as a stand-alone engineering discipline that can handle contemporary engineering challenges [27], [22]. Unfortunately, several SE techniques are unclear due to a lack of clear formalism and differentiation for key core ideas within the area [40]. In the past, the existence of new types of complex systems, including AI-enabled systems, may not have required such formalism, but the rise of these systems has put traditional techniques under pressure [7,24,25,34,35,36].

The connection between a system and a signal is intimately linked to the idea of transformation. The process of transforming an input signal into an output signal through a system's activity is known as transformation in the context of signal processing and systems. Signals can be altered using various techniques, including frequency-domain, time-domain, invertible, non-linear, and frequency-domain transformations. It is critical to comprehend the signal transformations that can be applied to signals in system design and analysis to control and analyse signals to yield the intended results. Next on our list of subjects is a discussion of signal transformations. Signal transformations are mathematical procedures that change a signal's representation, generally to make processing or analysis easier. Various commonly employed signal transformations consist of the Fourier transform, Laplace transform, $\mathcal{Z}$-transform, wavelet transform, Hilbert transform and wavelet transforms. The properties of the signal and the current analysis or processing task determine the precise transform required. Laplace lays the cornerstone of operational analysis and Fourier transforms. This discipline of mathematics has a wide range of applications in applied mathematics and other sciences like physics, engineering, astronomy, nonlinear occurrences, and mathematical approaches. Mathematicians have been interested in designing new integral transforms in the preceding few years. In the present era, numerous writers have been pulled to the expansion of this mathematical patience to investigate various issues in the sciences and daily life. New integral transforms such as the Sumudu transform (1993), Natural transform (2008), Elzaki transform (2011), Aboodh transform (2013), Tarig transform (2013), Srivastava transform (2015), Kamal transform (2016), Maghroub transform (Laplace–Carson) (2016), Mohand transform (2017), Shuhu transform (2019), and Sawi transform (2019) are among those that share this common goal. Recently, researchers have undertaken numerous studies to further explain the purposes and associations between certain transforms.

A few fundamental concepts of signals and systems are clearly discussed in [3,9]. An essential scientific tool for simulating sample-data control networks or other discrete-data systems is the $\mathcal{Z}$-transform. It is very interesting to examine the $\mathcal{Z}$-transform from non-classical perspectives using one of the numerous contemporary theories that currently offer a vast array of distinct infinities and infinitesimals, given that it is defined as a product of an infinite number of addends. [4] Suggested a new, straightforward applied method that was most recently presented by Y.D. Sergeyev and that provides it possible to perform mathematical operations using various sizes of infinite and infinitesimal integers with ease. In contrast to traditional analysis, where the bilateral $\mathcal{Z}$-transform frequently does not exist anywhere, this new method allows us to obtain a very different type of $\mathcal{Z}$-transform of a complex sequence (or, more accurately, a family of infinitely many $\mathcal{Z}$-transforms attached to the same sequence) whose existence is guaranteed almost everywhere on C. The method and technology employed to calculate the PTF of discrete digital filters are explained in Ciulla C [5]. The proof of concept has been effectively confirmed. This work introduces PTF-based two-dimensional $\mathcal{Z}$-space filtering. The focus of Chen B, Liu X, Liu K, and Lin C [6] is on adaptive fuzzy tracking control for certain types of nonlinear systems that have one input and one output. Unknown and desired control signals are directly simulated using fuzzy logic systems, and a unique direct adaptive fuzzy tracking controller is built via backstepping. The fundamental idea of the Laplace transforms its uses in signal processing and control systems, and the engineering applications of Laplace transforms were covered by Graf U [12]. Goel N and Singh K [13] present a simplified and easy-to-understand approach for a modified convolution and product theorem in the LCT domain, derived via a quantum mechanical representation transformation. When compared to the current convolution theorem for the LCT, it is discovered to be a more appropriate and superior proposal. Furthermore, we provide a filtering application using the generated results. After defining Huo H [14], we first present a novel idea of the canonical convolution operator and demonstrate its commutative, associative, and distributive characteristics, which might prove extremely beneficial in signal processing. Furthermore, we demonstrate that the new canonical convolution operator, when linked to the LCT, also satisfies the generalised Young's condition and the generalised convolution theorem. The proposed solution by Li Y, Li T, and Jing X [21] is studied through using fuzzy logic systems for dealing with unknown nonlinear functions and integrating adaptive fuzzy control design with the adaptive backstepping technique. In particular, in the closed-loop system, the number of online learning parameters is decreased to 2n. A generalised convolution theorem for the LCT was proposed by Shi J., Liu X., and Zhang N. [14], who also developed a similar product theorem related to the LCT. The ensuing results are shown to be special instances of the ordinary convolution theorem for the FT, the fractional convolution theorem for the FRFT, and several of the existing convolution theorems for the LCT. Additionally, a few uses for the deduced outcomes are showcased. The results of reduced adaptive fuzzy systems are found by Sun W, Su S, Wu Y, and Xia J [33]; these results are used to estimate the unknown function, which contains all of the system's state variables and guarantees that the back-stepping design approach for nonlinear systems with non-strict feedback functions normally. A novel convolution structure for the LCT is introduced by Wei Ran, Q Li, and Ma [38]. It maintains the convolution theorem for the Fourier transform and is simple to apply when creating filters. We demonstrate that our obtained conclusions are special instances of some of the well-known theorems regarding the convolution theorem in the fractional Fourier transform (FRFT) domain and the FT domain. Using an adaptive back-stepping control method, Wu C, Liu J, Jing X, Li H, and Wu L [39] tackle the issue of adaptive fuzzy control for a set of single-input, single-output nonlinear networked control systems that experience delays and data loss due to the network. Adaptive fuzzy tracking control for high-order nonlinear time-delay systems with full-state restrictions and input saturation is studied by Wu, Y., and Xie, X. [41]. A fuzzy approximation technique eliminates the often used growth assumptions on unknown system nonlinearities. Ze Ai, Huanqing Wang, and Haikuo Shen [42] study the issue of adaptive fuzzy fixed-time tracking control based on a class of nonlinear systems with unmodeled dynamics and dynamic disturbances. Using the idea of calculus, Jirakulchaiwong S, Nonlaopon K [16] constructed the analogues of Laplace-type integral transformations. Additionally, several features of the analogues of Laplace-type integral transforms are explored and used in the solution of a few differential equations. A new technique for solving linear and nonlinear non-homogeneous dispersive Korteweg–de Vries type equations is presented by Eltayeb H, Alhefthi R [2]. It is named the Sumudu–generalised method for Laplace transform decomposition. By combining the Adomian decomposition method and the Sumudu–generalised Laplace transform, it offers a potent method for solving complicated equations. Eltayeb H [11] examined how general two-dimensional singular pseudo-hyperbolic equations, subject to the initial conditions, can be solved using the technique developed by the double Sumudu transform when combined with a novel generalised Laplace transform decomposition method, referred to as the double Sumudu-generalised Laplace transform decomposition method. An approach to numerically modelling differential equations of fractional order is developed by Kamran Khan and Haque S. [18]. This approach is based on the Laplace transform and the numerical inverse Laplace transform. Eltayeb H and Alhefthi R [10] emphasized the importance of the G-Laplace transform. The G-transform varies from the Laplace, Sumudu, and Elzaki transforms because it can deal with an extensive range of differential equations having non-constant variables. As a result, it is a successful instrument for analysing differential equations with variable coefficients. The study provides reliable and precise solutions for homogeneous and non-homogeneous coupled Burgers' equations by utilising this groundbreaking approach. The concepts of Sumudu and Laplace transforms, both first and second kind, in quantum calculus are clarified for Tassaddiq A, Bhat A, and everyone [37] by using functions with multiple variables. The universal nature of the Aleph function's analogue has enabled the achievement of several new and renowned results for these transformations. Sadjang P N [28] introduces two Laplace transforms, explaining and demonstrating their respective features. Applications involve the solution of various linear difference equations. The definitions of Sumudu and Laplace transforms of the first and second sort in quantum calculus are introduced by Tassaddiq A., Bhat A.A., and others [37], utilising functions of many variables. Numerous fresh and well-known results for these transformations were achieved because of the universal nature of the analogue of the Aleph function. Furthermore, given these transformations, we derive several intriguing identities and correlations.

Discrete systems cannot be investigated using the standard Laplace or Fourier transforms because they require continuous functions; alternatively, they can be easily described using the $\mathcal{Z}$-transform. The $\mathcal{L}$-transform was first proposed by Laplace and then developed by Hurewicz as a controllable method of solving linear and constant-coefficient differential equations. One scientific method for resolving and converting differential equations is the Laplace transformation. In broad terms, solving ordinary or partial linear differential equations is efficient. It simplifies to an ordinary distinction equation in algebra. Regular linear differential equations with constant and variable coefficients can be readily solved using the Laplace transformation method without determining the complete solution or the arbitrary factor. Physical challenges employ this method. An integral and normal equation of differences is involved in this. Additionally, it can be applied as a quick and simple method to convert the signal network into a frequency domain. Along with the fundamental sciences and mathematics, it finds extensive use in various engineering and technological domains.
The $\mathcal{L}$-transform of a continuous time signal $\mathcal{X}_n(t)$ is given by the equation,

$$\mathcal{L}\{\mathcal{X}_n(t)\} = \mathcal{X}_t(\mathcal{L}) = \int_{-\infty}^{\infty} e^{-sn}\mathcal{X}_t(n)\, dn$$

The $\mathcal{Z}$-transform is one of the most popular mathematical methods for converting data sequences from the domain to the complex frequency domain, usually in discrete time. Originally conceived by W. Hurevicz (1947), the $\mathcal{Z}$-transform was subsequently designated as such in 1952 by the Colombia University sampled data control group, which included L.A. Zadeh, E.I. Jury, R.E. Kalman, and others and was supervised by Professor John R. Raggazini. It is essentially a discrete Fourier transform (DFT) extension, but it has a few significant differences that make it very beneficial for digital signal processing. The independent variable, which stands for the complex frequency domain, is the source of the name "$\mathcal{Z}$-transform". Simplifying the analysis and design of digital systems is its primary goal. One effective tool for digital systems is the $\mathcal{Z}$-transform. It is especially helpful for the analysis of linear time invariant (LTI) systems, which are extensively employed in audio processing, control systems, and telecommunication, among other fields. The $\mathcal{Z}$-transform provides a way to analyse the frequency response of a digital system, which is necessary for understanding its behaviour and performance. From a strictly mathematical standpoint, the $\mathcal{Z}$-transform is essentially a power series representation of a discrete-time sequence, and so it exists when the series converges. However, because it is defined as the sum of an infinite number of items, it becomes very interesting to consider the $\mathcal{Z}$-transform in relation to atypical numerical systems and non-standard theories that admit different types of countable (and uncountable) infinities and, recently, are experiencing a new impetus.

The -transform of a discrete-time signal $\mathcal{X}_t(n)$ is given by the equation,

$$\mathcal{Z}\{\mathcal{X}_t(n)\} = \mathcal{X}_t(\mathcal{Z}) = \sum_{n=-\infty}^{\infty} \mathcal{X}_t(n)\mathcal{Z}^{-n}$$

The numerator of the equation is the total of all the signal values scaled by z raised to the power of their index, where the complex frequency is represented by the variable. We may use this method to obtain the complex-valued function (z), which allows us to examine the signal's characteristics in the frequency domain.

The $\mathcal{Z}$-transform and $\mathcal{L}$-transform are helpful in digital signal processing because of their many significant features. The transform of a linear combination of signals is equal to the linear combination of each signal's unique transforms, making linearity one of the most significant characteristics. Because of this characteristic, complicated systems can be easily analysed by disassembling them into more manageable parts. The Laplace transform, which is used to convert continuous-time data from the time domain to the complex frequency domain, is connected to the transform. A discrete-time signal's -transform can be derived by evaluating the signal's Laplace transform at a particular point in s-space and replacing that result with z. A potent tool for analysing continuous-time systems in many branches of research and engineering is the Laplace transform. It offers a method for examining a system's behaviour in the frequency domain, which is crucial for comprehending its stability and performance. By utilising the relationship between the $\mathcal{Z}$ and Laplace transforms, we may do unified analyses of both discrete- and continuous-time systems. Digital signal processing uses the transform, a mathematical tool, to translate discrete-time signals into the frequency domain. With the help of this effective instrument, we may process signals by designing digital filters based on their frequency content analysis. The transform and the $\mathcal{L}$-transform are helpful tools because of their many qualities. The basic idea of this article comes from the convolution theorem. A generalised convolution theorem for the LCT was proposed by Shi J, Liu X, and Zhang N [30], who also derived a comparable product theorem related to the LCT. The consequent results are shown to be special situations of the ordinary convolution theorem for the FT, the fractional convolution theorem for the FRFT, and a number of the existing convolution theorems for the LCT. The convolution theorem states that the transform of the convolution of two functions is equal to the product of the Fourier transforms of the individual functions. Mathematically, the convolution theorem can be stated as follows: Let $f(t)$ and $g(t)$ be two functions, and let $F(\omega)$ and $G(\omega)$ be their respective Fourier transforms. Then, the Fourier transform of the convolution of $f(t)$ and $g(t)$, denoted as $(f * g)(t)$, is equal to the product of $F(\omega)$ and $G(\omega)$: $F\{(f * g)(t)\} = F(\omega) \cdot G(\omega)$ This means that convolution in the time domain corresponds to multiplication in the frequency domain, and vice versa.. The convolution theorem is widely used in various fields, such as signal processing, image processing, and linear systems theory, as it provides a powerful tool for simplifying and analysing convolution operations. This allows us to apply the stochastic order application to the randomised signal with ease.

Here's the structure of the paper: In Section 2, we explore $\mathcal{Z}$ and $\mathcal{L}$-transforms' mathematical elements and usage. The previously transformed has multiple uses in communications and control networks, which are explored in Section 3. The fourth segment examines the usage of the $\mathcal{Z}$-transform in control and signal processing systems. Part 5 shows an innovative use of the $\mathcal{Z}$-transform for fuzzy random variables and stochastic orders. The conclusion presents an overview of the primary facts.

2 Mathematical Properties and Applications of $\mathcal{Z}$ and $\mathcal{L}$- transforms

2.1 Linearity

The linearity property of the $\mathcal{Z}$-transform is one of the most important properties. It states that the $\mathcal{Z}$-transform is linear, which means that if we have two signals $\mathcal{X}_t(n)$ and $\mathcal{Y}_t(n)$ and their corresponding $\mathcal{Z}$-transforms $\mathcal{Z}\{\mathcal{X}_t(n)\}$ and $\mathcal{Z}\{\mathcal{Y}_t(n)\}$, then the $\mathcal{Z}$-transform of their sum

$$\mathcal{Z}\{\mathcal{X}_t(n) + \mathcal{Y}_t(n)\} = \mathcal{Z}\{\mathcal{X}_t(n)\} + \mathcal{Z}\{\mathcal{Y}_t(n)\}$$

is simply the sum of their individual $\mathcal{Z}$-transforms. This property is useful because it allows us to break down a complex signal into simpler signals that can be analyzed separately.

2.2 Time Shifting

The $\mathcal{Z}$-transform's time-shifting property states that a signal's $\mathcal{Z}$-transform is multiplied by Z^{-k} if it is shifted by a specific amount, k. This applies to signals $\mathcal{X}_t(n)$. This characteristic is helpful because it enables us to examine how variations in time impact the signal's frequency domain. For instance, a signal's frequency content will shift to the left if we move it to the right in time.

2.3 Scaling

According to the $\mathcal{Z}$-transform's scaling property, a signal's $\mathcal{Z}$-transform is multiplied by scalar α if it is multiplied by $\mathcal{X}_t\{1, 2, 3, ..., n\}$, a scalar α. This feature is helpful since it lets us modify the signal's amplitude in the frequency domain. For instance, we may easily multiply a signal by a scalar in the time domain to increase a certain frequency component of the signal. This will multiply the corresponding frequency component in the frequency domain.

2.4 Reversing Time

According to the $\mathcal{Z}$-transform's time reversal property, the $\mathcal{Z}$-transform of $\mathcal{X}_t\{1, 2, 3, ..., n\}$ is just mirrored along the complex plane if the samples' order is reversed. When evaluating symmetric signals and systems, this characteristic is helpful. For instance, we can make use of this attribute to streamline our analysis if our signal is symmetric about its middle.

2.5 Convolution

The $\mathcal{Z}$-transform of the convolution of two signals is just the product of their respective $\mathcal{Z}$-transforms, $K(\mathcal{Z})$ and $H(\mathcal{Z})$, according to the convolution property of the $\mathcal{Z}$-transform. When examining the frequency response of linear time-invariant systems, this characteristic is helpful. This feature can be used, for instance, to determine the frequency response of a system whose linear constant-coefficient-difference equation describes it. The above properties are satisfied for the $\mathcal{L}$-transform also.

3 Applications of Z-Transform

The $\mathcal{Z}$-transform is a mathematical tool used to analyze and design discrete-time systems. It has a wide range of applications in various fields such as signal processing, control systems, and digital filters. Let's dive deeper into some of these applications.

3.1 Signal Processing

Analyzing and modifying signals with digital methods is called digital signal processing. An important tool in digital signal processing is the $\mathcal{Z}$-transform. Digital filters, which are employed for signal augmentation, noise reduction, and other purposes, are analyzed and designed using this from a signal. We can evaluate a filter's frequency response and identify contemporary signal processing uses by using the $\mathcal{Z}$-transform.

3.2 Control Systems

Digital control system design and analysis also make use of the $\mathcal{Z}$-transform. Numerous industries, such as robotics, industrial automation, and aerospace, use digital control systems. With the $\mathcal{Z}$-transform, we may mimic a system's behavior in the frequency domain, examine a digital control system's stability, and create controllers with certain features. This creates and applies digital control mechanisms.

3.3 Digital Filters

$\mathcal{Z}$-transforms are frequently used in digital filters. They are employed to boost desired frequencies or eliminate undesired ones from a transmission. The $\mathcal{Z}$-transform can be used to create filters with the required frequency response characteristics. The $\mathcal{Z}$-transform, for instance, can be used to examine the stability of the filter's low pass response. Because of this, it's a crucial tool for creating and using digital filters.

3.4 System Stability Analysis

In many applications, a system's stability is a crucial factor to take into account. If the output of a system stays bounded when the input is bounded, the system is stable. Digital control systems and digital filters are examples of discrete-time systems whose stability can be examined using the $\mathcal{Z}$-transform. The transfer function's poles in the z-plane provide the basis for the first stability requirement. Designing and implementing stable systems requires the ability to examine a system's stability in the frequency domain, which the $\mathcal{Z}$-transform enables.

3.5 How $\mathcal{Z}$-transform working in signal processing and control

The $\mathcal{Z}$-transform plays a fundamental role in signal processing and control systems by providing a powerful tool for analyzing and manipulating discrete-time signals and systems. Here's how the $\mathcal{Z}$-transform works in these domains:

Signal representation:

We can express a discrete-time signal as a function of the complex variable z using the $\mathcal{Z}$-transform. The $\mathcal{Z}$-transform $\mathcal{X}_t(\mathcal{Z})$ of a discrete-time signal $\mathcal{X}_t(n)$ is defined as the product of $\mathcal{X}_t(n)$ and $\mathcal{Z}^{-n}$ where n is the discrete time index. Analyzing the frequency content and signal behavior in the z-domain is possible with the $\mathcal{Z}$-transforms.

Frequency analysis:

We can examine a discrete-time signal's frequency characteristics using the $\mathcal{Z}$-transforms. We can ascertain the frequency response of the signal by assessing the $\mathcal{Z}$-transform at different locations on the complex plane. This aids in our comprehension of the signal's behavior at various frequencies and makes it possible for us to create systems and filters that alter or extract particular frequency components.

System analysis and transfer functions:

The $\mathcal{Z}$-transform can be used to analyze the frequency characteristics of a discrete-time signal. By evaluating the $\mathcal{Z}$-transform at various points on the complex plane, we may determine the signal's frequency response. This helps us understand how the signal behaves at different frequencies and enables us to design systems and filters that modify or extract specific frequency components.

System design and frequency domain techniques:

We can create digital filters and systems in the discrete-time domain thanks to the $\mathcal{Z}$-transform. We can create filters with desired frequency responses, such as low-pass, high-pass, or band-pass filters, by modifying the $\mathcal{Z}$-transform representation of a system. Additionally, the $\mathcal{Z}$-transform makes it easier to apply frequency domain methods to comprehend the stability and behavior of discrete-time systems, such as pole-zero analysis.

Inverse $\mathcal{Z}$-transform:

A discrete-time signal can be recovered from its $\mathcal{Z}$-transform representation using the inverse $\mathcal{Z}$-transform, in the same way that a continuous-time signal can be recovered from its frequency representation using the inverse Fourier transform. We can recover the original discrete-time signal by going back from the $\mathcal{Z}$-domain to the time domain using the inverse $\mathcal{Z}$-transform.

Transfer function manipulation:

Transfer functions in the $\mathcal{Z}$-domain can be algebraically manipulated thanks to the $\mathcal{Z}$-transform. This enables the construction of controllers through the use of mathematical operations like addition, multiplication, and convolution, as well as the combining of various systems and system analysis. Complex control system design and analysis are made possible by these activities. To sum up, in signal processing and control systems, the $\mathcal{Z}$-transform offers a mathematical foundation for the analysis, design, and manipulation of discrete-time signals and systems. In addition to facilitating the application of numerous mathematical techniques in the $\mathcal{Z}$-domain, it permits frequency analysis, system representation, and system design.

4 Fresh Discoveries

To improve precision and lucidity, we will incorporate fuzzy logic into the $\mathcal{Z}$-transforms. We introduce a novel use of $\mathcal{Z}$-transforms for stochastic processes and fuzzy random variables.

4.1 Min and Max function ($\wedge \& \vee$)

The min and max functions are used in signal processing and control systems to find the lowest and highest values of a group of signals or variables. These functions are crucial for a number of tasks, including determining signal limits, identifying outliers, and locating peak values. The min and max functions in signal processing and control systems can be used in the following ways

Definition 4.1 ***$\mathcal{Z}$-transform of discrete fuzzy randomized signals***
Let $\mathcal{X}_t(\mathcal{Z})$ be a set of all signal values. Next, a fuzzy randomized signal $\tilde{A} = \{(x_t, \mu_A(x_t)) \,|\, x_t \in \mathcal{X}_t(\mathcal{Z})\}$ of $\mathcal{X}_t(\mathcal{Z})$ is determined by the role it plays in the membership function $\mu_A : \mathcal{X}_t(\mathcal{Z}) \to [t^L_\alpha, t^U_\beta]$. Let $\{x_t[n^L_\alpha, n^U_\beta]\}$ be the set of discrete fuzzy randomized signals and $\mathcal{X}_t(\mathcal{Z})$ be the $\mathcal{Z}$-transform of $\{x_t[n^L_\alpha, n^U_\beta]\}$. Then, the $\mathcal{Z}$-transform of the discrete signal $\{x_t[n^L_\alpha, n^U_\beta]\}$ is defined as

$$\mathcal{Z}\{x_t[n^L_\alpha, n^U_\beta]\} = \mathcal{X}_t(\mathcal{Z}) = \bigwedge_{0\leq\alpha\leq\beta\leq 1} \& \bigvee \sum_{n=-\infty}^{\infty} x_t[n^L_\alpha, n^U_\beta]\mathcal{Z}^{-[n^L_\alpha, n^U_\beta]}$$

A signal with a discrete range of defined amplitude values at all times. The discrete time signal $\{x_t[n^L_\alpha, n^U_\beta]\}$ in the equation that proceeds has been determined to be two-sided, and the transform is named the two-sided $\mathcal{Z}$-transform since the time index n is defined for both positive and negative values. One-sided $\mathcal{Z}$-transform is the $\mathcal{Z}$-transform that is applied to the signal $\{x_t[n^L_\alpha, n^U_\beta]\}$ if it is a one-sided signal. The following figure clearly shows the $\mathcal{Z}$-transform of discrete fuzzy randomized signals.

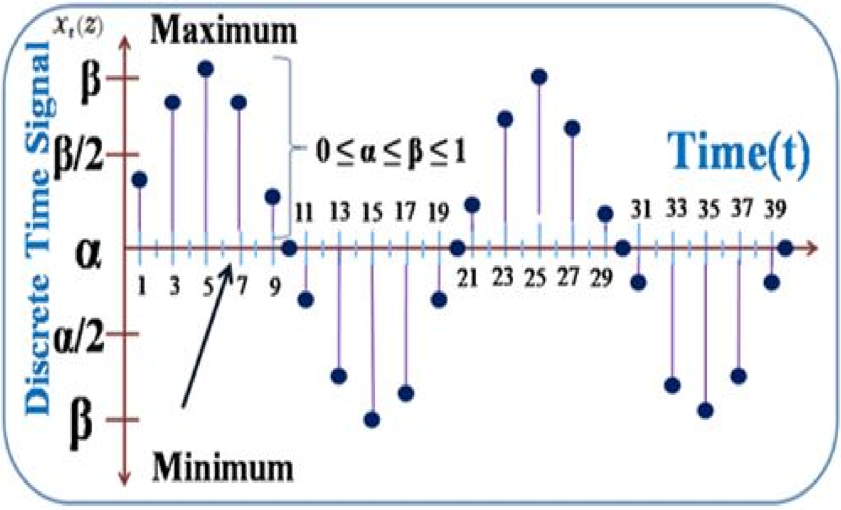

Figure 1: $\mathcal{Z}$-transform of discrete fuzzy randomized signals

Definition 4.2 ***$\mathcal{L}$- transform of continuous fuzzy randomized signals***
Let $\mathcal{X}_t(\mathcal{Z})$ be a set of all signal values. Next a fuzzy randomized signal $\tilde{A} = \{(x_t, \mu_A(x_t))/x_t \in \mathcal{X}_t(\mathcal{Z})\}$ of $\mathcal{X}_t(\mathcal{Z})$ is determined by the role it plays in membership function $\mu_A : \mathcal{X}_t(\mathcal{Z}) \to [t^L_\alpha, t^U_\beta]$ and let $\{x_n[t^L_\alpha, t^U_\beta]\}$ be set of continuous time fuzzy randomized signal and $\mathcal{X}_n(\mathcal{L})$ be a $\mathcal{L}$-transform of $\{x_n[t^L_\alpha, t^U_\beta]\}$. Then the $\mathcal{L}$-transform of continuous time signal $\{x_n[t^L_\alpha, t^U_\beta]\}$ is defined as

$$\mathcal{L}\{x_n[t^L_\alpha, t^U_\beta]\} = \mathcal{X}_n(\mathcal{L}) = \bigwedge_{0\leq\alpha\leq\beta\leq1} \& \bigvee \int_{-\infty}^{\infty} e^{-s[t^L_\alpha, t^U_\beta]}\{x_n[t^L_\alpha, t^U_\beta]\}\, dt$$

A signal with a continuous range of defined amplitude values at all times. The continuous time signal $\{x_n[t^L_\alpha, t^U_\beta]\}$ in the equation that proceeds has been determined to be two-sided, and the transform is named two-sided $\mathcal{L}$-transform since the time index n is defined for both positive and negative values. One-sided $\mathcal{L}$-transform is the $\mathcal{L}$-transform that is applied to the signal $\{x_n[t^L_\alpha, t^U_\beta]\}$ if it is a one-sided signal.

The following figure is clearly shown about $\mathcal{L}$-transform of continuous fuzzy randomized signals.

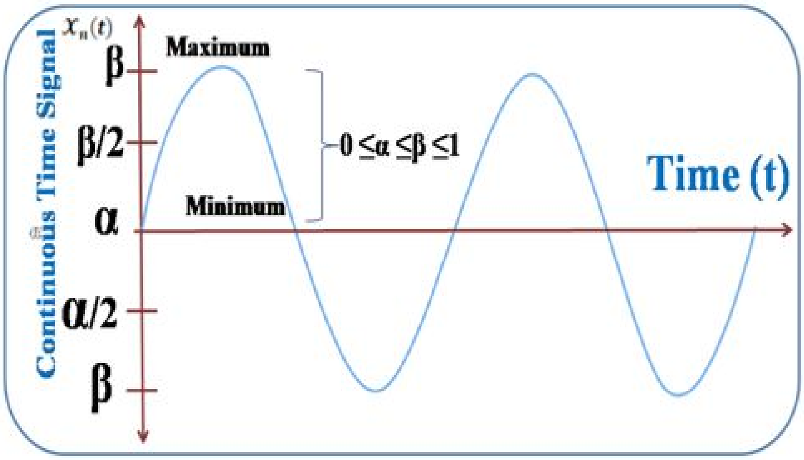

Figure 2: $\mathcal{L}$-transform of continuous fuzzy randomized signals

4.2 Life time of signals and systems

The term "signal and system life time" describes how long a signal lasts while it travels through a system. A signal's lifespan can be determined and defined in a few important ways: Up/Downtime, Period, Bandwidth, Propagation Delay, Simulation-Based Estimating, and Analytical Modeling. The type of signal, the test equipment that is available, and the needed level of precision all influence which approach is best. To completely characterize the signal lifetime in a system, a variety of methods are frequently applied. Determining a signal's lifetime is crucial for a number of reasons in electrical and electronic systems, including failure analysis, design optimization, power consumption, noise and electromagnetic interference (EMI), timing and synchronization, system performance, and signal integrity. This section's attention is on the causal connections that exist between RP order and other well-known stochastic orders. Furthermore, we go over the preservation qualities of the RP order under a few well-known dependability operations, including order data, combination, balanced distributions, and monotonic transformations. Before delivering our primary findings, we offer some definitions and key characteristics that will be used in the sequel for the sake of simplicity. Shaked and Shanthikumar [30], Di Crescenzo [8], and Lai and Xie [19] provide an extensive monograph on the definitions and characteristics of stochastic orders.

Definition 4.3 ***$\mathcal{Z}$-transform order of fuzzy discrete randomized signals*** *Let $\{x_t(n_\alpha^L, n_\beta^U)\}$ and $\{y_t(n_\alpha^L, n_\beta^U)\}$ be two sets of absolutely discrete-time non-negative fuzzy randomized signals with probability density functions f_t and g_t and survival functions $\overline{F}_t$ and $\overline{G}_t$. Then, the $\mathcal{Z}$-transform of $\{x_t(n_\alpha^L, n_\beta^U)\}$ is stochastically dominant over $\{y_t(n_\alpha^L, n_\beta^U)\}$, defined as*

$$\bigwedge_{0\le\alpha\le\beta\le1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \{x_t(n_\alpha^L, n_\beta^U)\}\, \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \le_{SD} \bigwedge_{0\le\alpha\le\beta\le1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \{y_t(n_\alpha^L, n_\beta^U)\}\, \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \quad (1)$$

The $\mathcal{Z}$-transform of $\{x_t(n_\alpha^L, n_\beta^U)\}$ is stochastically larger than $\{y_t(n_\alpha^L, n_\beta^U)\}$ and is denoted as $\mathcal{X}_t(\mathcal{Z}) \le_{HR} \mathcal{Y}_t(\mathcal{Z})$, and defined by

$$\bigwedge_{0\le\alpha\le\beta\le1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \left(\frac{f_t(\mathcal{X})}{\overline{F}_t(\mathcal{X})} \right) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \le_{HR} \bigwedge_{0\le\alpha\le\beta\le1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \left(\frac{g_t(\mathcal{Y})}{\overline{G}_t(\mathcal{Y})} \right) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \quad (2)$$

$\mathcal{Z}$-transform of $\{x_t(n_\alpha^L, n_\beta^U)\}$ is stochastically larger than $\{y_t(n_\alpha^L, n_\beta^U)\}$ and is denoted as $\mathcal{X}_t(\mathcal{Z}) \leq_{MRL}$ $\mathcal{Y}_t(\mathcal{Z})$, and defined by

$$\bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \left(\frac{\int_t^\infty \overline{F}_t(\mathcal{X})\, dX}{\overline{F}_t(\mathcal{X})} \right) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \leq_{MRL} \bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \left(\frac{\int_t^\infty \overline{G}_t(\mathcal{Y})\, dX}{\overline{G}_t(\mathcal{Y})} \right) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \tag{3}$$

$\mathcal{Z}$-transform of $\{x_t(n_\alpha^L, n_\beta^U)\}$ is stochastically larger than $\{y_t(n_\alpha^L, n_\beta^U)\}$ and is denoted as $\mathcal{X}_t(\mathcal{Z}) \leq_{MIT}$ $\mathcal{Y}_t(\mathcal{Z})$, and defined by

$$\bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \left(\frac{\int_t^\infty F_t(\mathcal{X})\, dX}{F_t(\mathcal{X})} \right) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \leq_{MIT} \bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \left(\frac{\int_t^\infty G_t(\mathcal{X})\, dX}{G_t(\mathcal{X})} \right) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \tag{4}$$

$\mathcal{Z}$-transform of $\{x_t(n_\alpha^L, n_\beta^U)\}$ is stochastically larger than $\{y_t(n_\alpha^L, n_\beta^U)\}$ and is denoted as $\mathcal{X}_t(\mathcal{Z}) \leq_{DO}$ $\mathcal{Y}_t(\mathcal{Z})$ and defined by, where $\mathcal{Y}_t(\mathcal{Z}) \geq 1$

$$\bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \overline{F}_t(\mathcal{X}) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \leq_{DO} \bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \Upsilon(\overline{G}_t(\mathcal{Y})) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \tag{5}$$

$\mathcal{Z}$-transform of $\{x_t(n_\alpha^L, n_\beta^U)\}$ is stochastically larger than $\{y_t(n_\alpha^L, n_\beta^U)\}$ and is denoted as $\mathcal{X}_t(\mathcal{Z}) \geq_{RPO}$ $\mathcal{Y}_t(\mathcal{Z})$, and defined by

$$\bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} f_t(\mathcal{X}) \overline{F}_t(\mathcal{X}) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \geq_{RPO} \bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} g_t(\mathcal{Y}) \overline{G}_t(\mathcal{Y}) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \tag{6}$$

$\mathcal{Z}$-transform of $\{x_t(n_\alpha^L, n_\beta^U)\}$ is stochastically larger than $\{y_t(n_\alpha^L, n_\beta^U)\}$ and is denoted as $\mathcal{X}_t(\mathcal{Z}) \leq_{PO}$ $\mathcal{Y}_t(\mathcal{Z})$, and defined by

$$\bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} f_t(\mathcal{X}-\mathcal{Y}) \overline{F}_t(\mathcal{X}-\mathcal{Y}) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\} \geq_{PO} 0.5 \tag{7}$$

Definition 4.4 *Let $x_t(n_\alpha^L, n_\beta^U)$ be the lifetime of a discrete-time non-negative fuzzy randomized signal with probability density function f_t and survival function $\overline{F}_t$. We say that the randomized signal $x_t(n_\alpha^L, n_\beta^U)$ is decreasing in mean residual life written as dmrl (imrl), whenever*

$$\bigwedge_{0\leq\alpha\leq\beta\leq 1}\bigvee \left\{ \sum_{n=-\infty}^{\infty} \left(\frac{\displaystyle\int_t^\infty \overline{F}_t(\mathcal{X})\, dX}{\overline{F}_t(\mathcal{X})} \right) \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \right\}$$

is decreasing (increasing) for $t \geq 0$, or equivalently if $\mathcal{X}_t^*(\mathcal{Z}) \leq_{HR}$ ($\geq_{HR}$) $\mathcal{X}_t(\mathcal{Z})$.

The above orderings are based on the basic concepts of the convolution theorem and its applications.

4.3 $\mathcal{Z}$-transform algorithm of discrete signal $\mathcal{X}_t(\mathcal{Z})$:

We know that

$$\begin{aligned}
\mathcal{Z}\{x_t(n_\alpha^L, n_\beta^U)\} &= \mathcal{X}_t(\mathcal{Z}) \\
&= \bigwedge_{0\le\alpha\le\beta\le 1}\bigvee \sum_{n=-\infty}^{\infty}\left(x_t[n_\alpha^L, n_\beta^U]\mathcal{Z}^{-[n_\alpha^L, n_\beta^U]}\right) \\
&= \sum_{n=1}^{\infty} x_t[n_\alpha^L, n_\beta^U]\frac{1}{\mathcal{Z}^{[n_\alpha^L, n_\beta^U]}} \\
&= \sum_{n=1}^{\infty} \frac{x_t[n_\alpha^L, n_\beta^U]}{\mathcal{Z}^{[n_\alpha^L, n_\beta^U]}} \\
&= 1 + \frac{x_t(n_\alpha^L, n_\beta^U)}{z} + \left(\frac{x_t(n_\alpha^L, n_\beta^U)}{z}\right)^2 + \left(\frac{x_t(n_\alpha^L, n_\beta^U)}{z}\right)^3 + \cdots \\
&= \left(\frac{z - x_t(n_\alpha^L, n_\beta^U)}{z}\right)^{-1}
\end{aligned}$$

$$\mathcal{Z}\{x_t(n_\alpha^L, n_\beta^U)\} = \left(\frac{z}{z - x_t(n_\alpha^L, n_\beta^U)}\right)^{-1}, \quad |z| > |x_t(n_\alpha^L, n_\beta^U)| \tag{20}$$

Example 4.1 *Find the $\mathcal{Z}$-transform of the sequence signal of the fuzzy random variable,*

$$\mathcal{X}_t(n_\alpha^L, n_\beta^U) = \{0.2, 0.4, 0.6, 0.8, 1\}$$

Solution: By the given condition $x_t(n_\alpha^L, n_\beta^U) = \{0.2, 0.4, 0.6, 0.8, 1\}$, we know that

$$\mathcal{Z}\{x_t(n_\alpha^L, n_\beta^U)\} = \mathcal{X}_t(\mathcal{Z}) = \bigwedge_{0\le\alpha\le\beta\le 1}\bigvee \sum_{n=-\infty}^{\infty} x_t[n_\alpha^L, n_\beta^U]\mathcal{Z}^{-[n_\alpha^L, n_\beta^U]}$$

$$\mathcal{X}_t(-3) = 0.2, \quad \mathcal{X}_t(-2) = 0.4, \quad \mathcal{X}_t(-1) = 0.6, \quad \mathcal{X}_t(0) = 0.8, \quad \mathcal{X}_t(1) = 1$$

$$\mathcal{Z}\{\mathcal{X}_t(0.2, 0.4, 0.6, 0.8, 1)\} = \bigwedge_{0\le\alpha\le\beta\le 1}\bigvee \sum_{n=-\infty}^{\infty}\left(\mathcal{X}_t(-3)\mathcal{Z}^3 + \mathcal{X}_t(-2)\mathcal{Z}^2 + \mathcal{X}_t(-1)\mathcal{Z}^1 + \mathcal{X}_t(0)\mathcal{Z}^0 + \mathcal{X}_t(1)\mathcal{Z}^{-1}\right)$$

$$\begin{aligned}
&= (0.2\cdot\mathcal{Z}^3) + (0.4\cdot\mathcal{Z}^2) + (0.6\cdot\mathcal{Z}^1) + (0.8\cdot\mathcal{Z}^0) + (1\cdot\mathcal{Z}^{-1}) \\
&= 0.2\mathcal{Z}^3 + 0.4\mathcal{Z}^2 + 0.6\mathcal{Z}^1 + 0.8\mathcal{Z}^0 + \mathcal{Z}^{-1}
\end{aligned}$$

Lemma 4.1 *Assume that δ is a real weighted measurement that is not always negative. Let ψ be a real-valued function that is not negative. If ψ is an increasing function and*

$$\int_t^{\infty} d\delta(\mathcal{X}_t(\mathcal{Z})) \ge 0 \quad \text{for every} \quad t \ge 0,$$

then

$$\int_0^{\infty} \psi(\mathcal{X}_t(\mathcal{Z}))d\delta(\mathcal{X}_t(\mathcal{Z})) \ge 0 \tag{*}$$

The first finding demonstrates the superior strength of the RPO over the dual order.

Theorem 4.1 *Let $\mathcal{X}_t(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_t(\mathcal{Z})$. Then, by the condition of residual probability order, we need to prove that*

$$\mathcal{X}_t(\mathcal{Z}) \leq_{DO} \mathcal{Y}_t(\mathcal{Z}).$$

Proof: For any $\mathcal{X}_t(\mathcal{Z}) \geq 0$, set (by Barlow and Proschan [7])

$$\delta(\mathcal{X}_t(\mathcal{Z})) = \int_X^\infty \left[\bar{F}_t(u)\bar{G}_t(u)\left(\frac{g_t(u)}{\bar{G}_t(u)} - \frac{f_t(u)}{\bar{F}_t(u)}\right)\right] du, \quad \text{for any } t \geq 0,$$

$$\int_t^\infty d\delta(\mathcal{X}_t) = \int_t^\infty \left[\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y})\left(\frac{f_t(\mathcal{X})}{\bar{F}_t(\mathcal{X})} - \frac{g_t(\mathcal{Y})}{\bar{G}_t(\mathcal{Y})}\right)\right] d(\mathcal{X}\mathcal{Y}).$$

According to remark 3 in Zardasht and Asadi [2],

$$\int_t^\infty d\delta(\mathcal{X}_t) \geq 0, \quad \text{for fixed } t \geq 0.$$

take

$$\psi(\mathcal{X}_t(\mathcal{Z})) = \begin{cases} \frac{1}{\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y})}, & \text{if } \mathcal{X}_t(\mathcal{Z}) > t, \\ 0, & \text{if } \mathcal{X}_t(\mathcal{Z}) \leq t. \end{cases}$$

We see that ψ is a non-negative and increasing function in $\mathcal{X}_t$, for any $t \geq 0$. So, because of the previous lemma, the non-negativity of

$$\int_0^\infty \psi(\mathcal{X}_t(\mathcal{Z}))d\delta(\mathcal{X}_t(\mathcal{Z}))$$

is guaranteed. That is,

$$\begin{aligned}\int_0^\infty \psi(\mathcal{X}_t(\mathcal{Z}))d\delta(\mathcal{X}_t(\mathcal{Z})) &= \int_t^\infty \left(\frac{1}{\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{X})}\right) d\delta(\mathcal{X}_t(\mathcal{Z})) \\ &= \int_t^\infty \left[\left(\frac{f_t(\mathcal{X})}{\bar{F}_t(\mathcal{X})} - \frac{g_t(\mathcal{Y})}{\bar{G}_t(\mathcal{Y})}\right)\right] d(\mathcal{X}\mathcal{Y}) \\ &= \ln\left[\lim_{n\to\infty}\left(\frac{\bar{G}_t(\mathcal{Y})}{\bar{F}_t(\mathcal{X})}\right)\right] - \ln\left[\left(\frac{\bar{G}_t(\mathcal{Y})}{\bar{F}_t(\mathcal{X})}\right)\right] \geq 0 \quad \text{for any } t \geq 0.\end{aligned}$$

By taking $\zeta = \lim_{n\to\infty}\left(\frac{\bar{G}_t(\mathcal{Y})}{\bar{F}_t(\mathcal{X})}\right)$, it follows that $\zeta \geq \frac{\bar{G}_t(\mathcal{Y})}{\bar{F}_t(\mathcal{X})}$, for all $t \geq 0$. Note that if we put $t = 0$ in the recent inequality, we see that $\zeta \geq 0$. Hence, there exists a $\zeta \geq 0$ for which $\bar{G}_t(\mathcal{Y}) \leq \zeta\bar{F}_t(\mathcal{X})$ for all $t \geq 0$. This is the complete proof. In the following result, we show that if the life times of two series randomized discrete signals with identically independent density functions are RP ordered, then their components are also RP ordered.

□

Theorem 4.2 *Let*

$$\bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \{\mathcal{X}_{t:1}(\mathcal{Z}), \mathcal{X}_{t:2}(\mathcal{Z}), \mathcal{X}_{t:3}(\mathcal{Z}), \ldots, \mathcal{X}_{t:n}(\mathcal{Z})\}$$

and

$$\bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \{\mathcal{Y}_{t:1}(\mathcal{Z}), \mathcal{Y}_{t:2}(\mathcal{Z}), \mathcal{Y}_{t:3}(\mathcal{Z}), \ldots, \mathcal{Y}_{t:m}(\mathcal{Z})\}$$

be two sets of fuzzy randomized signals from survival (density) functions $\bar{F}_t(f_t(\mathcal{X}))$ and $\bar{G}_t(g_t(\mathcal{Y}))$, respectively. Then,

$$\lim_{\alpha,\beta\to1}\bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \{\mathcal{X}_{t:1}(\mathcal{Z}), \mathcal{X}_{t:2}(\mathcal{Z}), \mathcal{X}_{t:3}(\mathcal{Z}), \ldots, \mathcal{X}_{t:n}(\mathcal{Z})\} \leq_{RP} \lim_{\alpha,\beta\to1}\bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \{\mathcal{Y}_{t:1}(\mathcal{Z}), \mathcal{Y}_{t:2}(\mathcal{Z}), \mathcal{Y}_{t:3}(\mathcal{Z}), \ldots, \mathcal{Y}_{t:m}(\mathcal{Z})\}$$

$$\Longrightarrow$$

$$\mathcal{X}_{t:1}(\mathcal{Z}) \leq_{RP} \mathcal{Y}_{t:1}(\mathcal{Z}).$$

Proof: We know that

$$\bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \{\mathcal{X}_{t:1}(\mathcal{Z}),\mathcal{X}_{t:2}(\mathcal{Z}),\mathcal{X}_{t:3}(\mathcal{Z}),\ldots,\mathcal{X}_{t:n}(\mathcal{Z})\} \leq_{RP} \bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \{\mathcal{Y}_{t:1}(\mathcal{Z}),\mathcal{Y}_{t:2}(\mathcal{Z}),\mathcal{Y}_{t:3}(\mathcal{Z}),\ldots,\mathcal{Y}_{t:m}(\mathcal{Z})\}$$

$$\Longleftrightarrow \int_t^\infty \left[\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y})\right]^{n-1}\left[\bar{F}_t(\mathcal{X})g_t(\mathcal{Y})\bar{G}_t(\mathcal{Y})f_t(\mathcal{X})\right]\, d(\mathcal{X}\mathcal{Y}), \quad \text{for all } t\geq 0.$$

Let us denote, for all $\mathcal{X}\mathcal{Y}\geq 0$,

$$\int_t^\infty \left[\bar{F}_t(u)\bar{G}_t(u)\right]^{n-1}\left[\bar{F}_t(u)g_t(u)\bar{G}_t(u)f_t(u)\right]\, d(u), \quad \text{for all } t\geq 0.$$

$$d\delta(\mathcal{X}_t(\mathcal{Z})) = \delta'(\mathcal{X}_t(\mathcal{Z}))\, dX = \left[\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y})\right]^{n-1}\left[\bar{G}_t(\mathcal{Y})f_t(\mathcal{X}) - \bar{F}_t(\mathcal{X})g_t(\mathcal{Y})\right].$$

Take

$$\psi(\mathcal{X}_t(\mathcal{Z})) = \begin{cases} \frac{1}{\left[\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y})\right]^{n-1}}, & \text{if } \mathcal{X}_t(\mathcal{Z}) > t,\\ 0, & \text{if } \mathcal{X}_t(\mathcal{Z}) \leq t.\end{cases}$$

This function ψ is non-negative and increasing in $\mathcal{X}_t(\mathcal{Z})$, for all $t\geq 0$. We have for any $t\geq 0$,

$$\int_0^\infty \psi(\mathcal{X}_t(\mathcal{Z}))\, d\delta(\mathcal{X}_t(\mathcal{Z})) = \int_t^\infty \frac{1}{\left[\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y})\right]^{n-1}}\, d\delta(\mathcal{X}_t(\mathcal{Z}))$$

$$= \int_t^\infty \left[\bar{G}_t(\mathcal{Y})f_t(\mathcal{X}) - \bar{F}_t(\mathcal{X})g_t(\mathcal{Y})\right]\, d(\mathcal{X}\mathcal{Y}), \quad \text{for all } t\geq 0.$$

Because of (*), $\int_0^\infty(\mathcal{X}_t)\, d\delta(\mathcal{X}_t) \geq 0$, for all $t\geq 0$. So, by the previous lemma, we can optain that $\int_0^\infty \psi(\mathcal{X}_t(\mathcal{Z}))\, d\delta(\mathcal{X}_t(\mathcal{Z})) \geq 0$, for all $t\geq 0$, which means that $\mathcal{X}_{t:1}(\mathcal{Z}) \leq_{RP} \mathcal{Y}_{t:1}(\mathcal{Z})$.

Let $\mathcal{X}_t\{(n_\alpha^L, n_\beta^U)\}$ be a nonnegative randomized signal with survival function $\bar{F}_t(\mathcal{X})$. The randomized signal $\mathcal{X}_{t(k)} = \frac{\mathcal{X}_t - k}{\mathcal{X}_t > t}$, for $n : \bar{F}_t(\mathcal{X}) > 0$, is well known in the literature as the residual lifetime of a randomized signal associated with $\mathcal{X}_t$. The randomized signal $\mathcal{Y}_{t(n)}\{(n_\alpha^L, n_\beta^U)\}$ represents the lifetime of a used device of age n. Similarly, define $\mathcal{Y}_{t(k)} = \frac{\mathcal{Y}_t - k}{\mathcal{Y}_t > t}$, for $n : \bar{G}_t(\mathcal{Y}) > 0$, in which $\bar{G}_t(\mathcal{Y})$ is the survival function of $\mathcal{Y}_t\{(n_\alpha^L, n_\beta^U)\}$. In the literature, the concept of residual life has been extended to the case where n is random. Suppose that $\mathcal{T}$ is a non-negative random variable with distribution function Ω such that $E(\bar{F}_t(\mathcal{T})) > 0$. Then the random variable $\mathcal{X}_{t(\mathcal{T})}(\mathcal{Z}) = \frac{\mathcal{X}_t(\mathcal{Z}) - \mathcal{T}}{\mathcal{X}_t(\mathcal{Z}) > \mathcal{T}}$ is known as the residual lifetime of $\mathcal{X}_t(\mathcal{Z})$ at the random time $\mathcal{T}$ (Nanda and Kundu [25], Abouammoh [1]). The following result establishes two characterizations of the RP order by means of the concepts of residual life and residual lifetime at random time.

□

Theorem 4.3 *For the two lifespans of discrete-time fuzzy randomized signals $\mathcal{X}_t(k)(\mathcal{Z})$ and $\mathcal{Y}_{t(k)}(\mathcal{Z})$, we have:*

1. $\mathcal{X}_t(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_t(\mathcal{Z}) \quad \Leftrightarrow \quad \mathcal{X}_t(k)(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_{t(k)}(\mathcal{Z})$, *for all* $k\geq 0$,
2. $\mathcal{X}_t(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_t(\mathcal{Z}) \quad \Leftrightarrow \quad \mathcal{X}_{t(\mathcal{T})}(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_{t(\mathcal{T})}(\mathcal{Z})$, *for all nonnegative fuzzy randomized signals $\mathcal{T}$ that are independent from $\mathcal{X}_t(k)(\mathcal{Z})$ and $\mathcal{Y}_t(k)(\mathcal{Z})$.*

Proof: (1) Consider the survivorship and density of $\mathcal{X}_{t(\mathcal{T})}$ and $\mathcal{Y}_{t(\mathcal{T})}$, which are respectively $\bar{F}_t(\mathcal{X})$ and $\bar{G}_t(g_t(\mathcal{Y}))$, $g_t(\mathcal{Y})$.

We know that $\mathcal{X}_{t(k)}(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_{t(k)}(\mathcal{Z})$ for any $t\geq 0$, if and only if

$$\int_t^\infty \left[\bar{G}_{t(k)}(\mathcal{Y})f_{t(k)}(\mathcal{X}) - \bar{F}_{t(k)}(\mathcal{X})g_{t(k)}(\mathcal{Y})\right]\, d(\mathcal{X}\mathcal{Y}) \geq 0, \quad \text{for all } t\geq 0.$$

We can see that

$$\begin{aligned}
&\int_t^{\infty} \left[\bar{G}_{t(k)}(\mathcal{Y}) f_{t(k)}(\mathcal{X}) - \bar{F}_{t(k)}(\mathcal{X}) g_{t(k)}(\mathcal{Y})\right] \, d(\mathcal{X}\mathcal{Y}) \\
&= \int_t^{\infty} \left(\frac{\bar{G}_t(k+\mathcal{Y}) f_t(k+\mathcal{X}) - \bar{F}_t(k+\mathcal{Y}) g_t(k+\mathcal{X})}{\bar{G}_t(\mathcal{Y})\bar{F}_t(\mathcal{X})} \right) d(\mathcal{X}\mathcal{Y}) \\
&= \frac{1}{\bar{G}_t(\mathcal{Y})\bar{F}_t(\mathcal{X})} \int_{t+\zeta}^{\infty} \left[f_t(\mathcal{X})\bar{G}_t(\mathcal{Y}) - g_t(\mathcal{Y})\bar{F}_t(\mathcal{X})\right] \, d(\mathcal{X}\mathcal{Y}).
\end{aligned}$$

This expression is non-negative for all $t \geq 0$ if and only if $\mathcal{X}_t(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_t(\mathcal{Z})$. □

Proof: (2) First, we assume that $\mathcal{X}_t(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_t(\mathcal{Z})$.

From Zardashat and Asadi [2], we know that $\mathcal{X}_t(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_t(\mathcal{Z})$ is equivalent to $R(\mathcal{T}) = P[\mathcal{X}_t \geq \mathcal{Y}_t] \leq 0.5$, for all $\mathcal{T} \geq 0$.

Also, it is derived that

$$R(\mathcal{T}) = \frac{P[\mathcal{X}_t \geq \mathcal{Y}_t \geq \mathcal{T}]}{\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y})}, \quad \text{for any } \mathcal{T} \geq 0.$$

Suppose $\mathcal{T}$ has distribution Ω.

Because $\mathcal{T}$ is independent from $\mathcal{X}_t$ and $\mathcal{Y}_t$, thus we can write

$$\begin{aligned}
P[\mathcal{X}_T \geq \mathcal{Y}_T] &= P\left(\frac{\mathcal{X}_t - \mathcal{T} > \mathcal{Y}_t - \mathcal{T}}{\mathcal{X}_t > T, \mathcal{Y}_t > T} \right) \\
&= \frac{\int_0^{\infty} P[\mathcal{X}_t > \mathcal{Y}_t > T] \, d\Omega(\mathcal{T})}{\int_0^{\infty} \bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y}) \, d\Omega(\mathcal{T})} \\
&= \frac{\int_0^{\infty} R(\mathcal{T})\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y}) \, d\Omega(\mathcal{T})}{\int_0^{\infty} \bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y}) \, d\Omega(\mathcal{T})} \\
&= \frac{\int_0^{\infty} \frac{1}{2}\bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y}) \, d\Omega(\mathcal{T})}{\int_0^{\infty} \bar{F}_t(\mathcal{X})\bar{G}_t(\mathcal{Y}) \, d\Omega(\mathcal{T})} \\
&= 0.5.
\end{aligned}$$

This means that, by the above equations, we have proved that $\mathcal{X}_{t(\mathcal{T})}(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_{t(\mathcal{T})}(\mathcal{Z})$.

Conversely, if we assume that $\mathcal{X}_{t(\mathcal{T})}(\mathcal{Z}) \leq_{RPO} \mathcal{Y}_{t(\mathcal{T})}(\mathcal{Z})$ for all nonnegative randomized signals $\mathcal{T}$, independent from $\mathcal{X}_t(\mathcal{Z})$ and $\mathcal{Y}_t(\mathcal{Z})$, then the reverse implication follows. □

To determine the lifetime and orderings of signals, utilize the previously solved theorems, lemmas, and definitions. The way continual and independent time signals are represented, manipulated, and evaluated in different applications influences how important it is to organize them. The combination of continual and independent time signals is crucial for the following main reasons: data representation and transmission, synchronization and timing, signal processing, system modeling and analysis, sampling and reconstruction, and numerical algorithms and simulations. To summarize, the significance of organizing discrete and continuous time signals emerges from their crucial roles in signal processing, representation, system analysis, synchronization, data transfer, and numerical calculations. In order to sustain the signals' integrity and promote meaningful interpretation in various kinds of applications, the samples ought to be kept in the correct sequence.

5 Integration of $\mathcal{Z}$-transform

Let $\{\mathcal{X}_t(z_1), \mathcal{X}_t(z_2), \mathcal{X}_t(z_3), \ldots, \mathcal{X}_t(z_m)\}$ be a sequence of $\mathcal{Z}$-transforms defined for $m = 1, 2, 3, \ldots, \infty$. If $X_t(z_m) = 0$, then its $\mathcal{Z}$-transform is defined to be

$$\bigwedge_{0 \leq \alpha \leq \beta \leq 1} \bigvee \int_{m=1}^{\infty} \mathcal{Z}\{x_t[n_\alpha^L, n_\beta^U]\} \, dx = \bigwedge_{0 \leq \alpha \leq \beta \leq 1} \bigvee \int_{m=1}^{\infty} \sum_{n=1}^{\infty} x_t[n_\alpha^L, n_\beta^U] \mathcal{Z}^{-[n_\alpha^L, n_\beta^U]} \, dt$$

where z is a random constant. A crucial mathematical tool in core dimensional signal processing, the integration of the $\mathcal{Z}$-transform allows one to link a continuous, discrete-time signal in two or more dimensions to the complex frequency domain. An analysis of the distinct signal and frequency content of signals in several dimensions can be done with this generalization of the one-dimensional $\mathcal{Z}$-transform.

5.1 Integration of a $\mathcal{Z}$-transform: stochastic order

To compare signals and control systems using the $\mathcal{Z}$-transform, you need to understand how the $\mathcal{Z}$-transform can represent signals and systems in the discrete-time domain. Here's a step-by-step guide on how to perform the comparison. Apply the $\mathcal{Z}$-transform to the given signal to obtain its representation in the z-domain. This step involves expressing the signal as a sequence $x_t(n_\alpha^L, n_\beta^U)$ and evaluating the $\mathcal{Z}$-transform using the formula mentioned above.

Obtaining the $\mathcal{Z}$-transform of the control system:
Let's assume that the control system's transfer function is z. To get the $\mathcal{Z}$-transform, we have to substitute the complex variable with z, its inverse. To obtain the inverse $\mathcal{Z}$-transform, apply the following algorithm.

$$\mathcal{H}(\mathcal{Z}) = \frac{\{\mathcal{X}_0+\mathcal{X}_1\mathcal{Z}^{-1}+\mathcal{X}_2\mathcal{Z}^{-2}+\cdots+\mathcal{X}_n\mathcal{Z}^{-m}\}}{\{1+\mathcal{Y}_1\mathcal{Z}^{-1}+\mathcal{Y}_2\mathcal{Z}^{-2}+\cdots+\mathcal{Y}_m\mathcal{Z}^{-n}\}}.$$

Where the denominators are $y_1, y_2, y_3, \ldots, y_m$ and the numerators are $x_1, x_2, x_3, \ldots, x_n$. The numerator and denominator have orders of m and n. The $\mathcal{Z}$-transform is thus obtained by simply replacing z in the transfer function with $\mathcal{Z}^{-1}$. To get at the $\mathcal{Z}$-transform, do the calculation that follows.

$$\mathcal{H}(\mathcal{Z}^{-1}) = \frac{\{\mathcal{X}_0+\mathcal{X}_1\mathcal{Z}^{-1}+\mathcal{X}_2\mathcal{Z}^{-2}+\cdots+\mathcal{X}_n\mathcal{Z}^{-m-1}\}}{\{1+\mathcal{Y}_1\mathcal{Z}^{-1}+\mathcal{Y}_2\mathcal{Z}^{-2}+\cdots+\mathcal{Y}_m\mathcal{Z}^{-n}\}}.$$

This is the control system's $\mathcal{Z}$-transform, stated in terms of the complex variable's inverse.

Comparing the $\mathcal{Z}$-transforms:
After obtaining the $\mathcal{Z}$-transforms of both the signal and the control system, you can compare them to analyze their properties. This comparison can involve various techniques, such as determining the poles and zeros, evaluating stability, analyzing frequency response, and examining other characteristics.

Pole-zero analysis:
To evaluate stability and system response, analyze the locations of poles and zeros in the z-plane. The pole zero analysis of the $\mathcal{Z}$-transform in a fuzzy environment is shown in the figure that follows,

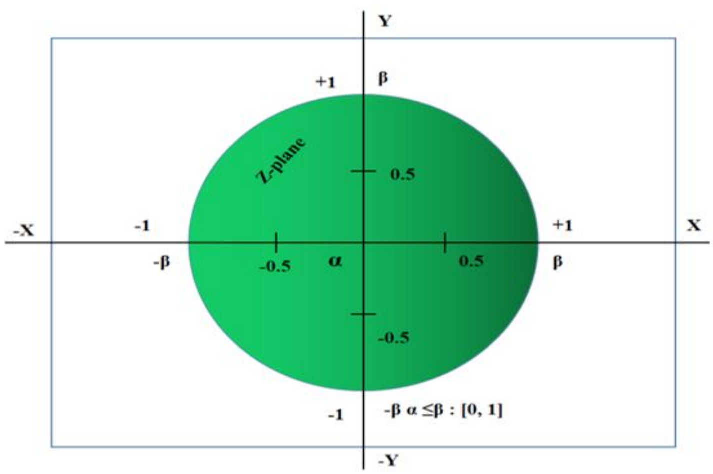

Figure 3: pole - zero analysis of $\mathcal{Z}$-transform

Frequency response:

Evaluate the $\mathcal{Z}$-transform at specific frequencies to analyze the system's behavior in the frequency domain.

System identification:

Compare the $\mathcal{Z}$-transforms to identify the system parameters, such as damping ratio, natural frequency, etc.

Remember that comparing signals and control systems using the $\mathcal{Z}$-transform is a complex process, and it requires a good understanding of the underlying theory and mathematics. It's recommended to consult textbooks, academic resources, or seek expert guidance to ensure accurate analysis and interpretation.

Let

$$\mathcal{Z}\{x_t(n_\alpha^L, n_\beta^U)\} = \{\mathcal{X}_t(z_1), \mathcal{X}_t(z_2), \mathcal{X}_t(z_3), \ldots, \mathcal{X}_t(z_n)\}$$

and

$$\mathcal{Z}\{y_t(n_\alpha^L, n_\beta^U)\} = \{\mathcal{Y}_t(z_1), \mathcal{Y}_t(z_2), \mathcal{Y}_t(z_3), \ldots, \mathcal{Y}_t(z_m)\}$$

be a sequence of fuzzy random variables, defined for $\{m = 1, 2, 3, \ldots, \infty\}$ and $x_n(n) = 0$, $y_n(n) = 0$. Then its $\mathcal{Z}$-transform $\mathcal{X}_t(\mathcal{Z})$ is stochastically dominant than $\mathcal{Y}_t(\mathcal{Z})$, denoted by

$$\mathcal{Z}\{x_t(n_\alpha^L, n_\beta^U)\} \leq_{SD} \mathcal{Z}\{y_t(n_\alpha^L, n_\beta^U)\}.$$

$$\bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \int_{-\infty}^{\infty} \sum_{n=1}^{m} \{x_t(n_\alpha^L, n_\beta^U)\}\mathcal{Z}^{-n}dx \leq_{SD} \bigwedge_{0\leq\alpha\leq\beta\leq1}\bigvee \int_{-\infty}^{\infty} \sum_{n=1}^{m} \{y_t(n_\alpha^L, n_\beta^U)\}\mathcal{Z}^{-n}dy$$

The Laplace transform also satisfies all of the aforementioned conditions and features.

6 Comparative Study

Recently, $\mathcal{Z}$-transforms have been playing an important role in understanding signal processing and control systems. More researchers came up with original solutions to the fundamental issues in control systems and signal processing. The creation of fractional $\mathcal{Z}$-transform algorithms and their uses in control, system identification, and signal processing have been the main topics of recent study. $\mathcal{Z}$-transform-based time-frequency analysis techniques have been investigated recently for speech processing, picture processing, and biological signal analysis, among other applications. Studies have been carried out to create sparse $\mathcal{Z}$-transform algorithms that take advantage of signal sparsity to accomplish effective signal processing and representation in the z-domain. The goal of multidimensional $\mathcal{Z}$-transform research is to create effective algorithms for multidimensional $\mathcal{Z}$-transform computation and investigate its uses in compression, analysis, and image and video processing. The goal of applications in data science and machine learning research is to investigate the use of $\mathcal{Z}$-transform based techniques for forecasting, anomaly detection, time series classification, and feature extraction.

7 Conclusion

$\mathcal{Z}$-transforms have proved crucial in recent years for comprehending control systems and signal processing. The basic problems in signal processing and control systems were solved by further researchers. Developing fractional-transform algorithms and applying them to signal processing, control, and system identification have been the primary subjects of contemporary research. But using the stochastic order approach, this article goes into great depth on the fuzzy randomised signal and its order. This technique is employed to ascertain the lifespan of a signal. A number of concepts, depending on the field of study, can be connected to the lifetime of signals, which is commonly discussed in relation to signal processing or communication systems. The following are possible explanations: Now is the time to thrive in networking. Duration, decay, and lifetime of signals in communication systems. Determining the context and parameters, including the operational environment and the physical characteristics of the signal, is essential to determining its lifetime. This paper introduces a whole new idea for control and signal processing systems. This article described the lifetime of the signal. A crucial task in communication engineering is figuring out the signal lifetime. This approach allows us to rapidly identify a moderate degree of signal loss in an electrical circuit. Finding the circuit's signal loss location and making the required corrections are quick and simple tasks for potential researchers.

Declaration:

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