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# An Infinite Integral of Three Constants: Boros Integral for Polynomial Families and Incomplete 𝕎-Functions

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ABSTRACT: In this paper, we explore the Boros integral with three parameters, which encompasses the incomplete 𝕎-functions and the Srivastava polynomial, a general class of polynomials. Our primary contribution is the derivation of the Boros integral for the product of a family of polynomials and the incomplete 𝕎-function. Furthermore, we present a novel family of special functions associated with the 𝕎-function, derived as a main result of our investigation. To expand the scope of our findings, we propose that the derived formulas can be generalized to incorporate classes of multivariable polynomials, thus opening the door for further exploration in the realm of special functions.

Key Words: Boros integral, Generalized family of polynomials, Incomplete gamma functions, Incomplete 𝕎-functions, Mellin Barnes type contour integral.

## Contents



## 1. Introduction and Preliminaries

The theory of special functions forms a vast and continually evolving branch of mathematics, driven by the emergence of complex problems across engineering, physics, and the applied sciences. As these disciplines advance, the demand for sophisticated analytical tools grows, positioning special functions as vital instruments for solving intricate differential equations of single or multiple variables. Their relevance continues to increase, not only due to their elegance and versatility but also because of their applicability to real-world phenomena. Consequently, the field has seen a surge in research, with numerous studies investigating new forms, properties, and applications of special functions [1,2,3].

Among these developments, incomplete special functions have emerged as potent tools due to their flexibility in modeling real-world systems characterized by partial or constrained behavior. Their applications span various domains, including probability theory, communication systems, heat conduction, groundwater hydrology, and engineering problems. Recent studies have explored higher transcendental incomplete functions and demonstrated their utility in solving boundary-layer problems and dynamic systems [4,5].

Classical special functions often fail to address specific real-world constraints in certain fields, such as astrophysics, biomedical modeling, and thermal conduction. This has motivated researchers to extend traditional forms and define new classes of incomplete functions, such as the incomplete hypergeometric function [6], incomplete Wright function [7], incomplete $H$-function [8], incomplete $I$-function [9], and the incomplete 𝕐-function [10]. These functions frequently incorporate incomplete gamma functions in their structure, offering broader flexibility for modeling partial processes or boundary-limited systems.

The present study builds on this trajectory by introducing a new analytical construct: the Boros integral involving the product of the incomplete 𝕎-function and the Srivastava polynomial. A new family of special functions associated with the 𝕎-function is proposed and explored in detail.

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To position this contribution within the broader literature, we note several recent advances in fractional integral operators, especially those involving products of generalized special functions. Notable works have introduced generalized fractional integrals involving combinations of $H$-functions and polynomial classes, laying foundational tools for modeling dynamic systems in applied mathematics [11,12]. The concept of incompleteness in special functions has also been expanded through studies on incomplete $H$-functions and $I$-functions, underscoring their significance in capturing singular behavior and boundary-dependent phenomena [13,14,15]. Applications of these functions have proven effective in biological modeling, including the dynamics of blood glucose regulation and internal blood pressure [16,17].

### 1.1. Mathematical Preliminaries

The functions $\gamma(\kappa,\tau)$ and $\Gamma(\kappa,\tau)$, known as the lower and upper incomplete gamma functions, respectively, are defined as follows:

$$\gamma(\kappa,\tau)=\int_0^\tau t^{\kappa-1}e^{-t}dt\,,\quad (\Re(\kappa)>0,\tau\geq 0),$$

and

$$\Gamma(\kappa,\tau)=\int_\tau^\infty t^{\kappa-1}e^{-t}dt\,,\quad (\Re(\kappa)>0,\tau\geq 0),$$

respectively.

$$\gamma(\kappa,\tau)+\Gamma(\kappa,\tau)=\Gamma(\kappa),\qquad (\Re(\kappa)>0).$$

Yang et al. [18] recently introduced the $\mathbb{W}$-function, and it is described as

$$\mathbb{W}_{r,s}^{m,n}[z;y;x]=\mathbb{W}_{r,s}^{m,n}\left[z;y;x\,\middle|\,\begin{matrix}(A_i,u_i,U_i)_1^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]=\int_\omega \Theta_{r,s}^{m,n}(\kappa)\;z^{-\kappa}\;e^{\kappa y}\kappa^x\;d\kappa, \tag{1.1}$$

where

$$\Theta_{r,s}^{m,n}(\kappa)=\frac{\left[\prod_{j=1}^m\{\Gamma(v_j-V_j\kappa)\}^{B_j}\right]\left[\prod_{i=1}^n\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}\right]}{\left[\prod_{j=m+1}^s\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\right]\left[\prod_{i=n+1}^r\{\Gamma(u_i-U_i\kappa\}^{A_i}\right]},$$

with complex variables $x$, $y$, $z$, and the contour $\omega$ in the complex plane, the orders $(m,n,r,s)$ are nonnegative integers such that $0\leq m\leq s$ and $0\leq n\leq r$. The parameters $U_i>0$ and $V_j>0$ are positive, while $u_i$ and $v_j$, for $i=1,\cdots,r$ and $j=1,\cdots,s$, are arbitrary complex numbers satisfying the following conditions:

$U_i(v_j+\rho)\neq V_j(u_i-\rho'-1),\;\; \rho,\rho'\in(\mathbb{N}_0(0,1,2,\cdots);\; i=1,2,\cdots,r; j=1,2,\cdots,s)$.

In their study, Kritika et al. [10] introduced a novel class of incomplete $\mathbb{W}$-functions, specifically ${}^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$ and ${}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$. These functions are defined in terms of incomplete gamma functions, as detailed in the study mentioned above.

$${}^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]={}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[z;y;x\,\middle|\,\begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]=\frac{1}{2\pi i}\int_\omega {}^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}\;e^{\kappa y}\kappa^x\;d\kappa, \tag{1.2}$$

where

$${}^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)=\frac{\{\gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\;\left[\prod_{j=1}^m\{\Gamma(v_j-V_j\kappa)\}^{B_j}\right]\left[\prod_{i=2}^n\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}\right]}{\left[\prod_{j=m+1}^s\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\right]\;\left[\prod_{i=n+1}^r\{\Gamma(u_i-U_i\kappa)\}^{A_i}\right]},$$

and

$$\begin{aligned}{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]&={}^{\Gamma}\mathbb{W}_{r,s}^{m,n}\left[z;y;x\,\middle|\,\begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]\\&=\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}\;e^{\kappa y}\kappa^x\;d\kappa,\end{aligned} \tag{1.3}$$

where

$$^{\Gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)=\frac{\{\Gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\ \left[\prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}^{B_j}\right]\left[\prod_{i=2}^{n}\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}\right]}{\left[\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\right]\ \left[\prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}^{A_i}\right]}.$$

The incomplete $\mathbb{W}$-functions $^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$ and $^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]$ in (1.2) and (1.3) exist for all $\tau \geq 0$ under the same contour and the same set of conditions as above. These definitions in (1.2) and (1.3) readily give the following decomposition formula:

$$^{\gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]+{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[z;y;x]=\mathbb{W}_{r,s}^{m,n}[z;y;x],$$

for the $\mathbb{W}$-function defined by (1.1).

The Srivastava [19,20] investigated a broader class of polynomials, which is summarised as follows:

$$S_V^U[t]=\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\,t^{\mathfrak{O}}, \tag{1.4}$$

where $U \in \mathbb{Z}^+$ and $A_{V,\mathfrak{O}}$ are real or complex numbers arbitrary constant. The notations $[k]$ indicates the floor function and $(\varrho)_\mu$ indicate the pochhammer symbol described by:

$$(\varrho)_0=1 \qquad \textit{and} \qquad (\varrho)_\mu=\frac{\Gamma(\varrho+\mu)}{\Gamma(\varrho)}, \qquad\qquad (\mu\in\mathbb{C})$$

in the form of the gamma function.

**Lemma 1.1** *Let $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$, Boros and Molls [21,22] introduced the integral depending on the three parameters which is given as follows:*

$$\int_0^\infty\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}}dh=\frac{B(\mathcal{P}-\frac{1}{2},\frac{1}{2})}{2^{\mathcal{P}+1/2}\sqrt{b}\left[a+\sqrt{bc}\right]^{\mathcal{P}-1/2}}; \tag{1.5}$$

*where B(m,n) denotes the beta function. Equation (1.5) can be expressed in the following way, by using the relation $B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$,*

$$\int_0^\infty\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}}dh=\frac{\sqrt{\pi}\Gamma(\mathcal{P}-\frac{1}{2})}{\Gamma(\mathcal{P})\sqrt{b}2^{\mathcal{P}+1/2}(a+\sqrt{bc})^{\mathcal{P}-1/2}}. \tag{1.6}$$

## 2. Main Results

We use the expression $\mathscr{X}=\left[\frac{h^2}{bh^4+2ah^2+c}\right]$ as the central term to obtain the main results of this study. Next, we present a sequence of results highlighting this structure's applicability and generality under various parameter constraints.

**Theorem 2.1** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$, then we get the following outcome:*

$$\int_0^\infty\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}}{}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[\mathscr{X}^e z;y;x]dh=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}$$
$$\times\,{}^{\Gamma}\mathbb{W}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e};y;x\,\middle|\,\begin{matrix}(A_1,u_1,U_1;\tau),(1,\frac{3}{2}-\mathcal{P},-e),(A_i,u_i,U_i)_2^r\\(1,1-\mathcal{P},-e),(B_j,v_j,V_j)_1^s\end{matrix}\right]. \tag{2.1}$$

**Proof:** The LHS of Equation (2.1) is:

$$L_1 = \int_0^\infty \left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}} {}^{\Gamma}\mathbb{W}^{m,n}_{r,s}[\mathscr{X}^e z; y; x]dh.$$

Replace the $\mathbb{W}$-function with (1.3), we obtain:

$$L_1 = \int_0^\infty \mathscr{X}^{\mathcal{P}}\left[\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Theta^{m,n}_{r,s}(\kappa,\tau)(\mathscr{X}^e z)^{-\kappa}e^{\kappa y}\kappa^x d\kappa\right]dh.$$

Interchange the order of integration in the above equation, we get:

$$L_1 = \frac{1}{2\pi i}\int_\omega \frac{\{\Gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\ \left[\prod_{j=1}^m\{\Gamma(v_j-V_j\kappa)\}^{B_j}\right]\left[\prod_{i=2}^n\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}\right]}{\left[\prod_{j=m+1}^s\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\right]\ \left[\prod_{i=n+1}^r\{\Gamma(u_i-U_i\kappa)\}^{A_i}\right]}$$
$$\times\ z^{-\kappa}e^{\kappa y}\kappa^x\int_0^\infty \mathscr{X}^{\mathcal{P}-e\kappa}d\kappa dh. \tag{2.2}$$

By applying Lemma (1.1), we can now evaluate the integral and get:

$$\int_0^\infty \mathscr{X}^{\mathcal{P}-e\kappa}dh = \frac{\sqrt{\pi}\left(\mathcal{P}-e\kappa-\frac{1}{2}\right)}{\Gamma(\mathcal{P}-e\kappa)\sqrt{b}2^{\mathcal{P}-e\kappa+\frac{1}{2}}\left[\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-e\kappa-\frac{1}{2}}}. \tag{2.3}$$

By placing (2.3) into (2.2), we get:

$$L_1 = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\frac{1}{2\pi i}$$
$$\int_\omega \frac{\{\Gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\ [\Gamma(\mathcal{P}-e\kappa-\frac{1}{2}]\ \left[\prod_{j=1}^m\{\Gamma(v_j-V_j\kappa)\}^{B_j}\right]\left[\prod_{i=2}^n\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}\right]}{\left[\prod_{j=m+1}^s\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\right]\ \left[\prod_{i=n+1}^r\{\Gamma(u_i-U_i\kappa)\}^{A_i}\right][\Gamma(\mathcal{P}-e\kappa)]}$$
$$\times\ z^{-\kappa}[2(a+\sqrt{bc})]^{-e\kappa}e^{\kappa y}\kappa^x d\kappa.$$

Change the order of integration, and after some adjustment of terms, we obtain the desired outcomes. □

**Theorem 2.2** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$, then the result is as follows.*

$$\int_0^\infty \left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}} {}^{\gamma}\mathbb{W}^{m,n}_{r,s}[\mathscr{X}^e z; y; x]dh = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}$$
$$\times\ {}^{\gamma}\mathbb{W}^{m,n+1}_{r+1,s+1}\left[z[2(a+\sqrt{bc})]^{-e}; y; x\ \middle|\ \begin{matrix}(A_1,u_1,U_1;\tau),(1,\frac{3}{2}-\mathcal{P},-e),(A_i,u_i,U_i)_2^r\\ (1,1-\mathcal{P},-e),(B_j,v_j,V_j)_1^s\end{matrix}\right].$$

**Proof:** Theorem 2.2 proved to be like Theorem 2.1 but with the same conditions. □

**Theorem 2.3** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^+$ then we have the following outcome:*

$$\int_0^\infty \left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}} S_V^U[t\mathscr{X}]\ {}^{\Gamma}\mathbb{W}^{m,n}_{r,s}[\mathscr{X}^e z; y; x]dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\,t^{\mathfrak{O}}\times\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}$$
$$\times\ {}^{\Gamma}\mathbb{W}^{m,n+1}_{r+1,s+1}\left[z[2(a+\sqrt{bc})]^{-e}; y; x\ \middle|\ \begin{matrix}(A_1,u_1,U_1;\tau),(1,1-\mathcal{P}-\mathfrak{O},-e),(A_i,u_i,U_i)_2^r\\ (1,1-\mathcal{P}-\mathfrak{O},-e),(B_j,v_j,V_j)_1^s\end{matrix}\right]. \tag{2.4}$$

**Proof:** The LHS of Equation (2.4) is

$$L_3 = \int_0^\infty \left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}} S_V^U[t\mathscr{X}]\ {}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[\mathscr{X}^e z; y; x]dh.$$

Replace the incomplete $\mathbb{W}$-function and Srivastava polynomial with (1.3) and (1.4) respectively, we obtain:

$$L_3 = \int_0^\infty \mathscr{X}^{\mathcal{P}} \sum_{\mathfrak{O}=0}^{[V/U]} \frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!} A_{V,\mathfrak{O}}\ (t\mathscr{X})^{\mathfrak{O}} \left[\frac{1}{2\pi i}\int_\omega {}^{\Gamma}\Phi_{r,s}^{m,n}(\kappa,\tau)(\mathscr{X}^e z)^{-\kappa} e^{\kappa y}\kappa^x d\kappa\right]dh.$$

Change the order of integration in the above equation, we get:

$$\begin{aligned} L_3 = &\sum_{\mathfrak{O}=0}^{[V/U]} \frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!} A_{V,\mathfrak{O}}\ t^{\mathfrak{O}}\ \times \frac{1}{2\pi i} \\ &\times \int_\omega \frac{\{\Gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\left[\prod_{j=1}^m \{\Gamma(v_j-V_j\kappa)\}^{B_j}\right]\left[\prod_{i=2}^n \{\Gamma(1-u_i+U_i\kappa)\}^{A_i}\right]}{\left[\prod_{j=m+1}^s \{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\right]\left[\prod_{i=n+1}^r \{\Gamma(u_i-U_i\kappa)\}^{A_i}\right]} \\ &\times z^{-\kappa} e^{\kappa y}\kappa^x \times \int_0^\infty \mathscr{X}^{\mathcal{P}+\mathfrak{O}-e\kappa} d\kappa dh. \end{aligned} \tag{2.5}$$

Using Lemma (1.1), we can now conclude:

$$\int_0^\infty \mathscr{X}^{\mathcal{P}+\mathfrak{O}-e\kappa}dh = \frac{\sqrt{\pi}\left(\mathcal{P}+\mathfrak{O}-e\kappa-\frac{1}{2}\right)}{\Gamma\left(\mathcal{P}+\mathfrak{O}-e\kappa\right)\sqrt{b}2^{\mathcal{P}+\mathfrak{O}-e\kappa+\frac{1}{2}}\left[\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}+\mathfrak{O}-e\kappa-\frac{1}{2}}}. \tag{2.6}$$

By placing (2.6) into (2.5), we get:

$$\begin{aligned} L_3 = &\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}} \times \sum_{\mathfrak{O}=0}^{[V/U]} \frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!} A_{V,\mathfrak{O}}\ t^{\mathfrak{O}} \times \frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}} \times \frac{1}{2\pi i} \\ &\int_\omega \frac{\{\Gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\ [\Gamma(\mathcal{P}+\mathfrak{O}-e\kappa-\frac{1}{2}]\ \left[\prod_{j=1}^m \{\Gamma(v_j-V_j\kappa)\}^{B_j}\right]\left[\prod_{i=2}^n \{\Gamma(1-u_i+U_i\kappa)\}^{A_i}\right]}{\left[\prod_{j=m+1}^s \{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\right]\ \left[\prod_{i=n+1}^r \{\Gamma(u_i-U_i\kappa)\}^{A_i}\right][\Gamma(\mathcal{P}+\mathfrak{O}-e\kappa)]} \\ &\times z^{-\kappa}[2(a+\sqrt{bc})]^{-e\kappa}e^{\kappa y}\kappa^x d\kappa. \end{aligned} \tag{2.7}$$

Now convert Equation (2.7) into an incomplete $\mathbb{W}$-function to obtain the desired results. □

**Theorem 2.4** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $P>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^+$ then we have the following outcome:*

$$\begin{aligned} &\int_0^\infty \left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}} S_V^U[t\mathscr{X}]\ {}^{\gamma}\mathbb{W}_{r,s}^{m,n}[\mathscr{X}^e z; y; x]dh \\ &= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}} \times \sum_{\mathfrak{O}=0}^{[V/U]} \frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!} A_{V,\mathfrak{O}}\ t^{\mathfrak{O}} \times \frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}} \\ &\times {}^{\gamma}\mathbb{W}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e}; y; x \,\middle|\, \begin{matrix} (A_1,u_1,U_1;\tau),(\frac{3}{2}-\mathcal{P}-\mathfrak{O},-e;1),(A_i,u_i,U_i)_2^r \\ (1-\mathcal{P}-\mathfrak{O},-e;1),(B_j,v_j,V_j)_1^s \end{matrix}\right]. \end{aligned}$$

**Proof:** Theorem 2.4 proved to be similar to Theorem 2.3 but with the same conditions. □

## 3. Special Cases

In this section, as a specific application of Theorem 2.3 and Theorem 2.4, we derive the Boros integral for the product of the Srivastava polynomial with the incomplete $\mathbb{Y}$-function, the incomplete $I$-function, the incomplete $\mathbb{U}$-function and the incomplete $\mathbb{O}$-function. Additionally, by assigning specific values to the Srivastava polynomial, we can obtain results in the form of Hermite and Laguerre polynomials. By adjusting the parameters according to particular features, we can derive these special cases to highlight the application of fundamental outcomes.

**(i) $\mathbb{Y}$-Function:** Setting $A_1 = A_2 = \cdots = A_r = 1$ and $B_1 = B_2 = \cdots = B_s = 1$ in Equation (1.1), the $\mathbb{W}$-function reduces to the $\mathbb{Y}$-function introduced by Yang et al. [18], leading to the following conclusion.

$$\mathbb{W}^{m,n}_{r,s}\left[z;y;x \,\middle|\, \begin{matrix}(1,u_i,U_i)_1^r\\(1,v_j,V_j)_1^s\end{matrix}\right] = \mathbb{Y}^{m,n}_{r,s}\left[z;y;x \,\middle|\, \begin{matrix}(u_i,U_i)_1^r\\(v_j,V_j)_1^s\end{matrix}\right] = \frac{1}{2\pi i}\int_{\omega} \Theta^{m,n}_{r,s}(\kappa,\tau) z^{-\kappa}\, e^{\kappa y}\kappa^{x}\, d\kappa,$$

where

$$\Theta^{m,n}_{r,s}(\kappa,\tau) = \frac{\prod_{j=1}^{m}\Gamma(v_j - V_j\kappa)\prod_{i=1}^{n}\Gamma(1-u_i+U_i\kappa)}{\prod_{j=m+1}^{s}\Gamma(1-v_j+V_j\kappa)\ \prod_{i=n+1}^{r}\Gamma(u_i-U_i\kappa)}.$$

Further, Kritika et al. [10] introduced incomplete $\mathbb{Y}$-function which is defined as:

$${}^{\gamma}\mathbb{Y}^{m,n}_{r,s}[z;y;x] = {}^{\gamma}\mathbb{Y}^{m,n}_{r,s}\left[z;y;x \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(u_i,U_i)_2^r\\(v_j,V_j)_1^s\end{matrix}\right] = \frac{1}{2\pi i}\int_{\omega} {}^{\gamma}\Theta^{m,n}_{r,s}(\kappa,\tau) z^{-\kappa}\, e^{\kappa y}\kappa^{x}\, d\kappa,$$

where

$${}^{\gamma}\Theta^{m,n}_{r,s}(\kappa,\tau) = \frac{\gamma(1-u_1+U_1\kappa,\tau)\ \prod_{j=1}^{m}\Gamma(v_j - V_j\kappa)\prod_{i=2}^{n}\Gamma(1-u_i+U_i\kappa)}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\ \prod_{i=n+1}^{r}\Gamma(u_i-U_i\kappa)}.$$

By applying the relationships specified in Theorems (2.3) and (2.4), we derive the following corollaries.

**Corollary 1** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{D}}$ are real or complex and $U\in\mathbb{Z}^{+}$ then we have the following outcome:*

$$\int_0^{\infty}\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}} S_V^U[t\mathscr{X}]\ {}^{\Gamma}\mathbb{Y}^{m,n}_{r,s}[\mathscr{X}^e z;y;x]dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}} \times \sum_{\mathfrak{D}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{D}}}{\mathfrak{D}!}A_{V,\mathfrak{D}}\, t^{\mathfrak{D}} \times \frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{D}}}$$
$$\times\, {}^{\Gamma}\mathbb{Y}^{m,n+1}_{r+1,s+1}\left[z[2(a+\sqrt{bc})]^{-e};y;x \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(\frac{3}{2}-\mathcal{P}-\mathfrak{D},-e),(u_i,U_i)_2^r\\(1-\mathcal{P}-\mathfrak{D},-e),(v_j,V_j)_1^s\end{matrix}\right].$$

**Corollary 2** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{D}}$ are real or complex and $U\in\mathbb{Z}^{+}$ then we have the following outcome:*

$$\int_0^{\infty}\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}} S_V^U[t\mathscr{X}]\ {}^{\gamma}\mathbb{Y}^{m,n}_{r,s}[\mathscr{X}^e z;y;x]dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}} \times \sum_{\mathfrak{D}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{D}}}{\mathfrak{D}!}A_{V,\mathfrak{D}}\, t^{\mathfrak{D}} \times \frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{D}}}$$
$$\times\, {}^{\gamma}\mathbb{Y}^{m,n+1}_{r+1,s+1}\left[z[2(a+\sqrt{bc})]^{-e};y;x \,\middle|\, \begin{matrix}(u_1,U_1;\tau),(1-\mathcal{P}-\mathfrak{D},-e),(u_i,U_i)_2^r\\(1-\mathcal{P}-\mathfrak{D},-e),(v_j,V_j)_1^s\end{matrix}\right].$$

**(ii) $I$-Function:** The $I$-function, introduced by Rathie [23] and built upon the work of Innayat Hussain [24], is defined through the Mellin-Barnes type integral:

$$I_{r,s}^{m,n}\left[z\,\middle|\,\begin{matrix}(u_i,U_i,A_i)_1^r\\(v_j,V_j,B_j)_1^s\end{matrix}\right]=\frac{1}{2\pi i}\int_\omega {}^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)z^{\kappa}\,d\kappa.$$

If we substitute $x=y=0$ and $\frac{1}{z}$ in place of $z$ in Equation (1.2), they get reduced to the incomplete $I$-functions [9]:

$$\begin{aligned}{}^{\gamma}\mathbb{W}_{r,s}^{m,n}\left[\frac{1}{z};0;0\,\middle|\,\begin{matrix}(A_1,u_1,U_1;\tau),(A_i,u_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]&={}^{\gamma}I_{r,s}^{m,n}\left[z\,\middle|\,\begin{matrix}(u_1,U_1,A_1;\tau),(u_i,U_i,A_i)_2^r\\(v_j,V_j,B_j)_1^s\end{matrix}\right]\\&=\frac{1}{2\pi i}\int_\omega {}^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)z^{\kappa}\,d\kappa,\end{aligned}$$

where

$${}^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)=\frac{\{\gamma(1-u_1+U_1\kappa,\tau)\}^{A_1}\ \prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}^{B_j}\prod_{i=2}^{n}\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\ \prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}^{A_i}}.$$

By applying the relationships specified in Theorems (2.3) and (2.4), we derive the following corollaries.

**Corollary 3** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^+$ then we have the following outcome:*

$$\begin{aligned}&\int_0^\infty\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}}S_V^U[t\mathscr{X}]\ {}^{\Gamma}I_{r,s}^{m,n}[\mathscr{X}^e z]dh\\&=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\,t^{\mathfrak{O}}\times\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}\\&\times{}^{\Gamma}I_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e}\,\middle|\,\begin{matrix}(u_1,U_1,A_1;\tau),(1-\mathcal{P}-\mathfrak{O},-e,1),(u_i,U_i,A_i)_2^r\\(1-\mathcal{P}-\mathfrak{O},-e,1),(v_j,V_j,B_j)_1^s\end{matrix}\right].\end{aligned}$$

**Corollary 4** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^+$ then we have the following outcome:*

$$\begin{aligned}&\int_0^\infty\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}}S_V^U[t\mathscr{X}]\ {}^{\gamma}I_{r,s}^{m,n}[\mathscr{X}^e z]dh\\&=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\,t^{\mathfrak{O}}\times\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}\\&\times{}^{\gamma}I_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e}\,\middle|\,\begin{matrix}(u_1,U_1,A_1;\tau),(1-\mathcal{P}-\mathfrak{O},-e,1),(u_i,U_i,A_i)_2^r\\(1-\mathcal{P}-\mathfrak{O},-e,1),(v_j,V_j,B_j)_1^s\end{matrix}\right].\end{aligned}$$

**(iii) $\mathbb{U}$-Function:** If we assign $x=0$ in Equation (1.2), the $\mathbb{W}$-function reduces to the $\mathbb{U}$-function proposed by Yang et al. [18] which is defined as follows:

$$\mathbb{W}_{r,s}^{m,n}\left[z;y;0\,\middle|\,\begin{matrix}(A_i,u_i,U_i)_1^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]=\mathbb{U}_{r,s}^{m,n}\left[z;y\,\middle|\,\begin{matrix}(A_i,U_i,U_i)_1^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right]=\frac{1}{2\pi i}\int_\omega\Theta_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}\,e^{\kappa y}\,d\kappa,$$

where

$$\Theta_{r,s}^{m,n}(\kappa)=\frac{\prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}^{B_j}\prod_{i=1}^{n}\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\ \prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}^{A_i}}.$$

In the sequence of the study of incomplete functions, here we are introducing incomplete $\mathbb{U}$-function by using the incomplete gamma functions $\gamma(\kappa,\tau)$ and $\Gamma(\kappa,\tau)$ and we reach the following conclusion:

$$^{\gamma}\mathbb{U}_{r,s}^{m,n}\left[z;y\left|\begin{matrix}(A_1,u_1,U_1;\tau),(A_i,U_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\omega}{}^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}\ e^{\kappa y}\ d\kappa,$$

where

$$^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)=\frac{\prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}^{B_j}\prod_{i=1}^{n}\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\ \prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}^{A_i}}.$$

By applying the relationships specified in Theorems (2.3) and (2.4), we derive the following corollaries.

**Corollary 5** *For $b>0,c\geq 0,a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^+$ then we have the following outcome:*

$$\int_0^{\infty}\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}}S_V^U[t\mathscr{X}]^{\Gamma}\mathbb{U}_{r,s}^{m,n}[\mathscr{X}^e z;y]dh$$
$$=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\ t^{\mathfrak{O}}\times\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}$$
$$\times\,^{\Gamma}\mathbb{U}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e};y\left|\begin{matrix}(A_1,u_1,U_1;\tau),(1,1-\mathcal{P}-\mathfrak{O},-e),(A_i,u_i,U_i)_2^r\\(1,1-\mathcal{P}-\mathfrak{O},-e),(B_j,v_j,V_j)_1^s\end{matrix}\right.\right].$$

**Corollary 6** *For $b>0,c\geq 0,a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^+$ then we have the following outcome:*

$$\int_0^{\infty}\left[\frac{h^2}{bh^4+2ah^2+c}\right]^{\mathcal{P}}S_V^U[t\mathscr{X}]^{\gamma}\mathbb{U}_{r,s}^{m,n}[\mathscr{X}^e z;y]dh$$
$$=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\ t^{\mathfrak{O}}\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}$$
$$\times\,^{\gamma}\mathbb{U}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e};y\left|\begin{matrix}(A_1,u_1,U_1;\tau),(1,1-\mathcal{P}-\mathfrak{O},-e),(A_i,u_i,U_i)_2^r\\(1,1-\mathcal{P}-\mathfrak{O},-e),(B_j,v_j,V_j)_1^s\end{matrix}\right.\right].$$

**(iv) $\mathbb{O}$-Function:** If we replace $y=0$ in the $\mathbb{W}$-function (1.2), it reduces to the $\mathbb{O}$-function [18], as follows:

$$\mathbb{W}_{r,s}^{m,n}\left[z;0;x\left|\begin{matrix}(A_i,u_i,U_i)_1^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right.\right]=\mathbb{O}_{r,s}^{m,n}\left[z;x\left|\begin{matrix}(A_i,U_i,U_i)_1^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\omega}\Theta_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}\ \kappa^{x}\ d\kappa,$$

where

$$\Theta_{r,s}^{m,n}(\kappa)=\frac{\prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}^{B_j}\prod_{i=1}^{n}\{\Gamma(1-u_i+U_i\kappa)\}^{A_i}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}^{B_j}\ \prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}^{A_i}}.$$

In the sequence of the study of incomplete functions, here we introduce the incomplete $\mathbb{O}$-function by using the incomplete gamma functions $\gamma(\kappa,\tau)$ and $\Gamma(\kappa,\tau)$ and we reach the following conclusion:

$$^{\gamma}\mathbb{O}_{r,s}^{m,n}\left[z;x\left|\begin{matrix}(A_1,u_1,U_1;\tau),(A_i,U_i,U_i)_2^r\\(B_j,v_j,V_j)_1^s\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\omega}{}^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)z^{-\kappa}\ \kappa^{x}\ d\kappa,$$

where

$$^{\gamma}\Theta_{r,s}^{m,n}(\kappa,\tau)=\frac{\{\gamma(1-u_1+U_1\kappa,\tau)\}\ \prod_{j=1}^{m}\{\Gamma(v_j-V_j\kappa)\}\prod_{i=2}^{n}\{\Gamma(1-u_i+U_i\kappa)\}}{\prod_{j=m+1}^{s}\{\Gamma(1-v_j+V_j\kappa)\}\ \prod_{i=n+1}^{r}\{\Gamma(u_i-U_i\kappa)\}}.$$

By applying the relationships specified in Theorems (2.3) and (2.4), we derive the following corollaries.

**Corollary 7** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^{+}$ then we have the following outcome:*

$$\int_{0}^{\infty}\left[\frac{h^{2}}{bh^{4}+2ah^{2}+c}\right]^{\mathcal{P}} S_{V}^{U}[t\mathscr{X}]^{\Gamma}\mathbb{O}_{r,s}^{m,n}[\mathscr{X}^{e}z;x]dh$$
$$=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\,t^{\mathfrak{O}}\times\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}$$
$$\times\,{}^{\Gamma}\mathbb{O}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e};x\,\middle|\,\begin{matrix}(A_{1},u_{1},U_{1};\tau),(1,1-\mathcal{P}-\mathfrak{O},-e),(A_{i},u_{i},U_{i})_{2}^{r}\\(1,1-\mathcal{P}-\mathfrak{O},-e),(B_{j},v_{j},V_{j})_{1}^{s}\end{matrix}\right].$$

**Corollary 8** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$ and the coefficient $A_{V,\mathfrak{O}}$ are real or complex and $U\in\mathbb{Z}^{+}$ then we have the following outcome:*

$$\int_{0}^{\infty}\left[\frac{h^{2}}{bh^{4}+2ah^{2}+c}\right]^{\mathcal{P}} S_{V}^{U}[t\mathscr{X}]^{\gamma}\mathbb{O}_{r,s}^{m,n}[\mathscr{X}^{e}z;x]dh$$
$$=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/U]}\frac{(-V)_{U\mathfrak{O}}}{\mathfrak{O}!}A_{V,\mathfrak{O}}\,t^{\mathfrak{O}}\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}$$
$$\times\,{}^{\gamma}\mathbb{O}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e};x\,\middle|\,\begin{matrix}(A_{1},u_{1},U_{1};\tau),(1,1-\mathcal{P}-\mathfrak{O},-e),(A_{i},u_{i},U_{i})_{2}^{r}\\(1,1-\mathcal{P}-\mathfrak{O},-e),(B_{j},v_{j},V_{j})_{1}^{s}\end{matrix}\right].$$

**(v) Hermite Polynomial:** If we set $A_{V,\mathfrak{O}}=(-1)^{\mathfrak{O}}$ and $U=2$ in (1.4), then $S_{V}^{2}[t]\rightarrow t^{V/2}H_{V}\left(\frac{1}{2\sqrt{t}}\right)$

$$\text{where,}\quad H_{V}(t)=\sum_{\mathfrak{O}=0}^{[V/2]}(-1)^{R}\frac{V!}{\mathfrak{O}!(V-2\mathfrak{O})!}(2t)^{V-2\mathfrak{O}}.$$

**Corollary 9** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$, then we get the following result:*

$$\int_{0}^{\infty}\left[\frac{h^{2}}{bh^{4}+2ah^{2}+c}\right]^{\mathcal{P}}[t\mathscr{X}]^{\frac{V}{2}}\;H_{V}\left(\frac{1}{2\sqrt{t\mathscr{X}}}\right){}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[\mathscr{X}^{e}z;y;x]dh$$
$$=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/2]}\frac{(-1)^{\mathfrak{O}}(V)!}{\mathfrak{O}!(V-2\mathfrak{O})!}\,t^{\mathfrak{O}}\times\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}$$
$$\times\,{}^{\Gamma}\mathbb{W}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e};y;x\,\middle|\,\begin{matrix}(A_{1},u_{1},U_{1};\tau),(1,1-\mathcal{P}-\mathfrak{O},-e),(A_{i},u_{i},U_{i})_{2}^{r}\\(1,1-\mathcal{P}-\mathfrak{O},-e),(B_{j},v_{j},V_{j})_{1}^{s}\end{matrix}\right].$$

**Corollary 10** *For $b>0, c\geq 0, a>-\sqrt{bc}$, and $\mathcal{P}>\frac{1}{2}$, then we get the following result:*

$$\int_{0}^{\infty}\left[\frac{h^{2}}{bh^{4}+2ah^{2}+c}\right]^{\mathcal{P}}[t\mathscr{X}]^{\frac{V}{2}}H_{V}\left(\frac{1}{2\sqrt{t\mathscr{X}}}\right){}^{\gamma}\mathbb{W}_{r,s}^{m,n}[\mathscr{X}^{e}z;y;x]dh$$
$$=\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}}\times\sum_{\mathfrak{O}=0}^{[V/2]}\frac{(-1)^{\mathfrak{O}}(V)!}{\mathfrak{O}!(V-2\mathfrak{O})!}\,t^{\mathfrak{O}}\times\frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{O}}}$$
$$\times\,{}^{\gamma}\mathbb{W}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e};y;x\,\middle|\,\begin{matrix}(A_{1},u_{1},U_{1};\tau),(1,1-\mathcal{P}-\mathfrak{O},-e),(A_{i},u_{i},U_{i})_{2}^{r}\\(1,1-\mathcal{P}-\mathfrak{O},-e),(B_{j},v_{j},V_{j})_{1}^{s}\end{matrix}\right].$$

**(vi) Laguerre Polynomial:** By setting $A_{V,\mathfrak{O}}\binom{V+\alpha}{V}\frac{1}{(\alpha+1)_{\mathfrak{O}}}$ and $U=1$ in (1.4) then $S_{V}^{1}[t]\rightarrow L_{V}^{(\alpha)}(t)$

$$L_{V}^{(\alpha)}(t)=\sum_{\mathfrak{O}=0}^{V}\binom{V+\alpha}{V-\mathfrak{O}}\frac{(-t)^{\mathfrak{O}}}{\mathfrak{O}!}.$$

**Corollary 11** *For $b > 0, c \geq 0, a > -\sqrt{bc}$, and $\mathcal{P} > \frac{1}{2}$, then we have the following outcome:*

$$\int_0^\infty \left[\frac{h^2}{bh^4 + 2ah^2 + c}\right]^{\mathcal{P}} L_V^{(\alpha)}(\mathscr{X} t)\ {}^{\Gamma}\mathbb{W}_{r,s}^{m,n}[\mathscr{X}^e z; y; x] dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a + \sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}} \times \sum_{\mathfrak{D}=0}^{V} \binom{V+\alpha}{V-\mathfrak{D}} \frac{(-t)^{\mathfrak{D}}}{\mathfrak{D}!} \times \frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{D}}}$$
$$\times {}^{\Gamma}\mathbb{W}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e}; y; x \,\middle|\, \begin{matrix} (A_1, u_1, U_1; \tau), (1, 1-\mathcal{P}-\mathfrak{D}, -e), (A_i, u_i, U_i)_2^r \\ (1, 1-\mathcal{P}-\mathfrak{D}, -e), (B_j, v_j, V_j)_1^s \end{matrix}\right].$$

**Corollary 12** *For $b > 0, c \geq 0, a > -\sqrt{bc}$, and $\mathcal{P} > \frac{1}{2}$, then we have the following outcome:*

$$\int_0^\infty \left[\frac{h^2}{bh^4 + 2ah^2 + c}\right]^{\mathcal{P}} L_V^{(\alpha)}(\mathscr{X} t)\ {}^{\gamma}\mathbb{W}_{r,s}^{m,n}[\mathscr{X}^e z; y; x] dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a + \sqrt{bc}\right)\right]^{\mathcal{P}-\frac{1}{2}}} \times \sum_{\mathfrak{D}=0}^{V} \binom{V+\alpha}{V-\mathfrak{D}} \frac{(-t)^{\mathfrak{D}}}{\mathfrak{D}!} \times \frac{1}{[2(a+\sqrt{bc})]^{\mathfrak{D}}}$$
$$\times {}^{\gamma}\mathbb{W}_{r+1,s+1}^{m,n+1}\left[z[2(a+\sqrt{bc})]^{-e}; y; x \,\middle|\, \begin{matrix} (A_1, u_1, U_1; \tau), (1, 1-\mathcal{P}-\mathfrak{D}, -e), (A_i, u_i, U_i)_2^r \\ (1, 1-\mathcal{P}-\mathfrak{D}, -e), (B_j, v_j, V_j)_1^s \end{matrix}\right].$$

**Remark**

1. Setting $U = 1, A_{V,\mathfrak{D}} = 1, \forall\ \mathfrak{D} \neq 0$ in Corollaries (1) and (2), yields the special case (in terms of incomplete $\mathbb{Y}$-function) of Theorems 2.1 and 2.2.
2. Setting $A_i = B_j = 1$ in Corollaries (5) and (6) gives incomplete $\mathbb{D}$-function which is the incomplete extension of the $\mathbb{D}$-function defined by Yang and Yu [18].
3. Setting $A_i = B_j = 1$ in Corollaries (7) and (8) gives incomplete $\mathbb{T}$-function which is the incomplete extension of the $\mathbb{T}$-function defined by Yang and Yu [18].

## 4. Conclusion

The significance of these results lies in their broad applicability. By appropriately selecting and adjusting the parameters in the incomplete $\mathbb{W}$-functions, we can derive various outcomes that involve useful special functions. These results can be expressed in terms of the $\mathbb{Y}$-function, $I$-function, $\mathbb{U}$-function, and $\mathbb{O}$-function. We also investigated the Boros integral for the product of the incomplete $\mathbb{Y}$-function and the Srivastava polynomial.

The Srivastava polynomial also generalizes several other polynomials, such as the Hermite, Jacobi, Laguerre, Gegenbauer, Legendre, Tchebycheff, Gould-Hopper polynomials, and many more. Our main findings are significant and can be used to evaluate various Boros integral forms related to different special functions and polynomials.

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