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IT-2 General Fuzzy Automata

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ABSTRACT: This research is dedicated to the investigation and development of interval type-2 general fuzzy automata (IT-2 GFA). It explores the connections between IT-2 general fuzzy automata and interval type-2 fuzzy regular grammars. The study also introduces and formalizes several operations on IT-2 fuzzy languages. The results indicate that the class of IT-2 fuzzy languages recognized by IT-2 general fuzzy automata is closed under union, intersection, concatenation, and Kleene closure operations, but not under complementation. To clarify the theoretical concepts and findings, the paper includes a number of illustrative examples

Key Words: General fuzzy automata, regular, grammars, language, interval type-2 fuzzy set, closure operator.

Contents

1. Introduction

The concept of type-2 fuzzy sets was introduced by Lotfi Zadeh [22], who aimed to model and emphasize the impact of uncertainty within fuzzy logic rule-based systems. In 2002, Jerry Mendel [10] noted that the membership function of type-1 fuzzy sets is entirely crisp, which limits their ability to effectively model certain ambiguities and uncertainties. In contrast, type-2 fuzzy sets can address such uncertainties due to their fuzzy membership functions, which are three-dimensional. This additional dimension allows for greater flexibility in modeling uncertainty compared to the two-dimensional membership functions found in type-1 fuzzy sets. Despite their advantages, type-2 fuzzy sets can be challenging to apply, leading to a preference for interval type-2 fuzzy sets in many practical applications, as they simplify computations [11]. Building on the theory of fuzzy sets, researchers such as Santos [13], Wee [18] and Wee and Fu [19] have explored fuzzy automata and languages, contributing to a rich body of literature on the subject parallel, classical algebraic foundations of automata theory are well documented by Holcombe [4], which provide the theoretical background for later fuzzy generalizations. Further, Malik, Mordeson and Sen [12] examined and developed the concept in more detail. Over the past few decades, numerous studies have been conducted on fuzzy automata and languages, demonstrating their potential not only for transforming classical automata into fuzzy automata but also for a wide array of applications. (cf., [1,2,5,6,8,9,14,15,16,17,21]). Accordingly, it has been found that fuzzy automata and fuzzy languages have gained not only conversion of classical automata to fuzzy automata but also a wide range of applications [2]. Fuzzy automata and fuzzy languages have primarily been based on type-1 fuzzy sets or specific lattice structures (cf., [5,6,9,21]). However, type-1 fuzzy sets often fall short in minimizing the uncertainties present in models. Mendel [11] has highlighted the benefits of using interval type-2 fuzzy sets in the context of computing with words, reinforcing their applicability. Additionally, Doostfatemeh and Kremer [3] proposed the concept of general fuzzy automata to address the shortcomings in existing

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literature regarding applications that depend on fuzzy automata for modeling and assigning membership values to active states. In their framework, a zero-weight transition does not necessarily imply no transition, which contrasts with traditional automata. More recently, Jing and Tang [7] have investigated IT-2 fuzzy automata and languages, laying the groundwork for modeling non-classical computations. In the current research, we provide a brief overview of studies conducted on interval type-2 (IT-2) general fuzzy automata and languages. Furthermore, we present specific operations on IT-2 fuzzy languages, demonstrating that these languages, defined by IT-2 general fuzzy automata, are closed under operations such as union, intersection, concatenation, and Kleene closure. However, it is important to note that they are not closed under the complement operation.
The motivation behind this research stems from the need to more effectively model and manage uncertainty in fuzzy logic systems, particularly in the context of automata and formal languages. Traditional type-1 fuzzy sets, while widely used, have inherent limitations due to their crisp membership functions, which restrict their ability to capture and represent complex uncertainties and ambiguities present in real-world applications. Type-2 fuzzy sets, introduced by Lotfi Zadeh, offer a more powerful framework by allowing the membership function itself to be fuzzy, thus providing a three-dimensional representation of uncertainty. However, the practical implementation of type-2 fuzzy sets is often hindered by their computational complexity. To address this, interval type-2 fuzzy sets (IT-2 FS) have been proposed as a computationally efficient alternative that retains much of the expressive power of general type-2 fuzzy sets. In the field of formal languages and automata theory, most existing research has focused on models based on type-1 fuzzy sets or specific lattice structures. These models, however, are often inadequate for minimizing the uncertainties inherent in many practical systems. The concept of general fuzzy automata (GFA) was introduced to overcome some of these limitations, allowing for more flexible state transitions and membership assignments. Recent studies have begun to explore the integration of interval type-2 fuzzy sets into automata models, leading to the development of interval type-2 general fuzzy automata (IT-2 GFA). This approach promises to enhance the modeling capabilities of fuzzy automata by more accurately capturing the range of uncertainties encountered in computational processes. The primary motivation of this research is to advance the theoretical foundation of IT-2 GFA and their associated languages, to investigate their algebraic properties, and to establish their relationships with interval type-2 fuzzy grammars. By doing so, the study aims to provide a robust framework for modeling, analyzing, and processing information in systems characterized by high levels of uncertainty, thereby broadening the practical applicability of fuzzy automata in various domains.

2. Preliminaries

In this section, we introduce and clarify several concepts related to type-2 fuzzy sets, interval type-2 (IT-2) fuzzy sets, IT-2 fuzzy relations, and general fuzzy automata. We will derive and present key results that will be essential for the discussions in the following sections. Throughout this study, we denote Σ as a nonempty set, with $I = [0, 1]$ representing the interval, and $[I] = \{[a, b] : a \leq b; a, b \in I\}$ indicating the collection of all closed intervals within I.

Definition 2.1 *[10] A type-2 fuzzy set, denoted as $\tilde{\mathcal{A}}$ is characterized by a type-2 membership function $\mu_{\tilde{\mathcal{A}}} : \Sigma \times J_a \to I$, $\forall a \in \Sigma$ where $J_a \subseteq I$. This can be expressed as:*

$$\tilde{\mathcal{A}} = \{((a, u), \mu_{\tilde{\mathcal{A}}}(a, u)) : a \in \Sigma, u \in J_a \subseteq I\},$$

with the condition that $0 \leq \mu_{\tilde{\mathcal{A}}}(a, u) \leq 1$. Additionally, $\tilde{\mathcal{A}}$ can be represented as:

$$\tilde{\mathcal{A}} = \int_{a\in\Sigma} \int_{u\in J_a} \mu_{\tilde{\mathcal{A}}}(a, u)/(a, u), J_a \subseteq I,$$

where the double integral signifies the union over all permissible values of a and u. In cases where the universe of discourse is discrete, the integral notation is replaced by the summation symbol $\sum$.

Definition 2.2 *[11] A type-2 fuzzy set $\tilde{\mathcal{A}}$ is classified as an interval type-2 (IT-2) fuzzy set if its type-2 membership function meets the condition $\mu_{\tilde{\mathcal{A}}}(a, u) = 1$ for every $a \in \Sigma$ and $\forall u \in J_a \subseteq I$. An IT-2 fuzzy*

set $\tilde{\mathcal{A}}$ can be represented in two forms:
1. As a set of ordered pairs:

$$\tilde{\mathcal{A}} = \{((a,u),1) : a \in \Sigma, u \in J_a\}$$

2. Using double integral notation:

$$\tilde{\mathcal{A}} = \int_{a\in\Sigma} \int_{u\in J_a} 1/(a,u), J_a \subseteq I.$$

Both forms indicate that the membership grade is consistently 1 for all pairs (a,u) that belong to the IT-2 fuzzy set $\tilde{\mathcal{A}}$.

Definition 2.3 *[10] Let $\tilde{\mathcal{A}}$ be a type-2 fuzzy set defined over the set Σ. For each element $a' \in \Sigma$, the vertical slice $\mu_{\tilde{\mathcal{A}}}(a')$ of $\tilde{\mathcal{A}}$ represents the intersection of the two-dimensional plane defined by the axes $u \in J_a$ and $\mu_{\tilde{\mathcal{A}}}(a',u)$ with the three-dimensional type-2 membership function $\tilde{\mathcal{A}}$. This can be expressed as:*

$$\mu_{\tilde{\mathcal{A}}}(a') \equiv \mu_{\tilde{\mathcal{A}}}(a=a',u) = \int_{u\in J_{a'}} f_{a'}(u)/u, J_{a'} \subseteq I,$$

where the condition $0 \le f_{a'}(u) \le 1$ holds. For an interval type-2 fuzzy set (IT-2 fuzzy set) $\tilde{\mathcal{A}}$, $\mu_{\tilde{\mathcal{A}}}(a=a',u)$ the vertical slice is characterized as:

$$\mu_{\tilde{\mathcal{A}}}(a') \equiv \mu_{\tilde{\mathcal{A}}}(a=a',u) = \int_{u\in J_{a'}} 1/u, J_{a'} \subseteq I,$$

For convenience, we will denote this vertical slice simply as $\mu_{\tilde{\mathcal{A}}}$ instead of $\mu_{\tilde{\mathcal{A}}}(a'), \forall a' \in \Sigma$. This vertical slice functions as a type-1 fuzzy set, which is referred to as the secondary membership function. The domain of this secondary membership function is known as the primary membership of a, which can also be referred to as the secondary set. In relation to the vertical slice, an IT-2 fuzzy set $\tilde{\mathcal{A}}$ can also be represented as:

$$\tilde{\mathcal{A}} = \{(a, \mu_{\tilde{\mathcal{A}}}(a)) : a \in \Sigma\}$$

or equivalently as:

$$\tilde{\mathcal{A}} = \int_{a\in\Sigma} \mu_{\tilde{\mathcal{A}}}(a)/a = \int_{a\in\Sigma} \left[\int_{u\in J_a} 1/u\right]/a, \text{ where } J_a \subseteq I$$

represents the primary membership of a.
In this study, we will denote the set of all IT-2 fuzzy sets over Σ as $IT-2F(\Sigma)$.
If both Σ and J_a are discrete, the IT-2 fuzzy set $\tilde{\mathcal{A}} \in IT-2F(\Sigma)$ can be represented as follows:

$$\begin{aligned}\tilde{\mathcal{A}} &= \sum_{a\in\Sigma}\left[\sum_{u\in J_a} 1/u\right]/a = \sum_{i=1}^{N}\left[\sum_{u\in J_a} 1/u\right]/a_i \\ &= \left[\sum_{k=1}^{S_1} 1/u_{1k}\right]/a_1 + \ldots + \left[\sum_{k=1}^{S_i} 1/u_{ik}\right]/a_i + \ldots + \left[\sum_{k=1}^{S_N} 1/u_{Nk}\right]/a_N,\end{aligned}$$

where the symbol $+$ denotes the union.
Throughout this study, we will consider both Σ and J_a as discrete. As noted in [16], the element a can be discretized into N values, and at each of these values, u can be discretized into S_i values. The discretization for each u_{ik} does not need to be uniform, but if they are the same across all u_{ik}, then we have $S_1 = S_2 = \ldots = S_N = S$.

Definition 2.4 *[7] An interval type-2 fuzzy grammar (IT-2 FG) is defined as a quadruple $G = (N, T, P, S)$, where:*
1. N represents a finite set of nonterminal symbols,
2. T denotes a finite set of terminal symbols, with the condition that $N \cap T = \phi$ (they are disjoint),
3. $S \in N$ is the starting nonterminal symbol,

4. P is a finite collection of IT-2 fuzzy productions over $N \cup T$ *specified as* $P = \{a \xrightarrow{\mathfrak{p}} b | a \in (N \cup T)^* N (N \cup T)^*, b \in (N \cup T)^*\}$, *where* $\mathfrak{p}$ *is a mapping from* $(N \cup T)^* \times (N \cup T)^*$ *to* $IT - 2F(T^*)$, *referred to as the IT-2 fuzzy transition function.*
In this context, $\mathfrak{p}(a, b)$ *represents the degree of membership indicating that the nonterminal a can be replaced by b, expressed as* $\mathfrak{p}(a, b) = \mathfrak{p}(a \to b)$. *For simplicity, the notation,* $a \xrightarrow{\mathfrak{p}} b$ *is often abbreviated to* $a \to b$ *in the set P.*

Definition 2.5 *[7] Let the production* $a \overset{\mathfrak{p}}{\Longrightarrow} b$ *denote a derivation. If c and d are elements of* $c, d \in (N \cup T)^*$, *then we can derive cbd from cad, which we represent as*

$$cad \overset{\mathfrak{p}}{\Longrightarrow} cbd.$$

For a sequence $a_i \in (N \cup T)^*$, *for* $i = 1, 2, 3, \ldots, m$ *and* a_{i+1} *can be directly derived from* a_i *for* $i = 1, 2, 3, \ldots, m-1$, *then* a_1 *is said to derive* a_m *in grammar G denoted as* $a_1 \underset{G}{\overset{\mathfrak{p}}{\Longrightarrow}} a_m$. *We can represent the derivation chain of* a_m *from* a_1 *as follows:*

$$a_1 \overset{\mathfrak{p}_1}{\Longrightarrow} a_2 \overset{\mathfrak{p}_2}{\Longrightarrow} a_3 \overset{\mathfrak{p}_3}{\Longrightarrow} \ldots \overset{\mathfrak{p}_{m-1}}{\Longrightarrow} a_m$$

In this context, we characterize p as the height of the intersection of the productions

$$\mathfrak{p} = height(\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \ldots \cap \mathfrak{p}_{m-1}).$$

Definition 2.6 *Let* $G = (N, T, P, S)$ *represent an interval type-2 fuzzy grammar (IT-2 FG). The IT-2 fuzzy language* $\mathcal{L}(G)$ *generated by G is defined as:*

$$\mathcal{L}(G, w) = height\{\mathfrak{p} : S \underset{G}{\overset{\mathfrak{p}}{\Longrightarrow}}^{*} w | w \in T^*\}.$$

In this expression, $IT - 2F(T^*)$ *denotes the collection of all interval type-2 fuzzy sets over the set of terminal strings* T^*. *Here,* $\mathcal{L}(G, w)$ *captures the height of the productions leading from the starting symbol S to any string w in the terminal set* T^*, *highlighting the fuzzy nature of the language generated by the grammar.*

Definition 2.7 *[3] A fuzzy set* μ_Q *defined on a set Q (discrete or continuous), has been a function mapping each element of Q to a unique element of the interval [0,1].*

$$\mu_Q : Q \to [0, 1]$$

Thenthe fuzzy power set of Q identified as $\tilde{P}(Q)$ *is the set of all fuzzy subsets* μ_Q, *which can be characterized on the set Q.*

$$\tilde{P}(Q) = \{\mu_Q | \mu_Q : Q \to [0, 1]\}$$

Definition 2.8 *[3] A general fuzzy automaton (GFA) is defined as:*

$$\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2),$$

where
(1) Q *is a finite set of states,* $Q = \{q_1, q_2, \ldots, q_n\}$,
(2) Σ *is a finite set of input symbols, denoted as* $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(3) $\tilde{R}$ *is the set of fuzzy start states, where* $\tilde{R} \subseteq \tilde{P}(Q)$,
(4) Z *is a finite set of output symbols, expressed as* $Z = \{b_1, b_2, \ldots, b_k\}$,
(5) ω *is the output function defined as* $\omega : Q \to Z$,
(6) $\tilde{\delta}$ *is the augmented transition function represented as* $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \to [0, 1]$.
(7) F_1 *is the membership assignment function defined as* $F_1 : [0, 1] \times [0, 1] \to [0, 1]$ *This function takes two parameters,* μ *and* δ, *where* μ *is the membership value of a predecessor state and* δ *is the weight of a transition.*

In this framework, the transition from state q_i to state q_j upon receiving input a_k is expressed as:

$$\mu^{t+1}(q_j) = \tilde{\delta}_{a_k}((q_i, \mu^t(q_i)), q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta_{a_k}(q_i, q_j)).$$

This equation indicates that the membership value (mv) of state q_j at time $t+1$ is calculated using the function F_1, which incorporates the membership value of state q_i at time t and the weight of the transition. The function $F_1(\mu, \delta)$ can take various forms, such as $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$, $\frac{\mu+\delta}{2}$, or other related mathematical functions. As indicated in the formulas, each fuzzy transition is associated with a membership value (mv) within the unit interval $[0, 1]$, which we refer to as the weight of the transition. The transition from the current state q_i to the next state q_j upon input a_k is denoted as $\delta_{a_k}(q_i, a_k, q_j)$, simplified to $\delta_{a_k}(q_i, q_j)$. This notation encompasses both the transition and its weight. When $\delta_{a_k}(q_i, q_j)$ is referenced, it pertains to the weight of the transition while also indicating the transition itself. The set of all transitions in a general fuzzy automaton $\tilde{F}$ is denoted as $\Delta_{\tilde{F}}$. In contexts where the subscript is understood, we simplify this to Δ.
(8) $F_2 : [0, 1]^* \to [0, 1]$, is called multi-membership resolution function. The multi-membership resolution function ascertains the multi-membership active states and assigns a single membership value to them. We let $Q_{act}(t_i)$ be the set of all active state at time t_i, $\forall_i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) | \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta_a(q', q) \in \Delta\}$, $\forall i \geq 1$. Given that $Q_{act}(t_i)$ is a fuzzy set, to demonstrate that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should demonstrate it as: $q \in Domain(Q_{act}(t_i))$ and $T \subseteq Domain(Q_{act}(t_i))$; hereafter, we simply denote them by: $q \in Q_{act}(t_i)$ and $T \subseteq Q_{act}(t_i)$.

Definition 2.9 *[3] The successor set of a state q_m on input symbol a_k denoted as $Q_{succ}(q_m, a_k)$, is the set of all states q_j which will be accomplished through transitions $\delta_{a_k}(q_m, q_j)$,*

$$Q_{succ}(q_m, a_k) = \{q_j | \delta_{a_k}(q_m, q_j) \in \triangle\}.$$

Correspondingly, we can characterize the predecessor set of a state set as follows:

$$Q_{pred}(q_m, a_k) = \{q_j | \delta_{a_k}(q_j, q_m) \in \triangle\}.$$

3. Language of IT-2 General Fuzzy Automata

In the following section, we propose and clarify the concept of IT-2 general fuzzy automata, establishing their relationship with IT-2 fuzzy languages.

Definition 3.1 *Let $\tilde{F} = (Q, \Sigma, \tilde{R}, \tilde{\delta}, Z, \omega, F_1, F_2)$ be a general fuzzy automaton. An IT-2 fuzzy set $Q_{act}(t)$ of Q is defined as follows:*

$$Q_{act}(t) = \{(q_m, \mu^t(q_m)) | \exists q_i \in Q_{act}(t-1), a_k \in \Sigma, q_m \in Q_{succ}(q_i, a_k)\}$$

or, as the following:

$$Q_{act}(t) = \int_{q_m \in Q_{succ}(q_i, a_k)} \mu^t(q_m)/q_m = \int_{q_m \in Q_{succ}(q_i, a_k)} \left[\int_{\mu \in J_{q_m}} 1/\mu\right] /q_m,$$

where $J_{q_m} \subseteq I$ has been the primary membership of q_m.

In this study, IT-2 $F(Q)$ will indicate the set of all IT-2 fuzzy sets in Q.
If both Q and J_{q_m} are discrete $Q_{act}(t) \in IT-2F(Q)$ can be stated as follows:

$$\begin{aligned} Q_{act}(t) &= \sum_{q_m \in Q_{succ}(q_i, a_k)} \left[\sum_{\mu \in J_{q_m}} 1/\mu\right] /q_m \\ &= \sum_{i=1}^{n} \left[\sum_{\mu \in J_{q_i}} 1/\mu\right] /q_i \\ &= \left[\sum_{k=1}^{M_1} 1/\mu_{1k}\right] /q_1 + \ldots + \left[\sum_{k=1}^{M_i} 1/\mu_{ik}\right] /q_i + \ldots + \left[\sum_{k=1}^{M_n} 1/\mu_{nk}\right] /q_n, \end{aligned}$$

where + denotes the union.
All through the present study, we regard both Q and J_q as discrete.

Definition 3.2 *The footprint of uncertainty for a type-2 fuzzy set $UQ_{act}(t)$, of a T-2 fuzzy set $Q_{act}(t)$ is defined as:*

$$UQ_{act}(t) = \bigcup_{q_m \in Q_{act}(t)} J_{q_m}.$$

Let $UQ_{act}(t) = J_{q_m}$ for all $q_m \in Q_{act}(t)$. Consequently, a type-2 fuzzy set $Q_{act}(t)$ can be represented as:

$$Q_{act}(t) = \int \int_{(q_m,\mu^t(q_m))\in UQ_{act}(t)} \mu(q_m, \mu^t(q_m))/(q_m, \mu^t(q_m)).$$

For a given type-2 fuzzy set $Q_{act}(t)$, there exist two type-1 membership functions that serve as the bounds of $UQ_{act}(t)$: the lower membership function, denoted as $\underline{\mu}$ corresponds to the lower bound, while the upper membership function, denoted as $\bar{\mu}$ corresponds to the upper bound of $UQ_{act}(t)$. For $Q_{act}(t) \in IT-2F(Q)$ and $q_m \in Q_{succ}(q_i, a_k)$, the membership function $\mu^t(q_m)$ is an interval type-1 fuzzy set on I. Therefore, we have: $UQ_{act}(t)(q_m) = [\underline{\mu}(q_m), \bar{\mu}(q_m)]$, where $\underline{\mu}(q_m)$ and $\bar{\mu}(q_m)$ represent the lower and upper membership functions, respectively, both of which are type-1 fuzzy sets. Consequently, the membership grade of each element in the type-2 fuzzy set is expressed as the interval $[\underline{\mu}(q_m), \bar{\mu}(q_m)]$.
Thus, any $Q_{act}(t) \in IT-2F(Q)$ can also be characterized as:

$$Q_{act}(t) = 1/UQ_{act}(t).$$

Definition 3.3 *Let $Q_{act}(t), Q_{act}(t+1) \in IT-2F(Q)$. Then*
1) the union of IT-2 fuzzy sets $Q_{act}(t), Q_{act}(t+1)$ is

$$Q_{act}(t) \cup Q_{act}(t+1) = 1/[\underline{\mu}(q) \vee \underline{\mu}(p), \bar{\mu}(q) \vee \bar{\mu}(p)],$$

where $p \in Q_{succ}(q, a)$ for $p, q \in Q$, $a \in \Sigma$;
2) the intersection of IT-2 fuzzy sets $Q_{act}(t), Q_{act}(t+1)$ is

$$Q_{act}(t) \cap Q_{act}(t+1) = 1/[\underline{\mu}(q) \wedge \underline{\mu}(p), \bar{\mu}(q) \wedge \bar{\mu}(p)],$$

where $p \in Q_{succ}(q, a)$ for $p, q \in Q$, $a \in \Sigma$;
3) the complement of IT-2 fuzzy set $Q_{act}(t)$ is

$$Q^c_{act}(t) = 1/[1 - \underline{\mu}(q), 1 - \bar{\mu}(q)],$$

for any $q \in Q_{act}(t)$;
4) the scale product of α and IT-2 fuzzy set $Q_{act}(t)$ is

$$\alpha.Q_{act}(t) = 1/[\underline{\alpha} \wedge \underline{\mu}(q), \bar{\alpha} \wedge \bar{\mu}(q)],$$

for any $q \in Q_{act}(t)$, where $\alpha = 1/[\underline{\alpha}, \bar{\alpha}]$, $[\underline{\alpha}, \bar{\alpha}] \in [0, 1]$ signifies the set of all closed subintervals of $[0, 1]$.
5) the height of IT-2 fuzzy set $Q_{act}(t)$ is

$$height(Q_{act}(t)) = 1/[\vee_{q\in Q_{act}(t)}\underline{\mu}(q), \vee_{q\in Q_{act}(t)}\bar{\mu}(q)].$$

Example 3.1 *Consider the GFA in Fig. 1, it has been indicated as $\tilde{F} = (Q, \Sigma, \tilde{R}, \tilde{\delta}, Z, \omega, F_1, F_2)$, where $Q = \{q_0, q_1, q_2\}$ has been the set of states, $\Sigma = \{a, b\}$ has been the set of all input alphabet, $\tilde{R} = \{(q_0, 1)\}$, $Z = \emptyset$ and ω has not been applicable. We verify operation of the GFA in Example 3.1 upon input "aba". If we select $F_1(\mu, \delta) = \delta, F_2() = \mu^{t+1}(q_m) = \wedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m))$, then we achieve (Table 1):*

$$
\begin{aligned}
&\mu^{t_0}(q_0) = 1,\\
&\mu^{t_1}(q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = \delta(q_0, a, q_1) = 0.2,\\
&\mu^{t_2}(q_0) = F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_0)) = \delta(q_1, b, q_0) = 0.6,\\
&\mu^{t_2}(q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, b, q_2)) = \delta(q_1, b, q_2) = 0.4,\\
&\mu^{t_3}(q_0) = F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_0)) = \delta(q_2, a, q_0) = 0.3,\\
&\mu^{t_3}(q_1) = F_1(\mu^{t_2}(q_0), \delta(q_0, a, q_1)), F_1(\mu^{t_2}(q_2), \delta(q_2, a, q_1))\\
&\qquad\quad = \delta(q_0, a, q_1) \wedge \delta(q_2, a, q_1) = 0.2 \wedge 0.1 = 0.1.
\end{aligned}
$$

By the Definition 3.1, we have an IT-2 fuzzy subset $Q_{act}(t_i)$ of Q for any $i \geq 0$ as follows:

$$Q_{act}(t_0) = \{(q_0, 1)\}, Q_{act}(t_1) = \{(q_1, 0.2)\}, Q_{act}(t_2) = \{(q_2, 0.4)\}, Q_{act}(t_3) = \{(q_0, 0.3), (q_1, 0.1)\};$$

or, as the following:

$$Q_{act}(t_0) = 1/1/q_0, Q_{act}(t_1) = 1/0.2/q_1, Q_{act}(t_2) = 1/0.6/q_0 + 1/0.4/q_2, Q_{act}(t_3) = 1/0.3/q_0 + 1/0.1/q_1.$$

Then

$$
\begin{aligned}
&UQ_{act}(t_0) = 1/1/q_0 + 1/0.6/q_0 + 1/0.3/q_0 = [0.3, 1],\\
&UQ_{act}(t_1) = 1/0.2/q_1 + 1/0.1/q_1 = [0.1, 0.2],\\
&UQ_{act}(t_2) = 1/1/q_0 + 1/0.6/q_0 + 1/0.4/q_2 = [0.4, 1],\\
&UQ_{act}(t_3) = 1/1/q_0 + 1/0.6/q_0 + 1/0.1/q_1 + 1/0.2/q_1 = [0.1, 1].
\end{aligned}
$$

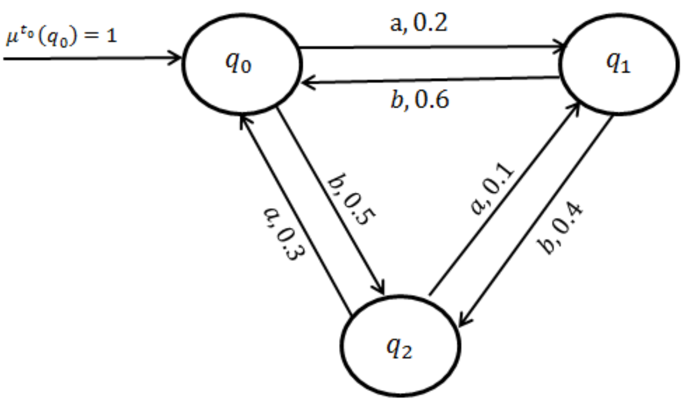

Figure 1: The GFA of Example 3.1

Table 1: Active states and their membership values (mv) at different times in Example 3.1

time	t_0	t_1	t_2	t_3
input	$\wedge$	a	b	a
$Q_{act}(t_i)$	q_0	q_1	q_0—q_2	q_0—q_1
mv	1	0.2	0.6—0.4	0.3—0.1

According to Definition 3.3, then we have:

$$\begin{aligned}
&Q_{act}(t_1)\cup Q_{act}(t_2)=1/[0.1\vee 0.4,0.2\vee 1]=1/[0.4,1],\\
&Q_{act}(t_1)\cap Q_{act}(t_2)=1/[0.1\wedge 0.4,0.2\wedge 1]=1/[0.1,0.2],\\
&Q^c_{act}(t_1)=1/[1-0.1,1-0.2]=1/[0.9,0.8],\\
&1/[0.3,0.4].Q_{act}(t_3)=1/[0.3\wedge 0.1,0.4\wedge 1]=1/[0.1,0.4].
\end{aligned}$$

Furthermore, $\underline{\mu}(\tilde{q_0})=0.3$, $\overline{\mu}(\tilde{q_0})=1$, $\underline{\mu}(\tilde{q_1})=0.1$, $\overline{\mu}(\tilde{q_1})=0.2$, $\underline{\mu}(\tilde{q_2})=0.4$, $\overline{\mu}(\tilde{q_2})=1$, $\underline{\mu}(\tilde{q_3})=0.1$, $\overline{\mu}(\tilde{q_3})=1$.

Definition 3.4 *An interval type-2 general fuzzy automaton (IT-2 GFA) has been ten-tuple* $\tilde{F}_{IT}=(\tilde{Q},\Sigma,\tilde{q}_0,\tilde{Z},\delta_{IT},\tilde{\delta}_{IT},\tilde{\omega},\tilde{q}_f,F_1,F_2)$, *where*
$\tilde{Q}$*: has been a set of all IT-2 fuzzy subsets* $Q_{act}(t_i)$ *of* Q*, for any* $i\geq 0$*. For simply we show that IT-2 fuzzy subset* $Q_{act}(t_i)$ *by* $\tilde{q}_i$*;*
Σ *: has been a finite set of input alphabet;*
$\tilde{q}_0$*: has been an IT-2 fuzzy subset* $Q_{act}(t_0)$*, recognized as the IT-2 fuzzy set of initial states;*
$\tilde{Z}$*: has been a finite set of output alphabet, in which* $\tilde{Z}$ *has been a power set of* Z*;*
$\delta_{IT}:\tilde{Q}\times\Sigma\to\tilde{Q}$*: has been the next state map characterized by* $\delta_{IT}(\tilde{q}_i,a)=\tilde{q}_{i+1}$*, such that* $\delta_{IT}(\tilde{q}_i,a)$ *has been an IT-2 fuzzy subset of* Q*, and it may be considered as the possibility distribution of the states that the automaton in state* $\tilde{q}_i$ *and with input* a *can enter;*
$\tilde{\delta}_{IT}:\tilde{Q}\times\Sigma\times\tilde{Q}\to[0,1]$*: is the transition function which has been applied to map a state (current state) into another state (next state) upon an input alphabet, assigning a value in the fuzzy interval* $[0,1]$ *as follows:*

$$\tilde{\delta}_{IT}(\tilde{q}_i,a,\tilde{q}_j)=(\underline{\mu}(\tilde{q}_i)\vee\underline{\mu}(\tilde{q}_j))\wedge(\overline{\mu}(\tilde{q}_i)\vee\overline{\mu}(\tilde{q}_j));$$

$\tilde{\omega}$*: is mapping from* $\tilde{Q}$ *into* $\tilde{Z}$*. Formally,* $\tilde{\omega}$ *is identified as follows:*

$$\begin{aligned}
&\tilde{\omega}:\tilde{Q}\to\tilde{Z}\\
&\tilde{\omega}(\tilde{q}_i)=\bigcup_{q_i\in Q_{act}(t_i)}\omega(q_i);
\end{aligned}$$

$\tilde{q}_f$*: has been an IT-2 fuzzy subset of* Q*, distinguished as the IT-2 fuzzy set of final states.*

Example 3.2 *Consider the general fuzzy automaton in Example 3.1. By the Definition 3.4 we have the IT-2 GFA in Fig. 2, where* $\tilde{Q}=\{\tilde{q}_0,\tilde{q}_1,\tilde{q}_2,\tilde{q}_3\}$, $\Sigma=\{a,b\}$, $\tilde{q}_0=1/1/q_0$, $\tilde{Z}=\emptyset$ *and* $\tilde{\omega}$ *has not been applicable.*

$$\begin{aligned}
\tilde{\delta}(\tilde{q}_0,a,\tilde{q}_1)&=(\underline{\mu}(\tilde{q}_0)\vee\underline{\mu}(\tilde{q}_1))\wedge(\overline{\mu}(\tilde{q}_0)\vee\overline{\mu}(\tilde{q}_1))\\
&=(0.3\vee 0.1)\wedge(1\vee 0.2)\\
&=0.3\wedge 1=0.3,\\
\tilde{\delta}(\tilde{q}_1,b,\tilde{q}_2)&=(\underline{\mu}(\tilde{q}_1)\vee\underline{\mu}(\tilde{q}_2))\wedge(\overline{\mu}(\tilde{q}_1)\vee\overline{\mu}(\tilde{q}_2))\\
&=(0.1\vee 0.4)\wedge(0.2\vee 1)\\
&=0.4\wedge 1=0.4,\\
\tilde{\delta}(\tilde{q}_2,a,\tilde{q}_3)&=(\underline{\mu}(\tilde{q}_2)\vee\underline{\mu}(\tilde{q}_3))\wedge(\overline{\mu}(\tilde{q}_2)\vee\overline{\mu}(\tilde{q}_3))\\
&=(0.4\vee 0.1)\wedge(1\vee 1)\\
&=0.4\wedge 1=0.4.
\end{aligned}$$

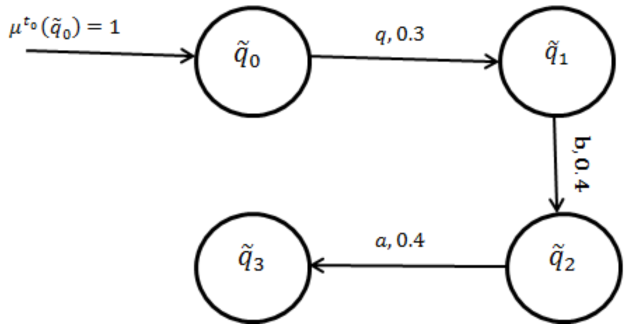

Figure 2: The IT-2 GFA of Example 3.2

Definition 3.5 *The mv of a string $w \in \Sigma^*$ indicated as $\mu(w)$, has been the maximum membership value among all its derivations, in which the mv of a derivation has been the minimum transition weight encountered in that derivation. Bearing in mind that w contains n derivations, and $w = a_1 a_2 \dots a_k a_{k+1} \dots a_m$, and that the ith derivation of w is:*

$$der_i(w) = \tilde{q}_{i_0}\tilde{q}_{i_1} \dots \tilde{q}_{i_k}\tilde{q}_{i_{k+1}} \dots \tilde{q}_{i_m},$$

where $\tilde{q}_{i_0} \in \tilde{q}_0$, the mv of $der_i(w)$ has been calculated as:

$$\mu(der_i(w)) = Min\{\delta_{IT}(\tilde{q}_{i_0}, a_1, \tilde{q}_{i_1}), \dots, \delta_{IT}(\tilde{q}_{i_{m-1}}, a_m, \tilde{q}_{i_m})\}$$

and the general formula to calculate the mv of w will be:

$$\mu(w) = \bigvee_{i=1}^{n}\{\mu(der_i(w))\} = \bigvee_{i=1}^{n}\{\bigwedge_{k=1}^{m}\{\delta_{IT}(\tilde{q}_{i_{k-1}}, a_k, \tilde{q}_{i_k})\}\}.$$

Definition 3.6 *Let $\tilde{F}_{IT} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{Z}, \delta_{IT}, \tilde{\delta}_{IT}, \tilde{\omega}, \tilde{q}_f, F_1, F_2)$ be an IT-2 general fuzzy automaton. An IT-2 Fuzzy set Σ^* of Σ is identified as follows:*

$$IT - 2F(\Sigma^*) = \{(w, \mu(w))|w \in \Sigma^*)\}.$$

The footprint of uncertainty, indicated by $UIT - 2F(\Sigma^*)$, of a T-2 fuzzy set Σ^* has been stated by

$$UIT - 2F(\Sigma^*) = \bigcup_{w \in \Sigma^*} J_w.$$

Definition 3.7 *An IT-2 fuzzy language $\tilde{\gamma} \in IT - 2F(\Sigma^*)$ is regarded to be recognizeed by an IT-2 GFA $\tilde{F}_{IT} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{Z}, \delta_{IT}, \tilde{\delta}_{IT}, \tilde{\omega}, \tilde{q}_f, F_1, F_2)$, if for any $w \in \Sigma^*$*

$$\tilde{\gamma}(w) = 1/[\bigvee\{\underline{\mu}_{\tilde{q}_0}(\tilde{q}_{i_0}) \wedge \underline{\mu}(w) \wedge \underline{\mu}_{\tilde{q}_f}(\tilde{q}_{i_m})|\tilde{q}_{i_0}, \tilde{q}_{i_m} \in \tilde{Q}\}, \bigvee\{\overline{\mu}_{\tilde{q}_0}(\tilde{q}_{i_0}) \wedge \overline{\mu}(w) \wedge \overline{\mu}_{\tilde{q}_f}(\tilde{q}_{i_m})|\tilde{q}_{i_0}, \tilde{q}_{i_m} \in \tilde{Q}\}].$$

Definition 3.8 *The active state set of aninput string w is the IT-2 fuzzy set of all active states, after string w has entered the IT-2 GFA*

$$Q_{act}(w) = \{(\tilde{q}_{i_m}, \mu^{t_0+|x|}(\tilde{q}_{i_m}))|\tilde{q}_{i_0}\tilde{q}_{i_1} \dots \tilde{q}_{i_k}\tilde{q}_{i_{k+1}} \dots \tilde{q}_{i_m} \in D_{der}(w)\},$$

where $D_{der}(w)$ is the set of all derivations of string w. We demonstrate that the IT-2 fuzzy set $Q_{act}(w)$ by $\tilde{q}_w$. For any $w \in \Sigma^*$, the degree to which w is identified by IT-2 GFA $\tilde{F}_{IT}$ is

$$\mathcal{L}(\tilde{F}_{IT}, w) = height(\tilde{q}_w \cap \tilde{q}_f).$$

An IT-2 fuzzy language $\mathcal{L}(\tilde{F}_{IT}) \in IT - 2F(\Sigma^*)$ accepted by $\tilde{F}_{IT}$ is distinguished as:

$$\mathcal{L}(\tilde{F}_{IT})(w) = \mathcal{L}(\tilde{F}_{IT}, w), \forall w \in \Sigma^*.$$

Theorem 3.1 *Let $G = (N, T, P, S)$ be an IT-2 fuzzy regular grammar, then there will be an IT-2 GFA $\tilde{F}_{IT} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{Z}, \delta_{IT}, \tilde{\delta}_{IT}, \tilde{\omega}, \tilde{q}_f, F_1, F_2)$, where $\tilde{Q} = \Sigma 1/[1,1]/N \cup q_f, \Sigma = T, \tilde{q}_0 = 1/[1,1]/S$. $\tilde{q}_f = 1/[1,1]/S + L_G(\wedge)/q_f$.*

$$\tilde{\delta}(1/[1,1]/A, a, 1/[1,1]/B) = \begin{cases} \mathfrak{p}(A \to aB), & A, B \in N \\ \mathfrak{p}(A \to a), & A \in N, B = q_f \\ 0, & otherwise \end{cases}$$

$\tilde{Z} = \emptyset$ and $\tilde{\omega}$ is not applicable.

Proof. For any $\alpha, \beta \in (N \cup T)^*, \mathfrak{p}(\alpha \to \beta) = height\{\mathfrak{p} : \alpha \overset{\mathfrak{p}}{\Longrightarrow}^* \beta\}$. We will present that $\mathcal{L}(\tilde{F}_{IT})(w) = \mathcal{L}(G)(w)$. Let $w \in V_T^*$ there will be two cases for w.
i) If $w = \wedge$, then $\mathcal{L}(\tilde{F}_{IT})(\wedge) = \mathcal{L}(G)(\wedge)$.
ii) If $w \neq \wedge$, let $w = a_1 a_2 \ldots a_n$. where $a_i \in T$ for $i = 1, 2, \ldots, n$.
If $S \overset{\mathfrak{p}}{\Longrightarrow}^* w$ there will be some derivation of w with the form

$$\begin{aligned} S &\overset{\mathfrak{p}_1}{\Longrightarrow} a_1 A_1 \\ &\overset{\mathfrak{p}_2}{\Longrightarrow} a_1 a_2 A_2 \\ &\vdots \\ &\overset{\mathfrak{p}_{n-1}}{\Longrightarrow} a_1 a_2 \ldots a_{n-1} A_{n-1} \\ &\overset{\mathfrak{p}_n}{\Longrightarrow} a_1 a_2 \ldots a_n, \end{aligned}$$

where $A_{i-1} \overset{\mathfrak{p}_i}{\Longrightarrow} a_i A_i \in P$ for $i = 1, 2, \ldots, n$ and $A_0 = S$. $A_n = a_n$ with $\mathfrak{p} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_n$.
Through the definition of $\tilde{q}_w$ of IT-2 GFA $\tilde{F}_{IT}$. we get

$$\begin{aligned} \tilde{q}_w(\tilde{q}_f) \supseteq & \tilde{\delta}_{IT}(1/[1,1]/S, a_1, 1/[1,1]/A_1) \wedge \ldots \\ & \wedge \tilde{\delta}_{IT}(1/[1,1]/A_{n-1}, a_n, \tilde{q}_f) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_n = \mathfrak{p}. \end{aligned}$$

Therefore, $\mathcal{L}(\tilde{F}_{IT})(w) = height(\tilde{q}_w \cap \tilde{q}_f) = \tilde{q}_w(\tilde{q}_f) \geq \mathfrak{p}$. Accordingly, we have illustrated that $\mathcal{L}(G)(w) \subseteq \mathcal{L}(\tilde{F}_{IT})(w)$ for any $w \in \Sigma^*$. that is $\mathcal{L}(G) \subseteq \mathcal{L}(\tilde{F}_{IT})$.
On the other hand, if $\mathcal{L}(\tilde{F}_{IT})(w) = \tilde{q}_w(\tilde{q}_f) \supseteq \tilde{\delta}_{IT}(1/[1,1]/S, a_1, 1/[1,1]/A_1) \wedge \tilde{\delta}_{IT}(1/[1,1]/A_1, a_2, 1/[1,1]/A_2) \wedge \ldots \wedge \tilde{\delta}_{IT}(1/[1,1]/A_{n-1}, a_n, \tilde{q}_f)$ for any $1/[1,1]/A_i \in \tilde{Q}$. let

$$\begin{aligned} &\tilde{\delta}_{IT}(1/[1,1]/S, a_1, 1/[1,1]/A_1) = \mathfrak{p}_1, \\ &\tilde{\delta}_{IT}(1/[1,1]/A_1, a_2, 1/[1,1]/A_2) = \mathfrak{p}_2, \\ &\quad \vdots \\ &\tilde{\delta}_{IT}(1/[1,1]/A_{n-1}, a_n, 1/[1,1]/A_n) = \mathfrak{p}_n, \end{aligned}$$

and let

$$\mathfrak{p} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_n,$$

subsequently, there will be corresponding derivation $S \overset{\mathfrak{p}}{\Longrightarrow}^* w$. That it demonstrates that if $\mathfrak{p} \subseteq \mathcal{L}(\tilde{F}_{IT})(w)$, then $\mathfrak{p} \subseteq \mathcal{L}(G)(w)$, then $\mathcal{L}(\tilde{F}_{IT}) \subseteq \mathcal{L}(G)$. Consequently, $\mathcal{L}(\tilde{F}_{IT}) = \mathcal{L}(G)$, that is IT-2 fuzzy languages can be recognizeed by IT-2 general Fuzzy automata. □

Theorem 3.2 *Let $\tilde{F}_{IT} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{Z}, \delta_{IT}, \tilde{\delta}_{IT}, \tilde{\omega}, \tilde{q}_f, F_1, F_2)$ be an IT-2 GFA, then there will be an IT-2 regular grammar $G = (N, T, P, S)$ such that $\mathcal{L}(G) = \mathcal{L}(\tilde{F}_{IT})$.*

Proof. We can propose an IT-2 regular grammar $G = (N, T, P, S)$, in which $T = \Sigma$, $N = \cup_{i=1}^{n} Q_{act}(t_i)$, $S = q_0$.The production rules in P has been recognized as follows:
1) For each $\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) \neq 0$ then $q_i \xRightarrow{\mathfrak{p}_1} aq_j \in P$ such that $\mathfrak{p}_1 = \tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j)$;
2) If $\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) \neq 0$ and $\tilde{q}_f(\tilde{q}_j) \neq 0$, then $q_i \xRightarrow{\mathfrak{p}_2} a \in P$, such that $\mathfrak{p}_2 = height_{\tilde{q}_j \in \tilde{Q}}(\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) \cap \tilde{q}_f(\tilde{q}_j))$;
3) If $\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) \neq 0$ and $\tilde{q}_0(\tilde{q}_i) \neq 0$, then $q_0 \xRightarrow{\mathfrak{p}_3} aq_j \in P$. such that $\mathfrak{p}_3 = \tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j)$;
4) If $\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) \neq 0$. $\tilde{q}_0(\tilde{q}_i) \neq 0$ and $\tilde{q}_f(\tilde{q}_j) \neq 0$, then $q_0 \xRightarrow{\mathfrak{p}_4} a \in P$, such that $\mathfrak{p}_4 = height_{\tilde{q}_j \in \tilde{Q}}(\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) \cap \tilde{q}_f(\tilde{q}_j))$.
We will illustrate that $\mathcal{L}(G) = \mathcal{L}(\tilde{F}_{IT})$. Let $w \in T^*$, then there will be two cases for w.
i) If $w = \wedge$, then $\mathcal{L}(G)(w) = \mathfrak{p}(q_0 \to \wedge) = \mathcal{L}(\tilde{F}_{IT})(w)$.
ii) when $w \neq \wedge$. let $w = a_1 a_2 \dots a_n$. where $a_i \in T$ for $i = 1, 2, \dots, n$. If $q_0 \xRightarrow{\mathfrak{p}}{}^{*} w$, there must be some derivation of w, having the form

$$\begin{aligned} q_0 &\xRightarrow{\mathfrak{p}_1} a_1 q_1 \\ &\xRightarrow{\mathfrak{p}_2} a_1 a_2 q_2 \\ &\vdots \\ &\xRightarrow{\mathfrak{p}_{n-1}} a_1 a_2 \dots a_{n-1} q_{n-1} \\ &\xRightarrow{\mathfrak{p}_n} a_1 a_2 \dots a_n = w, \end{aligned}$$

where $q_{i-1} \xRightarrow{\mathfrak{p}_i} a_i q_i \in P$ for $i = 1, 2, \dots, n$ and $q_0 = S$. $q_n = a_n$ with $\mathfrak{p} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_n$. Based on the definition of P. we recognized that

$$\begin{aligned} \mathcal{L}(\tilde{F}_{IT})(w) &= height(\tilde{q}_w(\tilde{q}_0) \cap \tilde{q}_f) \\ &\supseteq height(\tilde{\delta}_{IT}(\tilde{q}_0, a_1, \tilde{q}_1) \wedge \tilde{\delta}_{IT}(\tilde{q}_1, a_2, \tilde{q}_2) \wedge \dots \\ &\wedge \tilde{\delta}_{IT}(\tilde{q}_{n-1}, a_n, \tilde{q}_n) \cap \tilde{q}_f(\tilde{q}_n)) \\ &= \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_n = \mathfrak{p}. \end{aligned}$$

As a result, we have demonstrated that $\mathcal{L}(G, w) \subseteq \mathcal{L}(\tilde{F}_{IT}, w)$ for any $w \in \Sigma^*$, that is $\mathcal{L}(G) \subseteq \mathcal{L}(\tilde{F}_{IT})$. On the contrary, if

$$\begin{aligned} \mathcal{L}(\tilde{F}_{IT})(w) &= height(\tilde{q}_w(\tilde{q}_0) \cap \tilde{q}_f) \\ &= height(\tilde{\delta}_{IT}(\tilde{q}_0, a_1, \tilde{q}_1) \wedge \tilde{\delta}_{IT}(\tilde{q}_1, a_2, \tilde{q}_2) \wedge \dots \\ &\wedge \tilde{\delta}_{IT}(\tilde{q}_{n-1}, a_n, \tilde{q}_n) \cap \tilde{q}_f(\tilde{q}_n)) \\ &\supseteq height(\tilde{\delta}_{IT}(\tilde{q}_0, a_1, \tilde{q}_1) \wedge \tilde{\delta}_{IT}(\tilde{q}_1, a_2, \tilde{q}_2) \wedge \dots \\ &\wedge height_{\tilde{q}_n \in \tilde{Q}} \tilde{\delta}_{IT}(\tilde{q}_{n-1}, a_n, \tilde{q}_n) \cap \tilde{q}_f(\tilde{q}_n)). \end{aligned}$$

For any $\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n \in \tilde{Q}$. let $height(\tilde{\delta}_{IT}(\tilde{q}_0, a_1, \tilde{q}_1) \wedge \tilde{\delta}_{IT}(\tilde{q}_1, a_2, \tilde{q}_2) \wedge \dots \wedge height_{\tilde{q}_n \in \tilde{Q}} \tilde{\delta}_{IT}(\tilde{q}_{n-1}, a_n\ .\tilde{q}_n) \cap \tilde{q}_f(\tilde{q}_n) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_n = \mathfrak{p}$. subsequently, there will be corresponding derivation $q_0 \xRightarrow{\mathfrak{p}}{}^{*} w$. Hence, it is demonstrated that if $\mathfrak{p} \subseteq \mathcal{L}(\tilde{F}_{IT})(w)$, then $\mathfrak{p} \subseteq \mathcal{L}(G)(w)$, that is $\mathcal{L}(\tilde{F}_{IT})(w) \subseteq \mathcal{L}(G)(w)$ for any $w \in \Sigma^*$, that is $\mathcal{L}(\tilde{F}_{IT}) \subseteq \mathcal{L}(G)$. Therefore $\mathcal{L}(G) = \mathcal{L}(\tilde{F}_{IT})$, that is the languages accepted by IT-2 general fuzzy automata are IT-2 regular languages. □
Subsequently, in view of Theorem 3.1 and 3.2, there are two corollaries as follows:

Corollary 3.1 *IT-2 fuzzy regular grammars are said to be equivalent to IT-2 general fuzzy automata.*

Corollary 3.2 *IT-2 general fuzzy automata are said to be equivalent to IT-2 fuzzy regular grammars.*

4. Closure properties of IT-2 fuzzy language by IT-2 general fuzzy automata

An IT-2 fuzzy language $\mathcal{L}$ has been an IT- 2 fuzzy subset of Σ^* which has been characterized in Definition 3.6. The closure properties of IT-2 languages which have been identified by IT-2 general fuzzy automata are studied and elucidated in this section.

Definition 4.1 *Let $\mathcal{L}_1, \mathcal{L}_2 \in IT-2F(\Sigma^*)$. Then for any $w \in \Sigma^*$,*
1) the union of $\mathcal{L}_1$ and $\mathcal{L}_2$, indicated by $\mathcal{L}_1 \cup \mathcal{L}_2$, characterized as

$$\mathcal{L}_1 \cup \mathcal{L}_2 = \frac{1}{[\underline{\mu}_{\mathcal{L}_1}(w) \vee \underline{\mu}_{\mathcal{L}_2}(w), \overline{\mu}_{\mathcal{L}_1}(w) \vee \overline{\mu}_{\mathcal{L}_2}(w)]},$$

2) the intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$, indicated by $\mathcal{L}_1 \cap \mathcal{L}_2$, characterized as

$$\mathcal{L}_1 \cap \mathcal{L}_2 = \frac{1}{[\underline{\mu}_{\mathcal{L}_1}(w) \wedge \underline{\mu}_{\mathcal{L}_2}(w), \overline{\mu}_{\mathcal{L}_1}(w) \wedge \overline{\mu}_{\mathcal{L}_2}(w)]},$$

3) the complement of $\mathcal{L}_1$, indicated by $\mathcal{L}_1^c$, characterized as

$$\mathcal{L}_1^c = \frac{1}{[1 - \overline{\mu}_{\mathcal{L}_1}(w), 1 - \underline{\mu}_{\mathcal{L}_1}(w)]},$$

4) the concatenation of $\mathcal{L}_1$ and $\mathcal{L}_2$, indicated by $\mathcal{L}_1.\mathcal{L}_2$, characterized as

$$\mathcal{L}_1.\mathcal{L}_2 = \frac{1}{[\vee\{\wedge\{\underline{\mu}_{\mathcal{L}_1}(x), \underline{\mu}_{\mathcal{L}_2}(y)\}\}, \vee\{\wedge\{\overline{\mu}_{\mathcal{L}_1}(x), \overline{\mu}_{\mathcal{L}_2}(y)\}\}]},$$

in which $w = xy$,
5) the Kleene closure of $\mathcal{L}$, indicated by $\mathcal{L}^*$, characterized as

$$\tilde{\mathcal{L}}^* = \bigcup_{i=0}^{\infty} \mathcal{L}^i = \mathcal{L}^0 \cup \mathcal{L}^1 \cup \dots, \quad \textit{where}$$

$$\mathcal{L}^0 = \begin{cases} 1/[1,1], & x = \wedge \\ 1/[0,0], & x \neq \wedge. \end{cases}$$

Theorem 4.1 *Let $\tilde{F}_{1IT}$ and $\tilde{F}_{2IT}$ be two IT-2 GFAs. Then there will be an IT-2 GFA $\tilde{F}_{IT}$ such that*

$$\mathcal{L}(\tilde{F}_{IT}) = \mathcal{L}(\tilde{F}_{1IT}) \cup \mathcal{L}(\tilde{F}_{2IT}).$$

Proof. Let $\mathcal{L}(\tilde{F}_{1IT})$ and $\mathcal{L}(\tilde{F}_{2IT})$ be the IT-2 fuzzy languages accepted by IT-2 GFAs $\tilde{F}_{1IT} = (\tilde{Q}_1, \Sigma, \tilde{q}_{01}, \tilde{Z}, \delta_{1IT}, \tilde{\delta}_{1IT}, \tilde{\omega}_1, \tilde{q}_{1f}, F_1, F_2)$ and $\tilde{F}_{2IT} = (\tilde{Q}_2, \Sigma, \tilde{q}_{02}, \tilde{Z}, \delta_{2IT}, \tilde{\delta}_{2IT}, \tilde{\omega}_2, \tilde{q}_{2f}, F_1, F_2)$ respectively. Then we define an IT-2 GFA $\tilde{F}_{IT} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{Z}, \delta_{IT}, \tilde{\delta}_{IT}, \tilde{\omega}, \tilde{q}_f, F_1, F_2)$ as follows:

$$\tilde{Q} = \tilde{Q}_1 \cup \tilde{Q}_2, \quad \text{where} \quad \tilde{Q}_1 \cap \tilde{Q}_2 = \emptyset;$$

$$\tilde{q}_0 = \tilde{q}_{01} \cup \tilde{q}_{02};$$

$$\delta_{IT}(\tilde{q}_i, a) = \begin{cases} \delta_{1IT}(\tilde{q}_i, a) & \text{if } \tilde{q}_i \in \tilde{Q}_1, \\ \delta_{2IT}(\tilde{q}_i, a) & \text{if } \tilde{q}_i \in \tilde{Q}_2; \end{cases}$$

$$\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) = \begin{cases} \tilde{\delta}_{1IT}(\tilde{q}_i, a, \tilde{q}_j) & \text{if } \tilde{q}_i, \tilde{q}_j \in \tilde{Q}_1, \\ \tilde{\delta}_{2IT}(\tilde{q}_i, a, \tilde{q}_j) & \text{if } \tilde{q}_i, \tilde{q}_j \in \tilde{Q}_2, \\ 1/[0,0] & \text{otherwise;} \end{cases}$$

$$\tilde{\omega}(\tilde{q}_i) = \begin{cases} \bigcup_{q_i \in Q_1} \omega_1(q_i) & \text{if } \tilde{q}_i \in \tilde{Q}_1, \\ \bigcup_{q_i \in Q_2} \omega_2(q_i) & \text{if } \tilde{q}_i \in \tilde{Q}_2, \end{cases}$$

$$\tilde{q}_f(\tilde{q}_i) = \begin{cases} \tilde{q}_{1f}(\tilde{q}_i) & \text{if } \tilde{q}_i \in \tilde{Q}_1, \\ \tilde{q}_{2f}(\tilde{q}_i) & \text{if } \tilde{q}_i \in \tilde{Q}_2. \end{cases}$$

Consequently, for any $w \in \Sigma^*$,

$$\begin{aligned} \mathcal{L}(\tilde{F}_{IT}, w) &= height[\tilde{q}_w \cap \tilde{q}_f] \\ &= height[\{\tilde{q}_{1w} \cap \tilde{q}_{1f}\} \vee \{\tilde{q}_{2w} \cap \tilde{q}_{2f}\}] \\ &= height[\tilde{q}_{1w} \cap \tilde{q}_{1f}] \vee height[\tilde{q}_{2w} \cap \tilde{q}_{2f}] \\ &= \mathcal{L}(\tilde{F}_{1IT}, w) \vee \mathcal{L}(\tilde{F}_{2IT}, w). \end{aligned}$$

Therefore, $\mathcal{L}(\tilde{F}_{IT}) = \mathcal{L}(\tilde{F}_{1IT}) \cup \mathcal{L}(\tilde{F}_{2IT})$. □

Theorem 4.2 *Let $\tilde{F}_{1IT}$ and $\tilde{F}_{2IT}$ be two IT-2 GFAs. Subsequently, there will be an IT-2 GFA $\tilde{F}_{IT}$ such that*

$$\mathcal{L}(\tilde{F}_{IT}) = \mathcal{L}(\tilde{F}_{1IT}) \cap \mathcal{L}(\tilde{F}_{2IT}).$$

Proof. Let $\mathcal{L}(\tilde{F}_{1IT})$ and $\mathcal{L}(\tilde{F}_{2IT})$ be the IT-2 fuzzy languages accepted by IT-2 GFAs $\tilde{F}_{1IT} = (\tilde{Q}_1, \Sigma, \tilde{q}_{01}, \tilde{Z}, \delta_{1IT}, \tilde{\delta}_{1IT}, \tilde{\omega}_1, \tilde{q}_{1f}, F_1, F_2)$ and $\tilde{F}_{2IT} = (\tilde{Q}_2, \Sigma, \tilde{q}_{02}, \tilde{Z}, \delta_{2IT}, \tilde{\delta}_{2IT}, \tilde{\omega}_2, \tilde{q}_{2f}, F_1, F_2)$ respectively. Then we define an IT-2 GFA $\tilde{F}_{IT} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{Z}, \delta_{IT}, \tilde{\delta}_{IT}, \tilde{\omega}, \tilde{q}_f, F_1, F_2)$ as follows:

$$\begin{aligned} &\tilde{Q} = \tilde{Q}_1 \times \tilde{Q}_2 \text{ (direct product of } \tilde{Q}_1 \text{ and } \tilde{Q}_2\text{);} \\ &\tilde{q}_0 = (\tilde{q}_{01}, \tilde{q}_{02}); \\ &\delta((\tilde{q}_i, \tilde{q}_j), a) = (\delta_{1IT}(\tilde{q}_i, a), \delta_{2IT}(\tilde{q}_j, a)); \\ &\tilde{\delta}_{IT}((\tilde{q}_i, \tilde{q}_j), a, (\tilde{q}'_i, \tilde{q}'_j)) = \tilde{\delta}_{1IT}(\tilde{q}_i, a, \tilde{q}'_i) \wedge \tilde{\delta}_{2IT}(\tilde{q}_j, a, \tilde{q}'_j); \\ &\tilde{\omega}(\tilde{q}_i, \tilde{q}_j) = \left(\bigcup_{q_i \in Q_1} \omega_1(q_i) \right) \bigcup \left(\bigcup_{q_j \in Q_2} \omega_2(q_j) \right); \\ &\tilde{q}_f = \tilde{q}_{1f} \cap \tilde{q}_{2f}. \end{aligned}$$

Thus, for any $w \in \Sigma^*$,

$$\begin{aligned} \mathcal{L}(\tilde{F}_{IT}, w) &= height[\tilde{q}_w \cap \tilde{q}_f] \\ &= height[\tilde{q}_{1w} \cap \tilde{q}_{1f} \cap \tilde{q}_{2w} \cap \tilde{q}_{2f}] \\ &= height[\tilde{q}_{1w} \cap \tilde{q}_{1f}] \wedge height[\tilde{q}_{2w} \cap \tilde{q}_{2f}] \\ &= \mathcal{L}(\tilde{F}_{1IT}, w) \wedge \mathcal{L}(\tilde{F}_{2IT}, w). \end{aligned}$$

Accordingly, $\mathcal{L}(\tilde{F}_{IT}) = \mathcal{L}(\tilde{F}_{1IT}).\mathcal{L}(\tilde{F}_{2IT})$. □

Theorem 4.3 *Let $\tilde{F}_{1IT}$ and $\tilde{F}_{2IT}$ be two IT-2 GFAs. Therefore, there will be an IT-2 GFA $\tilde{F}_{IT}$ such that*

$$\mathcal{L}(\tilde{F}_{IT}) = \mathcal{L}(\tilde{F}_{1IT}).\mathcal{L}(\tilde{F}_{2IT}).$$

Proof. Let $\mathcal{L}(\tilde{F}_{1IT})$ and $\mathcal{L}(\tilde{F}_{2IT})$ be the IT-2 fuzzy languages accepted by IT-2 GFAs $\tilde{F}_{1IT} = (\tilde{Q}_1, \Sigma, \tilde{q}_{01}, \tilde{Z}, \delta_{1IT}, \tilde{\delta}_{1IT}, \tilde{\omega}_1, \tilde{q}_{1f}, F_1, F_2)$ and $\tilde{F}_{2IT} = (\tilde{Q}_2, \Sigma, \tilde{q}_{02}, \tilde{Z}, \delta_{2IT}, \tilde{\delta}_{2IT}, \tilde{\omega}_2, \tilde{q}_{2f}, F_1, F_2)$ respectively. Then we define an IT-2 GFA $\tilde{F}_{IT} = (\tilde{Q}, \Sigma, \tilde{q}_0, \tilde{Z}, \delta_{IT}, \tilde{\delta}_{IT}, \tilde{\omega}, \tilde{q}_f, F_1, F_2)$ as follows:

$$\tilde{Q} = \tilde{Q}_1 \cup \tilde{Q}_2, \text{ where } \tilde{Q}_1 \cap \tilde{Q}_2 = \emptyset \ ;$$
$$\tilde{q}_0 = \tilde{q}_{01} \cup \tilde{q}_{02};$$
$$\delta_{IT}(\tilde{q}_i, a) = \begin{cases} \delta_{1IT}(\tilde{q}_i, a) & \text{if } \tilde{q}_i \in \tilde{Q}_1, \\ \delta_{2IT}(\tilde{q}_i, a) & \text{if } \tilde{q}_i \in \tilde{Q}_2; \end{cases}$$
$$\tilde{\delta}_{IT}(\tilde{q}_i, a, \tilde{q}_j) = \begin{cases} \vee\{\wedge\{\tilde{\delta}_{1IT}(\tilde{q}_i, a, \tilde{q}_j), \tilde{\delta}_{2IT}(\tilde{q}_i, a, \tilde{q}_j)\}\}, \tilde{q}_i, \tilde{q}_j \in \tilde{Q}, \\ \vee\{\wedge\{\tilde{\delta}_{1IT}(\tilde{q}_i, a, \tilde{q}_j), \tilde{\delta}_{2IT}(\tilde{q}_i, a, \tilde{q}_j)\}\}, \tilde{q}_i, \tilde{q}_j \in \tilde{Q}_2; \end{cases}$$
$$\tilde{\omega}(\tilde{q}_i) = \begin{cases} \bigcup_{q_i \in Q_1} \omega_1(q_i) & \text{if } \tilde{q}_i \in \tilde{Q}_1, \\ \bigcup_{q_i \in Q_2} \omega_2(q_i) & \text{if } \tilde{q}_i \in \tilde{Q}_2; \end{cases}$$
$$\tilde{q}_f = \tilde{q}_{1f}.\tilde{q}_{2f}.$$

Hence, for any $w \in \Sigma^*$,

$$\begin{aligned} \mathcal{L}(\tilde{F}_{IT}, w) &= height[\tilde{q}_w \cap \tilde{q}_f] \\ &= \bigvee_{w = w_1 w_2} [height\{\tilde{q}_{1w_1} \cap \tilde{q}_{1f}\} \wedge height\{\tilde{q}_{2w_2} \cap \tilde{q}_{2f}\}] \\ &= \bigvee_{w = w_1 w_2} [\mathcal{L}(\tilde{F}_{1IT}, w_1).\mathcal{L}(\tilde{F}_{2IT}, w_2)]. \end{aligned}$$

Therefore, $\mathcal{L}(\tilde{F}_{IT}) = \mathcal{L}(\tilde{F}_{1IT}).\mathcal{L}(\tilde{F}_{2IT})$. □ The results can be easily proved by utilizing Theorems 4.1 and 4.3.

Theorem 4.4 *Let $\tilde{F}_{1IT}$ and $\tilde{F}_{2IT}$ be two IT-2 GFAs. Then*

$$\mathcal{L}(\tilde{F}_{1IT}) = \mathcal{L}(\tilde{F}_{2IT})^*.$$

Definition 4.2 *Let $\tilde{F}_{IT}$ be an IT-2 GFA and $\alpha = 1/[\underline{\alpha}, \overline{\alpha}]$, where $[\underline{\alpha}, \overline{\alpha}] \in [0, 1]$, set of all closed subintervals of $[0, 1]$. Then a α -IT-2 fuzzy language recognized by $\tilde{F}_{IT}$ with parameter α has been characterized as:*

$$\mathcal{L}(\tilde{F}_{IT}, \alpha) = \{w : \mathcal{L}_{\tilde{F}_{IT}}(w) \supseteq \alpha, w \in \Sigma^*\}.$$

Theorem 4.5 *Let $\tilde{F}_{IT}$ be an IT-2 GFA and $\mathcal{L}(\tilde{F}_{IT}, \alpha)$ a α -IT-2 fuzzy language recognized by $\tilde{F}_{IT}$. Then $\mathcal{L}(\tilde{F}_{IT}, \alpha)$ has not been closed under IT-2 fuzzy complement.*

Proof. Let $\tilde{F}_{IT}$ be an IT-2 GFA and $\mathcal{L}(\tilde{F}_{IT}, \alpha)$ a α -IT-2 fuzzy language accepted by $\tilde{F}_{IT}$. Then we have $\mathcal{L}(\tilde{F}_{IT}, \alpha_1) \supseteq \mathcal{L}(\tilde{F}_{IT}, \alpha_2)$ if $\alpha_1 \leq \alpha_2$. That is $\mathcal{L}(\tilde{F}_{IT}, \alpha)$ is non-increasing for α. On the other hand, if $\alpha_1 \leq \alpha_2$, then $\mathcal{L}^c(\tilde{F}_{IT}, \alpha_1) \subseteq \mathcal{L}^c(\tilde{F}_{IT}, \alpha_2)$ and thus, $\mathcal{L}^c(\tilde{F}_{IT}, \alpha)$ is said to be non-decreasing, and therefore a contradiction appears. □

5. Conclusions

In this study, we have introduced and developed the theory of interval type-2 general fuzzy automata (IT-2 GFA) and explored their relationship with interval type-2 fuzzy regular grammars. We have rigorously defined the structure and operational semantics of IT-2 GFA, and demonstrated how these automata can be used to recognize interval type-2 fuzzy languages. Our results show that the class of IT-2 fuzzy languages recognized by IT-2 GFA is closed under the operations of union, intersection, concatenation, and Kleene closure, but not under complement. Several illustrative examples were provided to clarify the concepts and validate the theoretical results.

There are several promising directions for future research based on the findings of this paper:

1. Minimization and Equivalence Algorithms: Developing efficient algorithms for the minimization of IT-2 GFA and investigating the equivalence problem for these automata could significantly enhance their

practical applicability.
2. Extension to Nondeterministic and Probabilistic Models: Extending the IT-2 GFA framework to encompass nondeterministic or probabilistic transitions may offer greater modeling power for real-world systems characterized by multiple sources of uncertainty.
3. Learning and Identification of IT-2 GFA: Investigating learning algorithms for IT-2 GFA from data, including grammatical inference and state identification methods, could open new avenues in machine learning and pattern recognition.
4. Applications in Natural Language Processing and Control Systems: Applying IT-2 GFA to fields such as natural language processing, speech recognition, and fuzzy control systems could demonstrate their effectiveness in handling complex uncertainties inherent in these domains.
5. Complexity Analysis and Optimization: Analyzing the computational complexity of basic operations on IT-2 GFA and developing optimization techniques for state reduction and transition simplification would be valuable for large-scale implementations.
6. Integration with Other Fuzzy Models: Exploring the integration of IT-2 GFA with other fuzzy models, such as fuzzy Petri nets or fuzzy Turing machines, could further enrich the theoretical landscape and practical utility of fuzzy computation.

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