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On the Powers of Companion Matrices via Linear Recursiveness. Application to Power of Matrices

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ABSTRACT: This paper concerns some explicit formulas for the entries of the powers of companion matrices and diagonal matrices with companion matrix blocks. The development of such formulas is based on the connection of the entries of the powers of the companion matrices and the recursive sequences of Fibonacci type. Indeed, their associated fundamental Fibonacci system and fundamental sequence will play a central role. Thus, several explicit formulas for these entries are established. In addition, the Frobenus decomposition permits to apply the former results for outlining the matrix powers in the general setting. Ilustrative special cases and examples are provided.

Key Words: Powers of companion matrix, diagonal matrix with companion matrix blocks, linear recursive recursiveness, analytic Binet representations, combinatorial expressions, power of matrices, power series of matrices.

Contents

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1. Introduction

It is well known that the matrix functions, especially the matrix powers function, play a significant role in many fields of pure and applied mathematics, engineering and applied sciences. Several methods and algorithms have been developed for computing the usual matrix functions (see,for example, [4,8,9,10, 11,12,17,19,21]). Particularly, for computing the matrix powers A^n $(n \geq r)$ various methods, techniques and algorithms have been improved in the literature (see,for example, [8,9,10,11,12,13,14,17]). Recently, it was shown in [5,6] that some properties of sequences defined by linear recurrence relations of constant coefficients (see [15,18,20], for example) are used as a fundamental tools for computing the entries of the companion matrix powers, as well as the matrix exponential, the matrix logarithm and matrix $p-th$ root. Such recursive sequences, known in the literature as as r-generalized Fibonacci sequences, appear naturally in the computation of the matrix powers (see [2,5]). These sequences $\{v_n\}_{n\geq 0}$ (of real or complex numbers) are defined by specified initial conditions $v_0 = \alpha_0, v_1 = \alpha_1, \cdots, v_{r-1} = \alpha_{r-1}$ $(r \geq 2)$ and the linear recurrence relation of order r

$$v_{n+1} = a_1 v_n + \cdots + a_r v_{n-r+1}, \quad \text{for} \quad n \geq r-1, \tag{1.1}$$

where the real (or complex) numbers $a_1, a_2, \cdots, a_r$ $(a_r \neq 0)$ are the coefficients (see, for example, [15,20]). The (characteristic) polynomial $P(z) = z^r - a_1 z^{r-1} - a_2 z^{r-2} - \ldots - a_r$, whose (characteristic) roots λ_k $(1 \leq k \leq h)$ of multiplicities m_k $(1 \leq k \leq h)$, plays a central role in the formulation of the general term v_n under the analytic Binet formula

$$v_n = \sum_{k=1}^{h} \left(\sum_{f=0}^{m_k-1} x_{k,f} n^f \right) \lambda_k^n, \tag{1.2}$$

where the scalars $x_{k,f}$ are furnished in terms of the initial data by solving the generalized Vandermonde linear system: $\sum_{k=1}^{h} \left(\sum_{f=0}^{m_k-1} x_{k,f} n^f \right) \lambda_k^n = \alpha_n$, for $0 \leq n \leq r-1$. In addition, the combinatorial expression of $\{v_n\}_{n\geq 0}$ is

$$v_n = \rho(n,r) w_0 + \rho(n-1,r) w_1 + \cdots + \rho(n-r+1,r) w_{r-1},$$

for every $n \geq r$, where $w_s = a_r v_s + \cdots + a_s v_{r-1}$ $(0 \leq s \leq r-1)$ and

$$\rho(n,r) = \sum_{k_1+2k_2+\cdots+rk_r=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_1! \cdots k_r!} a_1^{k_1} \cdots a_r^{k_r}, \tag{1.3}$$

with $\rho(r,r) = 1$ and $\rho(n,r) = 0$ if $n \leq r-1$ (see, for instance, [18,20] and references therein).

The purpose of this paper is to provide some explicit compact formulas for the entries $a_{js}(n)$ $(1 \leq j, s \leq r)$ of the matrix powers $\mathbf{A}^n$ $(n \geq 0)$, where $\mathbf{A}$ is a companion matrix or a diagonal matrix with diagonal companion matrices blocks. Our approach is based on the close relationship between the entries $a_{ij}(n)$ $(1 \leq i, j \leq r)$ and the family of sequences $\{v_n^{(s)}\}_{n\geq 0}$, indexed by s $(1 \leq s \leq r)$, of type (1.1) defined by

$$\begin{cases} v_{n+1}^{(s)} = a_0 v_n^{(s)} + \cdots + a_{r-1} v_{n-r+1}^{(s)}, & \text{for } n \geq r-1, \\ v_n^{(s)} = \delta_{s-1,n} & \text{for } 1 \leq n \leq r. \end{cases} . \tag{1.4}$$

The set $\{\{v_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$ is the so-called the *fundamental Fibonacci system*. Properties of the analytical Binet formulas and combinatorial expressions of the sequence $\{v_n^{(r)}\}_{n\geq 0}$, called the *fundamental sequence*, will play a central role in this study.

This study is organized as follows. Some preliminary considerations on the fundamental Fibonacci system and fundamental sequence related the sequences (1.1) are presented (Section 2). The entries $a_{ij}^{(n)}$ $(1 \leq i, j \leq r)$ of the powers $\mathbf{A}^n$, where $\mathbf{A}$ is a companion matrix or diagonal matrix with companion matrices blocks, are provided in terms of the fundamental sequence (Section 3). For reason of clarity and conciseness, some properties of sequences (1.1) are applied for providing the combinatorial and analytical expressions for the entries $a_{ij}^{(n)}$ ($1 \leq i, j \leq r$) of powers A^n ($n \geq 0$), where $\mathbf{A}$ is a companion matrix or a diagonal matrix with companion matrices blocks (Section 4). Two approaches for exhibiting some explicit formulas for the entries of the preceding matrix powers are provided, using some recent results of the literature concerning the scalars $x_{k,f}$ (Sections 5 and 6). In Section 7, we furnish an application to the analytic matrix functions. Concluding remarks and perspectives are stated in Section 8. Finally, illustrative examples and special cases are furnished.

2. Preliminary on the fundamental Fibonacci system

2.1. Combinatorial aspect

We denote by $\mathcal{E}_{\mathbb{K}}(a_1, a_2, \ldots, a_r)$ the $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ ot $\mathbb{C}$) of sequences (1.1), where a_1, a_2, $\ldots$, a_r are fixed coefficients. For every$\{v_n\}_{n\geq 0}$ in $\mathcal{E}_{\mathbb{K}}^{(r)}(a_1, \ldots, a_r)$ of initial data α_0, $\cdots$, α_{r-1}, we can verify that $v_n = \sum_{s=1}^{r} \alpha_s v_n^{(s)}$, for every $n \geq 0$. Moreover, the fundamental Fibonacci system $\{\{v_n^{(s)}\}_{n\geq 0}\ ,\ 1 \leq s \leq r\}$ represents a basis of the $\mathbb{K}$-vector space $\mathcal{E}_{\mathbb{K}}^{(r)}(a_1, \ldots, a_r)$, which is of finite dimension r. In addition, it was shown in [3,5] that each sequence $\{v_n^{(s)}\}_{n\geq 0}$ ($1 \leq s \leq r$) can be expressed in terms of the fundamental sequence $\{v_n^{(r)}\}_{n\geq 0}$. More precisely, we have

$$v_n^{(s)} = \sum_{t=1}^{s} a_{r-t+1} v_{n-s+t-1}^{(r)}, \text{ for every } n \geq 0. \tag{2.1}$$

We observe that Expression (2.1) also remains valid for $s = r$.

On the other side, it was proven in [18] that the sequence $\{w_n\}_{n\geq 0}$ defined by $w_n = \rho(n+1, r)$ satisfies the linear recursive relation Expression (1.1). Moreover, since the sequence $\{w_n\}_{n\geq 0}$ and the fundamental sequence $\{v_n^{(r)}\}_{n\geq 0}$ satisfy Expression (1.1) and own the same initial data, namely, $w_n = v_n^{(r)}$ for $0 \leq n \leq r-1$, we derive that the combinatorial formula the fundamental sequence is $v_n^{(r)} = \rho(n+1, r) = \sum_{k_1+2k_2+\cdots+rk_r=n-r+1} \frac{(k_1+\cdots+k_r)!}{k_1!\cdots k_r!} a_1^{k_1} \cdots a_r^{k_r}$. Moreover, the former combinatorial form of the fundamental sequence and Expression (2.1) imply that the combinatorial expression of every sequence $\{v_n^{(s)}\}_{n\geq 0}$ ($1 \leq s \leq r$) is

$$v_n^{(s)} = \sum_{t=1}^{s} a_{r-t+1} \rho(n-s+t, r), \quad \text{for every } n \geq r. \tag{2.2}$$

Given the close relationship between the powers of the companion matrices and the fundamental Fibonacci system (1.4), the formulas (1.3) and (2.2) will allow us to obtain the combinatorial form for the entries of the matrix powers of the companion matrices or diagonal matrices in blocks of companion matrices.

2.2. Analytical Binet aspect

For the fundamental sequence $\{v_n^{(r)}\}_{n\geq 0}$ the analytic Binet formula (1.2) is given under the form

$$v_n^{(r)} = \sum_{k=1}^{h} \left(\sum_{f=0}^{m_k-1} x_{k,f}^{(r)} n^f \right) \lambda_k^n, \tag{2.3}$$

where the λ_k are the characteristic roots of multiplicities m_k (respectively) and the scalars $x_{k,f}^{(r)}$ are obtained as the solution of the linear system $\sum_{k=1}^{h} \left(\sum_{f=0}^{m_k-1} x_{k,f}^{(r)} n^f \right) \lambda_k^n = \delta_{r-1,n}$ ($0 \leq n \leq r-1$). In addition, by substituting Expression (2.3) in Expression (2.1) we show that the analytic Binet formula of each sequence $\{v_n^{(s)}\}_{n\geq 0}$ $(1 \leq s \leq r-1)$ is

$$v_n^{(s)} = \sum_{k=1}^{h}\left[\sum_{f=0}^{m_k-1}\left(\sum_{t=1}^{s} a_{r-t+1}(n-s+t-1)^f \lambda_k^{-s+t-1}\right) x_{k,f}^{(r)}\right]\lambda_k^n,$$

for every $n \geq r$. And a straightforward computation permits us to write

$$v_n^{(s)} = \sum_{k=1}^{h}\left[\sum_{u=0}^{m_k-1}\Omega(s,k,u)n^u\right]\lambda_k^n, \tag{2.4}$$

where

$$\Omega(s,k,u) = \sum_{f=u}^{m_k-1}\left(\sum_{t=1}^{s} a_{r-t+1}\binom{f}{u}(-s+t-1)^{f-u}\lambda_k^{-s+t-1}\right)x_{k,f}^{(r)}. \tag{2.5}$$

Generally, the (characteristic) roots of the polynomial $P(z) = z^r - a_1 z^{r-1} - \cdots - a_r$ are not all simple. Suppose that $\Gamma_1 = \{k,\ m_k = 1\} \neq \emptyset$ and $\Gamma_2 = \{k,\ m_k \geq 2\} \neq \emptyset$, where m_k means the multiplicity of the root λ_k. Then, Expression (2.4) takes the form

$$v_n^{(r)} = \sum_{k\in\Gamma_1} x_k^{(r)}\lambda_k^n + v_n^{(r)} + \sum_{k\in\Gamma_2}\left(\sum_{f=0}^{m_k-1} x_{k,f}^{(r)} n^f\right)\lambda_k^n, \tag{2.6}$$

$$v_n^{(s)} = \sum_{k\in\Gamma_1}\left(\sum_{t=1}^{s} a_{r-t+1}\lambda_k^{s-t-1}\right)x_k^{(r)}\lambda_k^n + \sum_{k\in\Gamma_2}\left[\sum_{u=0}^{m_k-1}\Omega(s,k,u)n^u\right]\lambda_k^n, \tag{2.7}$$

where $\Omega(s,k,u)$ is given by formula (2.5). When all the roots λ_k $(1 \leq k \leq r)$ are simple we have $\Gamma_1 = \{k,\ m_k = 1\} = \{1,\ldots,r\}$ and $\Gamma_2 = \{k,\ m_k \geq 2\} = \emptyset$, which implies that

$$v_n^{(r)} = \sum_{k=1}^{r} x_k^{(r)}\lambda_k^n \qquad \text{and} \qquad v_n^{(s)} = \sum_{k=1}^{r}\left(\sum_{t=1}^{s} a_{r-t+1}\lambda_k^{-s+t-1}\right)x_k^{(r)}\lambda_k^n.$$

As stated before, given the existence of a close relationship between the powers of the companion matrices and the fundamental Fibonacci system (1.4) (see [2,3,5]), Expressions (2.4)-(2.5) will allow us to give explicit analytic formulas for the entries of the powers of the companion or diagonal matrices in companion matrix blocks.

3. Fundamental Fibonacci system and entries of the powers of the block square companion matrix

The matrix formulation of the sequence defined by (1.1) is given by $\mathbf{V}_{n+1} = \mathbf{A}\mathbf{V}_n$, for every $n \geq r-1$, where $\mathbf{V}_n$ is the vector column $\mathbf{V}_n = {}^t(v_n, v_{n-1}, \cdots, v_{n-r+1})$ and $\mathbf{A}$ is the companion matrix

$$\mathbf{A} = \mathbf{A}[a_1, a_2, \cdots, a_r] = \begin{pmatrix} a_1 & a_2 & \ldots & \ldots & a_r \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}. \tag{3.1}$$

Matrix formulation of sequences (1.1) has been considered in various papers of the literature, for example, it was used in [2,5,18] for studying several properties of sequences (1.1). The closed connection between the entries $a_{ij}(n)$ of the powers $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$ of the companion matrix (3.1) and the fundamental Fibonacci system (1.4), has been established in [3,5]. More precisely, the entries $a_{ij}(n)$ are given explicitly in terms of the fundamental Fibonacci system. Using an iterative process we get the matrix equation $\mathbb{V}_{n+r-1} = \mathbf{A}^n\mathbb{V}_{r-1}$, for $n \geq r-1$, where $\mathbb{V}_{r-1} = {}^t(v_{r-1},\ \ldots,\ v_0)$ is the vector column of the initial data.

More precisely, the vector $\mathbb{V}_{n+r-1}$ can be written under the matrix form $\mathbb{V}_{n+r-1} = \mathbb{M}_n\mathbb{V}_{r-1}$, where the entries $m_{ij}(n)$ of the matrix $\mathbb{M}_n = \mathbf{A}^n$ are stated under the form $m_{ij}(n) = v_{n+r-i}^{(r-j+1)}$ (see more details in [3,5]). In summary, we can formulate the following property.

Proposition 1 *Let $\{\{v_n^{(s)}\}_{n\geq 0}$, $1 \leq s \leq r\}$ be the fundamental Fibonacci system (1.4). Then, for every $n \geq 0$, the entries of the powers $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$ are imparted under the form*

$$a_{ij}(n) = v_{n+r-i}^{(r-j+1)} = \sum_{t=1}^{r-j+1} a_{r-t+1}v_{n-i+j+t-2}^{(r)}, \tag{3.2}$$

As a consequence of Proposition 1, each sequence $\{a_{ij}(n)\}_{n\geq 0}$ of the entries of the matrix powers $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$ is also a sequence of type (1.1), with some specific initial data.

Let us consider the diagonal matrix $\mathbf{A} = diag(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$, where each block $\mathbf{A}_d = \mathbf{A}[a_1^{(d)}, \ldots, a_{r_d}^{(d)}]$ $(1 \leq d \leq m)$ is a companion matrix of characteristic polynomial $P_d(z) = z^{r_d} - a_1^{(d)}z^{r_d-1} - \ldots - a_{r_d}^{(d)}$. For reason of simplicity, and without loss of generality, we suppose in the sequel that each matrix $\mathbf{A}_d$ $(1 \leq d \leq m)$ is of order $r_d \times r_d$ with $r_d \geq 2$. We set

$$S_1 = \{(i,j),\ 1 \leq i,j \leq r_1\} \text{ and } S_d = \{(i,j),\ \sum_{k=1}^{d-1} r_k + 1 \leq i,j \leq \sum_{k=1}^{d} r_k\}, \tag{3.3}$$

for $2 \leq d \leq m$. The fundamental Fibonacci system associated to each companion matrix $\mathbf{A}_d$ $(1 \leq d \leq m)$ consists of the sequences $\{v_n^{(d,s)}\}_{n\geq 0}$, with $1 \leq s \leq r_d$, given as follows

$$\begin{cases} v_{n+1}^{(d,s)} = a_1^{(d)}v_n^{(d,s)} + \cdots + a_{r_d}^{(d)}v_{n-r+1}^{(d,s)}, & \text{for } n \geq r_d - 1, \\ v_n^{(d,s)} = \delta_{s-1,n} & \text{for } 0 \leq n \leq r_d - 1. \end{cases} \tag{3.4}$$

Proposition 1 shows us that the entries $a_{ij,d}(n)$ of the matrix powers $\mathbf{A}_d^n = (a_{ij,d}(n))_{1\leq i,j\leq r_d}$ are imparted under the form $a_{ij,d}(n) = v_{n+r-i}^{(d,r_d-j+1)}$, for every $n \geq 0$. Then, using Expression (3.2) and Proposition 1 we can formulate the following proposition.

Proposition 2 *Let $\mathbf{A} = diag(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m) \in \mathbb{K}^{r\times r}$ be a diagonal matrix with companion matrices blocks, where each companion matrix $\mathbf{A}_d = \mathbf{A}_d[a_1^{(d)}, a_2^{(d)}, \cdots, a_{r_d}^{(d)}]$ $(d = 1, \ldots m)$ is of order $r_d \times r_d$. Then, for every $n \geq 0$, the entries of the powers $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$ are expressed under the form*

$$a_{ij}(n) = v_{n+r_d-i}^{(d,r_d-j+1)} \textit{ for } (i,j) \in S_d\ (1 \leq d \leq m), \quad \textit{and} \quad a_{ij}(n) = 0 \qquad \textit{eslwhere} \tag{3.5}$$

where the sequences $\{v_n^{(d,s)}\}_{n\geq 0}$ are given by (3.4), with $1 \leq s \leq r_d$.

For clarity and illustrative purpose, let consider the special case of $\mathbf{A} = diag(\mathbf{A_1}, \mathbf{A_2})$, such that $\mathbf{A_1} = \mathbf{A_1}[a_1^{(1)}, a_2^{(1)}, \cdots, a_{r_1}^{(1)}]$, $\mathbf{A_2} = \mathbf{A_2}[a_1^{(2)}, a_2^{(2)}, \cdots, a_{r_2}^{(2)}]$ are the companion matrices, where $a_j^{(d)}$ $(1 \leq j \leq r_d)$ are in $\mathbb{K}$. Then, for every $n \geq 0$, the entries $a_{ij}(n)$ of of $\mathbf{A}^n$ are as

$$a_{ij}(n) = v_{n+r_d-i}^{(d,r_d-j+1)} \text{ for } (i,j) \in S_d,\ \ d = 1,\, 2, \quad \text{and} \quad a_{ij}(n) = 0 \qquad \text{eslwhere} \tag{3.6}$$

Follows here an illustrative example of the preceding results.

Example 1 *Let $A = diag(\mathbf{A_1}, \mathbf{A_2})$ be the 4×4, with $\mathbf{A_1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{A_2} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ The two sequences of the fundamental Fibonacci system related to the companion matrices $\mathbf{A_d}$ $(d = 1,\, 2)$ are $\{v_n^{(1,d)}\}_{n\geq 0}$, $\{v_n^{(2,d)}\}_{n\geq 0}$, where $v_{n+1}^{(1,d)} = a_1^{(d)}v_n^{(1,d)} + a_2^{(d)}v_{n-1}^{(1,d)}$, for $n \geq 1$, with $v_n^{(s,d)} = \delta_{s-1,n}$ for $n = 0, 1$. Here, we have $a_1^{(1)} = 2$, $a_2^{(1)} = 1$ and $a_1^{(2)} = 3$, $a_2^{(2)} = 2$. Then, the matrix powers $\mathbf{A}^n$ of the 2-diagonal block matrix $\mathbf{A}$ are given by $\mathbf{A}^n = diag(\mathbf{A_1}^n, \mathbf{A_2}^n)$, where $\mathbf{A_1}^n = \begin{pmatrix} v_{n+1}^{(1,2)} & v_{n+1}^{(1,1)} \\ v_n^{(1,2)} & v_n^{(1,1)} \end{pmatrix}$ and $\mathbf{A_2}^n = \begin{pmatrix} v_{n+1}^{(2,2)} & v_{n+1}^{(2,1)} \\ v_n^{(2,2)} & v_n^{(2,1)} \end{pmatrix}$, for every $n \geq 0$.*

Proposition 2 and Expression (3.6) show that throughout the properties of sequences (1.1), we can exhibit some explicit formulas for the entries $a_{ij}(n)$ $(1 \leq i,\ j \leq r)$ of the powers $\mathbf{A}^n$, whose diagonal blocks are companion matrices.

4. Combinatorial formula and analytic representation for the entries of the companion matrix and diagonal matrix with companion matrices blocks

By considering the combinatorial formula and the analytic Binet representation of the fundamental sequence, we provide in this section some formulas for the entries of the powers of a companion matrix and a diagonal matrix with companion matrices blocks.

4.1. Combinatorial Formula for the entries of the companion matrix and the diagonal matrix with companion matrices blocks

Proposition 1 shows that for a companion matrix (3.1), the entries $a_{ij}(n)$ of the powers $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$ are imparted under the form $a_{ij}(n) = v_{n+r-i}^{(r-j+1)}$. In addition, Expression (2.2) implies the following property.

Proposition 3 *The combinatorial formula for the entries $a_{ij}(n)$ are given by*

$$a_{ij}(n) = v_{n+r-i}^{(r-j+1)} = \sum_{t=1}^{r-j+1} a_{r-t+1}\rho(n-i+j+t-1, r),\ \textit{for every } n \geq 0. \tag{4.1}$$

where the $\rho(n, r)$ are as in Expression (1.3).

For a diagonal matrix with companion matrices blocks $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$, Proposition 2 and Expressions (3.5), (4.1) imply that the combinatorial formulas for the entries $a_{ij,d}(n)$ of each matrix powers $\mathbf{A}_d^n$ is

$$a_{ij,d}(n) = v_{n+r_d-i}^{(d,r_d-j+1)} = \sum_{t=1}^{r_d-j+1} a_{r-t+1}^{(d)}\rho(n-i+j+t-1, r_d),\ \text{for every } n \geq 0. \tag{4.2}$$

By considering Expressions (3.6) and (4.2), we can start the following proposition concerning the combinatorial formulas for the entries of the powers $\mathbf{A}^n$, for a given matrix $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$.

Proposition 4 *Let $\mathbf{A} = diag(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ be an element $\mathbb{K}^{r\times r}$, where each companion matrix $\mathbf{A}_d = \mathbf{A}_d[a_1^{(d)}, a_2^{(d)}, \cdots, a_{r_d}^{(d)}]$ is of order $r_d \times r_d$. Then, for every $n \geq 0$, the combinatorial formulas of the entries $a_{ij}(n)$ of $\mathbf{A}^n$ the powers of the matrix $\mathbf{A}$ are given by*

$$a_{ij}(n) = \sum_{t=1}^{r_d-j+1} a_{r_d-t+1}^{(d)}\rho(n-i+j+t-1, r_d),\ \textit{when } (i,j) \in S_d,\ \textit{and } a_{ij}(n) = 0 \textit{ eslwhere}$$

where the sets S_d are given by (3.3) and $\rho(n, r)$ are as in Expression (1.3).

4.2. Analytic Binet formula for the entries of the companion matrix and the diagonal matrix with matrix companion blocks

Let $P(z) = z^r - a_1 z^{r-1} - \ldots - a_r = \prod_{k=1}^{h}(z-\lambda_k)^{m_k}$ be the characteristic polynomial of the generalized Fibonacci sequence (1.1), where m_k is the multiplicity of the root λ_k $(1 \leq k \leq h)$. For the fundamental sequence $\{v_n^{(r)}\}_{n\geq 0}$ of the fundamental Fibonacci system (1.4), the analytic Binet formula is given by (2.3). Using the decomposition formula (2.1), it was shown in Expressions (2.4)-(2.5) that the analytic Binet formula of the sequences $\{v_n^{(s)}\}_{n\geq 0}$ $(1 \leq s \leq r-1)$ defined by (1.4), can be written under the form $v_n^{(s)} = \sum_{k=1}^{h}\left[\sum_{u=0}^{m_k-1}\Omega(s,k,u)n^u\right]\lambda_k^n$, where $\Omega(s,k,u) = \sum_{f=u}^{m_k-1}\left(\sum_{t=1}^{s} a_{r-t+1}\binom{f}{u}(-s+t-1)^{f-u}\lambda_k^{-s+t-1}\right)x_{k,f}^{(r)}$.

Proposition 1 shows that the entries $a_{ij}(n)$ of the powers $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$ are imparted under the form $a_{ij}(n) = v_{n+r-i}^{(r-j+1)}$, for every $n \geq 0$. Therefore, a direct application of Expressions (2.4)-(2.5), with $n+r-i$ (instead of n) and $s = r-j+1$, implies that we have

$$a_{ij}(n) = v_{n+r-i}^{(r-j+1)} = \sum_{k=1}^{h}\left[\sum_{u=0}^{m_k-1}\Omega(r-j+1,k,u)(n+r-i)^u\right]\lambda_k^{n+r-i},$$

where $\Omega(r-j+1,k,u) = \displaystyle\sum_{f=u}^{m_k-1}\left(\sum_{t=1}^{r-j+1} a_{r-t+1}\binom{f}{u}(-r+j+t)^{f-u}\lambda_k^{-r+j+t}\right)x_{k,f}^{(r)}$. Taking into the formula $(n+r-i)^u = \displaystyle\sum_{l=0}^{u}\binom{u}{l}(r-i)^{u-l}n^l$, we have

$$\sum_{u=0}^{m_k-1}\Omega(r-j+1,k,u)(n+r-i)^u = \sum_{l=0}^{m_k-1}\left[\sum_{u=l}^{m_k-1}\Omega(r-j+1,e,u)\binom{u}{l}(r-i)^{u-l}\right]n^l.$$

We write $\Omega(r-j+1,k,u)\binom{u}{l}(r-i)^{u-l} = \sum_{f=u}^{m_k-1}\Phi(i,j,f,u,l)x_{k,f}^{(r)}$, where the scalars $\Phi(i,j,f,u,l)$ are given by $\Phi(i,j,f,u,l) = \displaystyle\sum_{t=1}^{r-j+1} a_{r-t+1}\binom{f}{u}\binom{u}{l}(r-i)^{u-l}(-r+j+t)^{f-u}\lambda_k^{-r+j+t}$. Therefore, we get

$$a_{ij}(n) = \sum_{k=1}^{h}\left[\sum_{l=0}^{m_k-1}\left(\sum_{u=l}^{m_k-1}\sum_{f=u}^{m_k-1}\Phi(i,j,f,u,l)\lambda_k^{r-i}x_{k,f}^{(r)}\right)n^l\right]\lambda_k^n.$$

On the other side, we have

$$\sum_{u=l}^{m_k-1}\sum_{f=u}^{m_k-1}\Phi(i,j,f,u,l)\lambda_k^{r-i}x_{k,f}^{(r)} = \sum_{t=1}^{r-j+1}\sum_{u=l}^{m_k-1}\sum_{f=u}^{m_k-1}\Re_{l,k,u}(i,j)x_{k,f}^{(r)},$$

where $\Re_{l,k,t,u}(i,j) = a_{r-t+1}\binom{f}{u}\binom{u}{l}(r-i)^{u-l}(-r+j+t)^{f-u}\lambda_k^{-i+j+t}$. In summary, we have the following proposition.

Proposition 5 *Let* $\mathbf{A}$ *be the companion matrix defined in (3.1) and* $P(z) = z^r - a_1z^{r-1} - \ldots - a_r = \prod_{k=1}^{h}(z-\lambda_k)^{m_k}$ *its characteristic polynomial. Then, for every* $n \geq 0$, *the analytic formula for the entries of the matrix powers* $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$ *is given by*

$$a_{ij}(n) = \sum_{k=1}^{h}\left[\sum_{l=0}^{m_k-1}\Delta^{(r)}(i,j,l,k)n^l\right]\lambda_k^n, \tag{4.3}$$

where

$$\Delta^{(r)}(i,j,l,k) = \sum_{t=1}^{r-j+1}\sum_{u=l}^{m_k-1}\sum_{f=u}^{m_k-1} a_{r-t+1}\binom{f}{u}\binom{u}{l}(r-i)^{u-l}(-r+j+t)^{f-u}\lambda_k^{-i+j+t}x_{k,f}^{(r)}, \tag{4.4}$$

Especially, when the roots of $P(z)$ *are not all simple, we get*

$$a_{ij}(n) = \sum_{k\in\Gamma_1}\Delta_1^{(r)}(i,j,k)\lambda_k^n + \sum_{k\in\Gamma_2}\left[\sum_{l=0}^{m_k-1}\Delta_2^{(r)}(i,j,l,k)n^l\right]\lambda_k^n, \tag{4.5}$$

where $\Delta_2^{(r)}(i,j,l,k)$ *is as in (4.4) and* $\Delta_1^{(r)}(i,j,k) = \displaystyle\sum_{t=1}^{r-j+1} a_{r-t+1}\lambda_k^{-i+j+t}x_k^{(r)}$, *with* $\Gamma_1 = \{k,\ m_k = 1\} \neq \emptyset$ *and* $\Gamma_2 = \{k,\ m_k \geq 2\} \neq \emptyset$.

Let $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ be element of $\mathbb{K}^{r\times r}$ and suppose that each companion matrix $\mathbf{A}_d = \mathbf{A}_d[a_1^{(d)}, a_2^{(d)}, \cdots, a_{r_d}^{(d)}]$ $(d = 1, \ldots m)$ is of order $r_d \times r_d$, with $r_d \geq 2$. Let $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \ldots - a_{r_d}^{(d)} = \prod_{k=1}^{h_d}(z-\lambda_k^{(d)})^{m_k^{(d)}}$ be the characteristic polynomial of the matrix $\mathbf{A}_d$. Then, a direct application of Proposition 5 allows us to calculate explicitly the analytic formulas of the entries $a_{ij}(n)$ $(1 \leq i, j \leq r)$ of the powers $\mathbf{A}^n$.

Proposition 6 *Let $\mathbf{A} = diag(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ in $\mathbb{K}^{r\times r}$ be a given diagonal matrix with companion matrices blocks, under the preceding data the entries $a_{ij}(n)$ $(1 \leq i, j \leq r)$ of the matrix powers $\mathbf{A}^n = diag(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n)$ take the form*

$$a_{ij}(n) = a_{ij,d}(n), \text{ for } (i,j) \in S_d, \quad 1 \leq d \leq m, \qquad \text{and} \qquad a_{ij}(n) = 0 \text{ elsewhere},$$

for every $n \geq 0$, where the sets S_d are given by (3.3) and

$$a_{ij,d}(n) = v_{n+r_d-i}^{(d,r_d-j+1)} = \sum_{k=1}^{h_d}\left[\sum_{l=0}^{m_k^{(d)}-1} \Delta^{(r_d)}(i,j,l,k)n^l\right](\lambda_k^{(d)})^n, \tag{4.6}$$

such that

$$\Delta^{(r_d)}(i,j,l,k) = \sum_{t=1}^{r_d-j+1}\sum_{u=l}^{m_k^{(d)}-1}\sum_{f=u}^{m_k^{(d)}-1} a_{r_d-t+1}^{(d)} K_{i,j}(f,u,t)(\lambda_k^{(d)})^{-i+j+t} x_{k,f}^{(r_d)}, \tag{4.7}$$

with $K_{i,j}(f,u,t) = \binom{f}{u}\binom{u}{l}(r_d - i)^{u-l}(-r_d + j + t)^{f-u}$.

Suppose that the roots of $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \ldots - a_{r_d}^{(d)} = \prod_{k=1}^{h_d}(z-\lambda_k^{(d)})^{m_k^{(d)}}$, the characteristic polynomial of the matrix $\mathbf{A}_d$, are not all simple, and set $\Gamma_1^{(d)} = \{k,\ m_k^{(d)} = 1\} \neq \emptyset$ and $\Gamma_2^{(d)} = \{k,\ m_k^{(d)} \geq 2\} \neq \emptyset$. Then, using Expressions (2.6)-(2.7) we derive that the entries of the powers $\mathbf{A}^n = \mathrm{diag}(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n)$ are under the form

$$a_{ij}(n) = a_{ij,d}(n), \text{ for } (i,j) \in S_d, \quad 1 \leq d \leq m, \qquad \text{and} \qquad a_{ij}(n) = 0 \text{ elsewhere},$$

where the sets S_d are given by (3.3) and the entries $a_{ij,d}(n)$ are given under the form

$$a_{ij,d}(n) = \sum_{k\in\Gamma_1^{(d)}} \Delta_1^{(r_d)}(i,j,k)(\lambda_k^{(d)})^n + \sum_{k\in\Gamma_2^{(d)}}\left[\sum_{l=0}^{m_k^{(d)}-1} \Delta_2^{(r_d)}(i,j,l,k)n^l\right](\lambda_k^{(d)})^n, \tag{4.8}$$

with $\Delta_2^{(r_d)}(i,j,l,k)$ is as in (4.7) and $\Delta_1^{(r_d)}(i,j,k) = \displaystyle\sum_{t=1}^{r_d-j+1} a_{r_d-t+1}^{(d)}(\lambda_k^{(d)})^{-i+j+t} x_k^{(r_d)}$. If all the roots of $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \ldots - a_{r_d}^{(d)}$ are simple we have $a_{ij,d}(n) = \displaystyle\sum_{k=1}^{r_d} \Delta_1^{(r_d)}(i,j,k)(\lambda_k^{(d)})^n$.

5. Some explicit analytic formulas for the entries of the powers of a diagonal matrix in blocks of companion matrices: First Approach

In this section we are interested in a first approach for the analytic formulas for the entries of the powers of the companion matrices and the diagonal matrices with companion matrices blocks, through some explicit analytic expressions of the scalars $x_k^{(r)}$ and $x_{k,f}^{(r)}$, of Expressions (1.2) and (2.3), obtained without solving the Vandermonde system (see, for instance, [2,3,5], and references therein).

5.1. Case of simple eigenvalues

Suppose that the roots of $P(z) = z^r - a_1 z^{r-1} - \cdots - a_r$, the characteristic polynomial of the companion matrix (3.1) are simple, which implies $P(z) = \prod_{i=1}^{r}(z - \lambda_i)$. It was established that the analytic Binet form of the term $v_n^{(r)} = \rho(n+1, r)$ can be expressed in terms of the simple roots (see [2,3,5]). That is, we have the lemma.

Lemma 1 *(Rachidi et al.) Suppose that the roots $\lambda_1, \ldots, \lambda_r$ of characteristic polynomial $P(z) = z^r - a_1 z^{r-1} - \ldots - a_{r-1} z - a_r$ are simple. Then, we have*

$$v_n^{(r)} = \rho(n+1, r) = \sum_{k=1}^{r} \frac{1}{P'(\lambda_k)} \lambda_k^n = \sum_{k=1}^{r} \frac{1}{\prod\limits_{f \neq k} (\lambda_k - \lambda_f)} \lambda_k^n, \text{ for every } n \geq r, \tag{5.1}$$

where $P'(z) = \frac{dP}{dz}(z)$ and the $\rho(n, r)$ are as in (1.3). In other terms, we have $v_n^{(r)} = \sum\limits_{k=1}^{r} x_k^{(r)} \lambda_k^n$, where $x_k^{(r)} = \frac{1}{P'(\lambda_k)} = \frac{1}{\prod\limits_{f \neq k} (\lambda_k - \lambda_f)}$.

By considering Expressions (4.3)-(4.4), we can furnish the following property concerning the analytic formula for entries of the powers of a companion matrix (3.1).

Proposition 7 *Let $\mathbf{A}$ be the companion matrix defined in (3.1). Let $P(z) = z^r - a_1 z^{r-1} - \ldots - a_r = \prod_{k=1}^{r}(z - \lambda_k)$ be the characteristic polynomial of the generalized Fibonacci sequence (1.1), where the roots λ_k $(1 \leq k \leq r)$ are all simple. Then, for every $n \geq 0$, the analytic formula for the entries $a_{ij}(n)$ of the powers $\mathbf{A}^n = (a_{ij}(n))_{1 \leq i,j \leq r}$, is given by*

$$a_{ij}(n) = \sum_{k=1}^{r} \sum_{k=1}^{r} \sum_{t=1}^{r-j+1} a_{r-t+1} \frac{\lambda_k^{-i+j+t}}{\prod\limits_{f \neq k} (\lambda_k - \lambda_f)} \lambda_k^n. \tag{5.2}$$

Application of Proposition 7 to a diagonal matrix with companion matrices blocks $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ permits us to get the following proposition.

Proposition 8 *Under the data of Proposition 6 suppose that for every d $(1 \leq d \leq m)$ the characteristic polynomial $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \ldots - a_{r_d}^{(d)} = \prod_{k=1}^{r_d}(z - \lambda_k^{(d)})$ of the matrix $\mathbf{A}_d$ are simple. Then, for every $n \geq 0$, the entries $a_{ij}(n)$ $(1 \leq i, j \leq r)$ of the powers $\mathbf{A}^n = diag(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n)$ take the form*

$$a_{ij}(n) = a_{ij,d}(n), \text{ for } (i,j) \in S_d, \quad 1 \leq d \leq m, \qquad \text{and} \qquad a_{ij}(n) = 0 \quad \text{elsewhere},$$

where $a_{ij,d}(n) = \sum\limits_{k=1}^{r_d} \left[\sum_{t=1}^{r_d - j+1} a_{r_d - t}^{(d)} \frac{(\lambda_k^{(d)})^{-i+j+t}}{\prod\limits_{f \neq k} (\lambda_k^{(d)} - \lambda_f^{(d)})} \right] (\lambda_k^{(d)})^n$.

5.2. Case of multiple eigenvalues

Suppose that the roots λ_i $(1 \leq i \leq h)$ of the characteristic polynomial $P(z) = z^r - a_1 z^{r-1} - \cdots - a_r = \prod_{i=1}^{h}(z - \lambda)^{m_i}$ are all of multiplicity $m_i \geq 2$ $(1 \leq i \leq h)$. For every $m_e \geq 2$ we set $\mathcal{E}_p^{[e]} = \{(n_1, \ldots, n_h) \in \mathbb{N}^{l-1};\ n_1 + \cdots + n_{e-1} + n_{e+1} + \cdots + n_h = m_e - p - 1\}$. In [6, Section 4.1] the following expression was considered

$$\gamma_p^{[e]}(\lambda_1, \ldots, \lambda_h) = (-1)^{r - m_e} \sum_{\mathcal{E}_p^{[e]}} \left(\prod_{1 \leq g \neq e \leq h} \frac{\binom{n_g + m_g - 1}{n_g}}{(\lambda_g - \lambda_e)^{n_g + m_g}} \right), \tag{5.3}$$

for $0 \leq p \leq m_e - 1$ and $1 \leq e \leq h$. The analytic formula of the fundamental solution $v_n^{(r)}$ related to the fundamental Fibonacci system (1.4), can be expressed in terms of the roots λ_i $(1 \leq i \leq h)$ of the polynomial $P(z) = z^r - a_1 z^{r-1} - \cdots - a_r$, by considering Expression (5.3) of the $\gamma_p^{[e]}(\lambda_1, \ldots, \lambda_h)$. Result of [7, Theorem 2.2] implies that we have $v_n^{(r)} = \sum_{e=1}^{h} \left(\sum_{p=0}^{m_e-1} \binom{n}{k} \gamma_p^{[e]}(\lambda_1, \ldots, \lambda_h) \right) \lambda_p^{n-p}$, for all $n \geq r$. And since $\frac{n!}{(n-p)!} = n(n-1)\cdots(n-p+1) = \sum_{q=0}^{p} s(p,q) n^q$, where the $s(p,q)$ are the Stirling numbers of the first kind, we get $v_n^{(r)} = \sum_{e=1}^{h} \left(\sum_{p=0}^{m_e-1} \left(\sum_{q=p}^{m_e-1} s(q,p) \frac{\gamma_q^{[e]}(\lambda_1, \ldots, \lambda_h)}{q! \lambda_e^q} \right) n^p \right) \lambda_e^n$, for every $n \geq r$. In summary, we have the following result.

Lemma 2 *The analytic expression of the fundamental solution $v_n^{(r)}$ is given by the following formula $v_n^{(r)} = \sum_{e=1}^{h} \left(\sum_{p=0}^{m_e-1} x_{e,p}^{(r)} n^p \right) \lambda_e^n$, for every $n \geq r$, where*

$$x_{e,p}^{(r)} = \sum_{q=p}^{m_e-1} s(q,p) \frac{\gamma_q^{[e]}(\lambda_1, \ldots, \lambda_h)}{q! \lambda_e^q}, \tag{5.4}$$

such that the $s(q,p)$ are the Stirling numbers of the first kind and the $\gamma_q^{[e]}(\lambda_1, \ldots, \lambda_h)$ are given by (5.3).

Combining Proposition 5 and Lemma 2 we can formulate the following proposition.

Proposition 9 *Let $\mathbf{A}$ be the companion matrix defined in (3.1) and $P(z) = z^r - a_1 z^{r-1} - \ldots - a_r = \prod_{k=1}^{h} (z - \lambda_k)^{m_k}$ its characteristic polynomial. Then, for every $n \geq 0$, the analytic formula for the entries of the powers $\mathbf{A}^n = (a_{ij}(n))_{1 \leq i,j \leq r}$, is given by*

$$a_{ij}(n) = \sum_{k=1}^{h} \left[\sum_{l=0}^{m_k-1} \Delta^{(r)}(i,j,l,k) n^l \right] \lambda_k^n, \tag{5.5}$$

where

$$\Delta^{(r)}(i,j,l,k) = \sum_{t=1}^{r-j+1} \sum_{u=l}^{m_k-1} \sum_{q=f}^{m_k-1} \frac{s(q,f)}{q!} L_{i,j,k}(f,u,t) \gamma_q^{[k]}(\lambda_1, \ldots, \lambda_h) \lambda_k^{-q-i+j+t}, \tag{5.6}$$

with $L_{i,j,k}(f,u,t) = \sum_{f=u}^{m_k-1} a_{r-t+1} \binom{f}{u} \binom{u}{l} (r-i)^{u-l} (-r+j+t)^{f-u}$ and $\gamma_q^[k]$ are as in 5.3.

More generally, suppose that the roots of the polynomial $P(z) = z^r - a_1 z^{r-1} - \cdots - a_r = \prod_{k=1}^{h_d} (z - \lambda_k^{(d)})^{m_k^{(d)}}$ are not all simple and set $\Gamma_1 = \{k,\ m_k = 1\} \neq \emptyset$ and $\Gamma_2 = \{k,\ m_k \geq 2\} \neq \emptyset$. Similarly to (4.8), the the combination of Expression (5.2) and Expressions (5.5)-(5.6) permit to get the analytic formula of the entries of the powers $\mathbf{A}^n = (a_{ij}(n))_{1 \leq i,j \leq r}$ under the form

$$a_{ij}(n) = \sum_{k \in \Gamma_1} \sum_{t=1}^{r-j+1} a_{r-t+1} \frac{\lambda_k^{-i+j+t}}{\prod_{f \neq k} (\lambda_k - \lambda_f)} \lambda_k^n + \sum_{k \in \Gamma_2} \left[\sum_{l=0}^{m_k-1} \Delta^{(r)}(i,j,l,k) n^l \right] \lambda_k^n, \tag{5.7}$$

where $\Delta^{(r)}(i,j,l,k)$ is given as in (5.6).

In the general setting, for a matrix $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ suppose that some roots of the polynomial $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \ldots - a_{r_d}^{(d)} = \prod_{k=1}^{h_d} (z - \lambda_k^{(d)})^{m_k^{(d)}}$ are not all simple, and set $\Gamma_1^{(d)} = \{k,\ m_k^{(d)} = 1\} \neq \emptyset$, $\Gamma_2^{(d)} = \{k,\ m_k^{(d)} \geq 2\} \neq \emptyset$. Then, by applying the formula (5.7) to the entries of $\mathbf{A}_d^n$ the powers of the matrix $\mathbf{A}_d$, we can formulate the following property.

Proposition 10 *Under the preceding data, the entries $a_{ij}(n)$ of the powers $\mathbf{A} = diag(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ are given by*

$$a_{ij}(n) = a_{ij,d}(n), \text{ for } (i,j) \in S_d, \quad 1 \leq d \leq m, \qquad \text{and} \qquad a_{ij}(n) = 0 \ \text{ elsewhere},$$

for every $n \geq 0$, where the sets S_d are given by (3.3) and

$$a_{ij,d}(n) = \sum_{k \in \Gamma_1^{(r_d)}} \Delta_1^{(r_d)}(i,j,k)(\lambda_k^{(r_d)})^n + \sum_{k \in \Gamma_2^{(r_d)}} \left[\sum_{l=0}^{m_k^{(r_d)}-1} \Delta_2^{(r_d)}(i,j,l,k) n^l \right] (\lambda_k^{(r_d)})^n,$$

with $\Delta_1^{(r_d)}(i,j,k) = \sum_{t=1}^{r_d-j+1} a_{r_d-t+1}^{(r_d)} \frac{\lambda_k^{-i+j+t}}{\prod_{f \neq k} (\lambda_k^{(r_d)} - \lambda_f^{(r_d)})}$ and $\Delta_2^{(r_d)}(i,j,l,k)$ is as in (5.6).

Especially, we have the following corollary.

Corollary 1 *Under the data od Proposition 10, suppose that the roots of each characteristic polynomial $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \ldots - a_{r_d}^{(d)}$, where $1 \leq d \leq m$, are simple. Then, for every $n \geq 0$, the entries $a_{ij}(n)$ of the powers $\mathbf{A}^n$ of the diagonal square matrix $\mathbf{A} = diag(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ with diagonal companion matrices blocks are*

$$a_{ij}(n) = \begin{cases} \sum_{k=1}^{r_d} \left[\sum_{t=1}^{r_d-j+1} a_{r_d-t}^{(r_d)} \frac{\lambda_k^{-i+j+t}}{\prod_{f \neq k} (\lambda_k^{(r_d)} - \lambda_f^{(r_d)})} \right] (\lambda_k^{(d)})^n, & \text{for } (i,j) \in S_d \ 1 \leq d \leq m, \\ 0, & \text{eslwhere.} \end{cases}$$

6. Some explicit analytic formulas for the entries of the powers of a diagonal matrix with diagonal companion matrices blocks: Second Approach

In this section we are interested in a second approach for the analytic formulas for the entries of the powers of the companion matrices, and the diagonal matrices with diagonal companion matrices blocks, by establishing some explicit analytic expression for the scalars $x_k^{(r)}$ and $x_{k,f}^{(r)}$.

6.1. The derivative formula for the entries of the powers of a diagonal matrix in blocks of companion matrices

Let λ_k $(1 \leq k \leq h)$ be the roots of the polynomial $P(z) = z^r - a_1 z^{r-1} - \cdots - a_r$, of multiplicities m_1, m_2,..., m_h (respectively). Using the divided difference method and Newton interpolation techniques, it was proved in [2, Theorem 3.1] that fundamental sequence $v_n^{(r)} = \rho(n+1, r)$ can be formulated as follows

$$v_n^{(r)} = \rho(n+1, r) = \sum_{p=1}^{h} \frac{f_{p,n+1}^{(m_p-1)}(\lambda_p)}{(m_p-1)!}, \text{ for every } n \geq r, \tag{6.1}$$

with $\rho(r,r) = 1$, $\rho(n,r) = 0$ for $0 \leq n \leq r-1$ and $f_{p,n+1}(x) = \dfrac{x^n}{\prod_{b=1,\, b \neq p}^{h} (x - \lambda_b)^{m_b}}$, where $f_{p,n+1}^{(k)}(x)$ means the derivative of order k of the function $f_{p,n+1}$. Especially, when the roots λ_p $(1 \leq p \leq r)$ are simple, Expression (6.1) takes the form

$$v_n^{(r)} = \rho(n+1, r) = \sum_{p=1}^{r} \frac{f_{p,n+1}^{(m_p-1)}(\lambda_p)}{(m_p-1)!} = \sum_{p=1}^{r} \frac{\lambda_p^n}{\prod_{b=1,\, b \neq p}^{r} (\lambda_p - \lambda_b)}. \tag{6.2}$$

We can observe that Expression (6.2) is nothing else but Expression (5.1) of Lemma 1. On the other side, the combinatorial formula for the entries of the powers $\mathbf{A}^n$ of the companion matrix $\mathbf{A}$ are given by formula (4.1), namely, $a_{ij}(n) = v_{n+r-i}^{(r-j+1)} = \sum_{t=1}^{r-j+1} a_{r-t+1}\rho(n-i+j+t-1, r)$, for every $n \geq 0$. Therefore, Expression (6.1) shows that

$$a_{ij}(n) = \sum_{t=1}^{r-j+1} a_{r-t+1}\rho(n-i+j+t-1,r) = \sum_{p=1}^{h}\sum_{t=1}^{r-j+1} a_{r-t+1}\frac{f_{p,n-i+j+t-1}^{(m_p-1)}(\lambda_p)}{(m_p-1)!}, \tag{6.3}$$

for every $n \geq r$. When all the roots of the polynomial $P(z) = z^r - a_1z^{r-1} - \cdots - a_r$ are simple Expression (6.2) is given by

$$a_{ij}(n) = \sum_{p=1}^{r}\sum_{t=1}^{r-j+1} \frac{a_{r-t+1}\lambda_p^{-i+j+t}}{\prod_{b=1,\, b\neq p}^{r}(\lambda_p - \lambda_b)}\lambda_p^n. \tag{6.4}$$

Mor generally, suppose that some roots of the polynomial $P(z) = z^r - a_1z^{r-1} - \cdots - a_r$ are not all simple and suppose that $\Gamma_1 = \{k,\ m_k = 1\} \neq \emptyset$ and $\Gamma_2 = \{k,\ m_k \geq 2\} \neq \emptyset$, where m_k means the multiplicity of λ_k. Then, the superposition of Expressions (6.3) and (6.4), allows us to obtain the derivative analytic formulas for the entries $a_{ij}(n)$ of the powers $\mathbf{A}^n = (a_{ij}(n))_{1\leq i,j\leq r}$. More precisely, we have the proposition.

Proposition 11 *Under the preceding data, suppose that some roots λ_k of the characteristic polynomial $P(z) = z^r - a_1z^{r-1} - \cdots - a_r$ of the sequence (1.1) are not all simple. Then, the analytic derivative formula for the entries $a_{ij}(n)$ $(1 \leq i,j \leq r)$ of the powers $\mathbf{A}^n$ of the companion matrix $\mathbf{A}$ given by (3.1), in terms of the derivative form is*

$$a_{ij}(n) = \sum_{p\in\Gamma_1}\sum_{t=1}^{r-j+1} a_{r-t+1}\frac{\lambda_p^{-i+j+t}}{\prod_{f\neq p}(\lambda_p - \lambda_f)}\lambda_p^n + \sum_{p\in\Gamma_2}\sum_{t=1}^{r-j+1} a_{r-t+1}\frac{f_{p,n-i+j+t-1}^{(m_p-1)}(\lambda_p)}{(m_p-1)!}, \tag{6.5}$$

for every $n \geq r$, where $\Gamma_1 = \{k,\ m_k = 1\}$ and $\Gamma_2 = \{k,\ m_k \geq 2\}$ are no empty sets.

In the general setting, for a diagonal matrix $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ with companion matrices blocks, suppose that some roots of the characteristic polynomial $P_d(z) = z^{r_d} - a_1^{(d)}z^{r-1} - \ldots - a_{r_d}^{(d)} = \prod_{k=1}^{h_d}(z - \lambda_k^{(d)})^{m_k^{(d)}}$ of the matrix $\mathbf{A}_d$ are not all simple. We set $\Gamma_1^{(d)} = \{k,\ m_k^{(d)} = 1\} \neq \emptyset$ and $\Gamma_2^{(d)} = \{k,\ m_k^{(d)} \geq 2\} \neq \emptyset$. Then, Proposition 11, namely, Expression (6.5), allows us to get the following corollary.

Corollary 2 *Under the preceding data, we show that the entries $a_{ij}(n)$ of the matrix powers $\mathbf{A}^n = diag(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n)$ are given by*

$$a_{ij}(n) = a_{ij,d}(n),\ \text{for } (i,j) \in S_d,\quad 1 \leq d \leq m, \qquad \text{and} \qquad a_{ij}(n) = 0 \ \text{ elsewhere},$$

for every $n \geq r$, where the sets S_d are given by (3.3) and

$$a_{ij,d}(n) = v_{n+r_d-i}^{(d,r_d-j+1)} =$$

$$\sum_{p\in\Gamma_1^{(d)}}\sum_{t=1}^{r_d-j+1} a_{r_d-t+1}^{(d)}\frac{\lambda_p^{-i+j+t}}{\prod_{f\neq p}(\lambda_p^{(d)} - \lambda_f^{(d)})}\left(\lambda_p^{(d)}\right)^n + \sum_{p\in\Gamma_2^{(d)}}\sum_{t=1}^{r_d-j+1} a_{r-t+1}^{(d)}\frac{f_{p,n-i+j+t-1}^{(m_p^{(d)}-1)}(\lambda_p^{(d)})}{(m_p^{(d)}-1)!}, \tag{6.6}$$

for every $n \geq r$, where $\Gamma_1^{(d)} = \{k,\ m_k^{(d)} = 1\}$ and $\Gamma_2^{(d)} = \{k,\ m_k^{(d)} \geq 2\}$ are no empty sets.

6.2. From the derivative formula to the analytic expression of the entries of the powers of a diagonal matrix with companion matrices blocks

To achieve the analytic formula of the entries of the powers of a companion matrix (3.1) or a diagonal matrix with diagonal companion matrices blocks, we need to compute the explicit formula of the derivative of order $m_p - 1$ (for $m_p \geq 2$) of the function $f_{p,n+1}(x) = \dfrac{x^n}{\prod_{k=1,\, k\neq p}^{s}(x-\lambda_k)^{m_k}}$. To reach this goal, we will proceed in two steps. First, let f, g two functions admitting derivatives of order $m \geq 1$ on a non-empty subset of $\mathbb{R}$. It is well known that we have $(fg)^{(m)} = \sum_{d=0}^{m}\binom{m}{d} f^{(d)}g^{(m-d)}$ (generalized Leibniz's rule). Application of this former formula to $f_{p,n+1}(x) = q_n(x)H_p(x)$, where $q_n(x) = x^n$ and $H_p(x) = \dfrac{1}{\prod_{k=1,\, k\neq p}^{s}(x-\lambda_k)^{m_k}}$, allows us to obtain $f_{p,n+1}^{(m)}(x) = \sum_{d=0}^{m}\binom{m}{d} q_n^{(d)}(x)H_p^{(m-d)}(x) = \sum_{d=0}^{m}\binom{m}{d}(n)_d x^{n-d}H_p^{(m-d)}(x)$, for every $n \geq r$, where $(n)_d = n(n-1)\cdots(n-d+1)$. It is well known that we have $(n)_d = \sum_{h=0}^{d} s(d,h)n^h$, where the $s(d,h)$ are the Stirling numbers of the first kind, we derive $f_{p,n+1}^{(m)}(x) = \sum_{d=0}^{m}\binom{m}{d}\left(\sum_{h=0}^{d} s(d,h)n^h\right)H_p^{(m-d)}(x)x^{n-d}$. for every $n \geq r$. Now with the aid of the identity $\sum_{d=0}^{m}\sum_{h=0}^{d} z_{d,h} = \sum_{h=0}^{m}\sum_{d=h}^{m} z_{d,h}$, for a bi-indexed sequence $\{z_{d,h}\}_{n\geq 0}$, we obtain $f_{p,n+1}^{(m)}(x) = \sum_{h=0}^{m}\left(\sum_{d=h}^{m} s(d,h)\binom{m}{d}H_p^{(m-d)}(x)x^{-d}\right)n^h x^n$. Thus, for $x = \lambda_i$ and $m = m_i - 1$, we arrive to get

$$f_{p,n+1}^{(m_p-1)}(\lambda_p) = \sum_{h=0}^{m_p-1}\left(\sum_{d=h}^{m_p-1} s(d,h)\binom{m_p-1}{d}H_p^{(m_p-d-1)}(\lambda_p)\lambda_p^{-d}\right)n^h\lambda_p^n, \tag{6.7}$$

for every $n \geq r$.

Second, for improving Expression (6.7) we need the explicit form of the $m-th$ derivative of the function $H_p(x)$. To this aim, we use the following known formula $(f_1\cdot f_2\cdots f_l)^{(m)} = \sum_{k_1+\cdots+k_l=m}\binom{m}{k_1\ \ldots\ k_l}\prod_{k=1}^{l} f_k^{(k_l)}$, where $\binom{m}{k_1\ \ldots\ k_l} = \dfrac{m!}{k_1!k_2!\cdots k_l!}$ and $f_l : I \to \mathbb{R}$ $(1 \leq l \leq h)$, where I is an interval of $\mathbb{R}$, are n times differentiable, and $f^{(s)}$ means the derivative of order s of the function f (see, for instance, [1,16]). Let us apply the above formulas to the function $H_p(x)$, written under the form $H_p(x) = \dfrac{1}{\prod_{g=1,\, g\neq p}^{h}(x-\lambda_g)^{m_g}} = \prod_{g=1,\, g\neq p}^{h} f_g(x)$, with $f_g(x) = (x-\lambda_g)^{-m_g}$. Thus, we get $H_p^{(k)}(x) = \sum_{\hat{\varepsilon}_k^{[p]}}\binom{k}{s_1\ \ldots\ s_h}\prod_{g=1,\, g\neq p}^{h} f_g^{(s_g)}(x)$, where $\hat{\varepsilon}_k^{[p]} = \{(s_1, s_2, ..., s_h) \in \mathbb{N}^{h-1};\ s_1+\cdots+s_{p-1}+s_{p+1}+\cdots+s_h = k\}$ and $f_g^{(s_g)}(x) = (-1)^{s_g}\dfrac{(m_g+s_g-1)!}{(m_g-1)!}(x-\lambda_g)^{-m_g-s_g}$, for every g $(1 \leq g \neq p \leq h)$. Therefore, we derive the following lemma.

Lemma 3 *For every $k \geq 1$ and $1 \leq p \leq s$, we have,*

$$H_p^{(k)}(x) = (-1)^k\sum_{\hat{\varepsilon}_k^{[p]}}\binom{k}{p_1\ldots p_h}\prod_{g=1,\, g\neq p}^{s}\frac{(m_g+p_g-1)!}{(m_g-1)!}(x-\lambda_g)^{-m_g-p_g}. \tag{6.8}$$

Summarizing, using Expressions (6.1) and (6.7)-(6.8) we can formulate the lemma.

Lemma 4 *(see [3]) Let λ_i $(1 \leq i \leq s)$ be the roots of the characteristic polynomial $P(z) = z^r - a_1z^{r-1} - \cdots - a_r$ of multiplicities m_1, m_2,..., m_s, respectively. Suppose that for every root λ_i the associated* *multiplicity $m_i \geq 2$ $(1 \leq i \leq s)$. Then, the analytic formula of the fundamental sequence is given as follows*

$$v_n^{(r)} = \sum_{p=1}^{h} \frac{1}{(m_p-1)!} \sum_{g=0}^{m_p-1} \left(\sum_{f=g}^{m_p-1} s(f,g) \binom{m_p-1}{f} H_p^{(m_p-f-1)}(\lambda_p) \lambda_p^{-f} \right) n^g \lambda_p^n, \tag{6.9}$$

for every $n \geq r$, where $s(f,g)$ are the Stirling numbers of the first kind and the $H_p^{(k)}(x)$ are as in (6.8).

When all the roots $\lambda_i (1 \leq i \leq r)$ are simple, Expression (6.9) implies that, for every $n \geq r$, we have $v_n^{(r)} = \sum_{p=1}^{r} \frac{\lambda_p^n}{\prod_{k=1,\, k\neq p}^{r} (\lambda_p - \lambda_k)}$. Taking into account Expressions (6.5)- (6.6), we show that Expression (6.9) can permit us to provide another analytic formula for the entries $a_{ij}(n)$ $(1 \leq i, j \leq r)$ of the powers of the companion matrix (3.1) and a diagonal matrix $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ with companion matrices blocks. That is, Expression (3.2) shows that $a_{ij}(n) = v_{n+r-i}^{(r-j+1)} = \sum_{t=1}^{r-j+1} a_{r-t+1} v_{n-i+j+t-2}^{(r)}$, and using Expression (6.5) we have $v_{n-i+j+t-2}^{(r)} = \sum_{p=1}^{h} \sum_{g=0}^{m_p-1} \Theta(g,p)(n-i+j+t-2)^g \lambda_p^n$, where

$$\Theta(g,p) = \frac{1}{(m_p-1)!} \sum_{f=g}^{m_p-1} s(f,g) \binom{m_p-1}{f} H_p^{(m_p-f-1)}(\lambda_p) \lambda_p^{2r-i-j-t-f}.$$

Following the identity $(n-i+j+t-2)^g = \sum_{l=0}^{g} \binom{g}{l} n^l (-i+j+t-2)^{g-l}$, we derive that we have $v_{n-i+j+t-2}^{(r)} = \sum_{p=1}^{h} \sum_{l=0}^{m_p-1} \Omega_{l,p,t}(i,j) n^l \lambda_p^n$, where

$$\Omega_{l,p,t}(i,j) = \sum_{g=l}^{m_p-1} \sum_{f=g}^{m_p-1} \frac{\binom{g}{l}\binom{m_p-1}{f}}{(m_p-1)!} s(f,g)(-i+j+t-2)^{g-l} H_p^{(m_p-f-1)}(\lambda_p) \lambda_p^{-i+j+t-f-2}.$$

In addition, Expression (2.2) shows that $a_{ij}(n) = \sum_{p=1}^{h} \sum_{l=0}^{m_p-1} \sum_{t=1}^{r-j+1} a_{r-t+1} \Omega_{l,p,t}(i,j,t) n^l \lambda_p^n$. In summary, we have the following proposition.

Proposition 12 *Under the preceding data, suppose that roots λ_p $(1 \leq p \leq h)$ of multiplicities m_p $(1 \leq p \leq h)$, of the polynomial $P(z) = z^r - a_1 z^{r-1} - \cdots - a_r$ are not all simple. Then, the analytic formula for the entries $a_{ij}(n)$ $(1 \leq i, j \leq r)$ of the powers $\mathbf{A}^n$ of the companion matrix $\mathbf{A}$ given by (3.1), in terms of the roots λ_p $(1 \leq p \leq h)$, is*

$$a_{ij}(n) = \sum_{p \in \Gamma_1} \sum_{t=1}^{r-j+1} a_{r-t+1} \frac{\lambda_p^{-i+j+t}}{\prod_{f \neq p} (\lambda_p - \lambda_f)} \lambda_p^n + \sum_{p \in \Gamma_2} \sum_{l=0}^{m_p-1} \sum_{t=1}^{r-j+1} a_{r-t+1} \Omega_{l,p,t}(i,j) n^l \lambda_p^n,$$

for every $n \geq r$, where $\Gamma_1 = \{k,\ m_k = 1\}$ and $\Gamma_2 = \{k,\ m_k \geq 2\}$ are no empty sets and

$$\Omega_{l,p,t}(i,j) = \sum_{g=l}^{m_p-1} \sum_{f=g}^{m_p-1} \frac{\binom{g}{l}\binom{m_p-1}{f}}{(m_p-1)!} s(f,g)(-i+j+t-2)^{g-l} \lambda_p^{-i+j+t-f-2} H_p^{(m_p-f-1)}(\lambda_p), \tag{6.10}$$

where $s(f,g)$ are the Stirling numbers of the first kind and $H_p^{(k)}(x)$ is given by (6.8).

For a diagonal matrix $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ with companion matrices blocks, we suppose that the roots of the characteristic polynomial $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \ldots - a_{r_d}^{(d)} = \prod_{k=1}^{h_d}(z - \lambda_k^{(d)})^{m_k^{(d)}}$ are not all simple, and set $\Gamma_1^{(d)} = \{k,\ m_k^{(d)} = 1\} \neq \emptyset$, $\Gamma_2^{(d)} = \{k,\ m_k^{(d)} \geq 2\} \neq \emptyset$. Hence, the formula (6.5) can be applied for computing the entries $a_{ij,d}(n)$ of the powers $\mathbf{A}_d^n$ of each companion matrix $\mathbf{A}_d$ $(1 \leq d \leq m)$. Therefore, Propositions 11 and 12 allow us to have the following corollary.

Corollary 3 *Under the preceding data, suppose that the sets* $\Gamma_1^{(d)} = \{k,\ m_k^{(d)} = 1\}$, $\Gamma_2^{(d)} = \{k,\ m_k^{(d)} \geq 2\}$ *are no empty and consider the set* S_d *are given by* (3.3). *Then, for every* $n \geq r$, *the entries* $a_{ij}(n)$ *of the matrix powers* $\mathbf{A}^n = diag(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n)$ *are given by*

$$a_{ij}(n) = a_{ij,d}(n),\ \text{for } (i,j) \in S_d,\quad 1 \leq d \leq m, \qquad \text{and} \qquad a_{ij}(n) = 0 \text{ elsewhere},$$

with $a_{ij,d}(n) = \sum_{p \in \Gamma_1^{(d)}} \bigwedge_d(i,j) \left(\lambda_p^{(d)}\right)^n + \sum_{p \in \Gamma_2^{(d)}} \sum_{l=0}^{m_p^{(d)}-1} \sum_{t=1}^{r_d-j+1} a_{r-t+1}^{(d)} \Omega_{l,p,t}^{(d)}(i,j) n^l \left(\lambda_p^{(d)}\right)^n$, *where* $\bigwedge_d(i,j) = \sum_{t=1}^{r_d-j+1} a_{r_d-t+1}^{(d)} \frac{\lambda_p^{-i+j+t}}{\prod_{f \neq p}(\lambda_p^{(d)} - \lambda_f^{(d)})}$ *and* $\Omega_{l,p,t}^{(d)}(i,j,t)$ *is as in* (6.10).

7. Application: Some formulas of power of matrices

The results of the preceding sections can be used as another approach for studying some standard matrix functions such as the powers of matrices and power series matrix functions. In this section, we consider the matrix power function $F(A)$, where Let $F(z) = z^n$.

For a given matrix $M \in \mathbb{K}^{r \times r}$ the Frobenius decomposition shows that it is similar to a diagonal matrix $\mathbf{A} = \mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ with companion matrices block, where $\mathbf{A}_d = [a_1^{(d)}, \cdots, a_{r_d}^{(d)}]$ $(1 \leq d \leq m)$ is a companion matrix of order $r_d \times r_d$ $(1 \leq r_m \leq \cdots \leq r_1)$ for more details (see, for example, [10,11]). Therefore, there exists an invertible matrix $P \in \mathbb{K}^{r \times r}$ such that $M = P.\mathrm{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m).P^{-1}$. Hence, for every $n \geq 0$, the powers M^n of the matrix M are given by $M^n = P.\mathrm{diag}(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n).P^{-1}$. This former expression shows that the linear, the combinatorial and the analytic properties elaborated in the preceding sections will allow us to establish other properties of the powers of every square matrix.

Recall that Expression (3.2) shows that the entries of the powers ${\mathbf{A_d}}^n = (a_{ij,d}(n))_{1 \leq i,j \leq r}$ are imparted under the form

$$a_{ij,d}(n) = v_{n+r_d-i}^{(d,r_d-j+1)} = \sum_{t=1}^{r_d-j+1} a_{r_d-t+1} v_{n-i+j+t-2}^{(d,r_d)}, \tag{7.1}$$

for every $n \geq 0$, where the set of sequences $\{v_n^{(d,r_d-s+1)}\}_{n \geq 0}$ $(1 \leq s \leq r_d)$ represents the fundamental Fibonacci system associated with the companion matrix A_d and $\{v_n^{(d,r_d)}\}_{n \geq 0}$ is its related fundamental sequence. Without lost of generality, we suppose in this section that $2 \leq d \leq m$ and we set $P = (m_{ij})_{1 \leq i,j \leq r}$ and $P^{-1} = (b_{ij})_{1 \leq i,j \leq r}$. Therefore, for every $n \geq 0$ and $1 \leq i \leq r$, the entries of the the matrix product $P.A^n = (c_{ij}(n))_{1 \leq i,j \leq r}$ are given by $c_{ij}(n) = \displaystyle\sum_{k=r_{d-1}+1}^{r_d} m_{ik} a_{kj,d}(n)$, for $r_{d-1}+1 \leq j \leq r_d$ and $1 \leq d \leq m$, where $d_0 = 0$. A straightforward computation shows that the entries $h_{ij}(n)$ $(1 \leq i,\ j \leq r)$ of the matrix powers $M^n = P.\mathrm{diag}(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n).P^{-1}$ are as

$$h_{ij}(n) = \sum_{p=1}^{r} \left[\sum_{k=r_{d-1}+1}^{r_d} m_{ik} a_{kp,d}(n) \right] b_{pj},\ \text{for } r_{d-1}+1 \leq j \leq r_d \text{ and } 1 \leq d \leq m, \tag{7.2}$$

for every $n \geq 0$ and $1 \leq i \leq r$, where $d_0 = 0$. Taking into account Expressions (7.1)-(7.2) the entries $h_{ij}(n)$ $(1 \leq i,\ j \leq r)$ of the matrix powers $M^n = P.\mathrm{diag}(\mathbf{A}_1^n, \mathbf{A}_2^n, \ldots, \mathbf{A}_m^n).P^{-1}$ we can be expressed in terms of the family of fundamental Fibonacci systems $\{\{v_n^{(d,s)}\}_{n \geq 0}\ ,\ 1 \leq s \leq r_d\}$ $(1 \leq d \leq m)$ associated to the companion matrices A_d $(2 \leq r_m \leq \cdots \leq r_1)$ and their fundamental sequences $\{v_n^{(d,r_d)}\}_{n \geq 0}$. More precisely, we get the following results.

Proposition 13 *Let M be an element of $\mathbb{K}^{r\times r}$, whose Frobenius decomposition is under the form $M = P.diag(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_m)P^{-1}$, with companion matrices blocks $\mathbf{A}_d = [a_1^{(d)}, \cdots, a_{r_d}^{(d)}]$ of order $r_d \times r_d$, such that $2 \le r_m \le \cdots \le r_1$. Set $P = (m_{ij})_{1\le i,j\le r}$ and $P^{-1} = (b_{ij})_{1\le i,j\le r}$. Then, for every $n \ge 0$, the entries $h_{ij}(n)$ $(1 \le i,\ j \le r)$ of the matrix powers M^n are given by*

$$h_{ij}(n) = \sum_{p=1}^{r} \left[\sum_{k=r_{d-1}+1}^{r_d} m_{ik} v_{n+r_d-k}^{(d,r_d-p+1)} \right] b_{pj}, \ \text{for } r_{d-1}+1 \le j \le r_d, \tag{7.3}$$

for every $1 \le i \le r$, with $1 \le d \le m$ and $d_0 = 0$, where $\{v_n^{(d,r_d-s+1)}\}_{n\ge 0}$ $(1 \le s \le r_d)$ represents the fundamental Fibonacci system associated with the companion matrix A_d $(1 \le d \le m)$. Moreover, for every $n \ge 0$, the expression the entries $h_{ij}(n)$ $(1 \le i,\ j \le r)$ in terms of the fundamental sequences $\{v_n^{(d,r_d)}\}_{n\ge 0}$ related to the companion matrices A_d $(1 \le d \le m)$ is as follows

$$h_{ij}(n) = \sum_{p=1}^{r} \left[\sum_{k=r_{d-1}+1}^{r_d} \sum_{t=1}^{r_1-p+1} m_{ik} a_{r_d-t+1}^{(d)} v_{n+r_d-k+p+t-1}^{(d,r_d)} \right] b_{pj}, \ \text{for } r_{d-1}+1 \le j \le r_d, \tag{7.4}$$

for every $1 \le i \le r$, where $1 \le d \le m$ and with $d_0 = 0$.

Expressions (7.2), (7.3) and (7.4), will play a central role for establishing the combinatorial and analytical formulas for the entries $h_{ij}(n)$ of the matrix powers M^n, of a given square matrix $M \in \mathbb{K}^{r\times r}$. Indeed, the combinatorial expression of the entries $h_{ij}(n)$ can be obtained from results of Subsection 4.1. That is, Expression (4.2) combined with Expression (7.3) allow us to get the following result.

Proposition 14 *Under the data of Proposition 13, for every $1 \le i \le r$, with $1 \le d \le m$ and $d_0 = 0$, the combinatorial expression of the entries $h_{ij}(n)$ of the matrix power M^n are as*

$$h_{ij}(n) = \sum_{p=1}^{r} \Theta_{kp}(n) b_{pj}, \ \text{for } r_{d-1}+1 \le j \le r_d, \tag{7.5}$$

where $\Theta_{kp}(n) = \sum_{k=r_{d-1}+1}^{r_d} \sum_{t=1}^{r_d-p+1} m_{ik} a_{r-t+1}^{(d)} \rho(n+r_d-k+p+t-1, r_d)$, such that $\rho(n, r_d) = \displaystyle\sum_{k_1+2k_2+\cdots+rk_r=n-r_d} \frac{(k_1+\cdots+k_r)!}{k_1!\cdots k_r!} (a_1^{(d)})^{k_1} \cdots (a_r^{(d)})^{k_r}$.

For the analytic formula of the entries of the matrix powers $M^n = P.\text{diag}(\mathbf{A}_1^n, \mathbf{A}_2^n, \dots, \mathbf{A}_m^n).P^{-1}$, Expressions (7.3)-(7.4) combined with results of Subsection 4.2 and Sections 5-6 will permit us to derive several analytical formulas of the entries $h_{ij}(n)$ $(1 \le i,\ j \le r)$.

Suppose that each companion matrix $\mathbf{A}_d = \mathbf{A}_d[a_1^{(d)}, a_2^{(d)}, \cdots, a_{r_d}^{(d)}]$ $(d = 1, \dots m)$ is of order $r_d \times r_d$, with $r_d \ge 2$. Let $P_d(z) = z^{r_d} - a_1^{(d)} z^{r-1} - \dots - a_{r_d}^{(d)} = \prod_{k=1}^{h_d} (z - \lambda_k^{(d)})^{m_k^{(d)}}$ be the characteristic polynomial of the matrix $\mathbf{A}_d$. Following Proposition 6, namely, Expressions (4.6)-(4.7), and formula (7.2) we get the result.

Proposition 15 *Under the data of Proposition 13, for every $1 \le i \le r$, with $1 \le d \le m$ and $d_0 = 0$, the analytical expression of the entries $h_{ij}(n)$ of the matrix power M^n are given by*

$$h_{ij}(n) = \sum_{p=1}^{r} \left[\sum_{k=r_{d-1}+1}^{r_d} m_{ik} a_{kp,d}(n) \right] b_{pj}, \ \text{for } r_{d-1}+1 \le j \le r_d, \tag{7.6}$$

where $a_{kp,d}(n) = v_{n+r_d-k}^{(d,r_d-p+1)} = \displaystyle\sum_{s=1}^{h_d} \left[\sum_{l=0}^{m_s^{(d)}-1} \Delta^{(r_d)}(k,p,l,s) n^l \right] (\lambda_s^{(d)})^n$, such that the scalars $\Delta^{(r_d)}(k,p,l,s)$ are as follows $\Delta^{(r_d)}(k,p,l,s) = \displaystyle\sum_{t=1}^{r_d-p+1} \sum_{u=l}^{m_s^{(d)}-1} \sum_{f=u}^{m_s^{(d)}-1} a_{r_d-t+1}^{(d)} K_{k,p}(f,u,t) (\lambda_s^{(d)})^{-k+p+t} x_{s,f}^{(r_d)}$, *with* $K_{i,j}(f,u,t) = \binom{f}{u}\binom{u}{l}(r_d - i)^{u-l}(-r_d + j + t)^{f-u}$ *and* $x^{r_d}_{s,f}$ *are the scalars of the analytic Binet formula of the fundamental sequence* $\{v_n^{(d,r_d)}\}_{n\geq 0}$ *(see formula (2.3)).*

More analytic formulas for the matrix powers can be established with the aid of results of [2,6,7]. In addition the knowledge of the matrices P and P^{-1} allows us to obtain the explicit formula for the entries $h_{ij}(n)$ of the matrix powers M^n.

8. Concluding remarks and perspective

In this study we had established some compact formulas for the powers of the companion matrix and a diagonal matrix $\mathbf{A} = \text{diag}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m)$ with companion matrices blocks. More precisely, the entries of the powers of these type of matrices are exhibited in terms of linear and combinatorial representations, as well as in terms of analytic formulas. In addition, we had provided some applications for computing the matrix powers in the general setting. The results of this study are not known in the literature under this form.

Based on the results of the present paper, the next step is to study some known matrix functions, which admit an analytical development in series. Finally, our approach can also be considered for studying some topics in the additive number theory. Some preliminary results have already been established on these this topic.

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