Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–23.
ISSN-0037-8712
doi:10.5269/bspm.76462

Blow-up phenomena for a pseudo-parabolic equation with variable exponents

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ABSTRACT: This study will concentrate on a class of pseudo-parabolic equations with variable exponents. We will present a threshold result concerning the finite-time blow-up of solutions and introduce a new criterion for blow-up. Furthermore, we will derive estimates for the lifespan of solutions and establish both upper and lower bounds for the blow-up time.

Key Words: Pseudo-parabolic equation, positive initial energy, blow-up time, bounds for the blow-up time, Sobolev spaces with variable exponents nonlinearities.

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1. Introduction

In recent years, considerable mathematical effort has been dedicated to studying nonlinear wave equations that involve a variable source nonlinearity term, along with (potentially strong) linear damping effects, due to their extensive applications in physics, engineering, and material sciences. This paper examines a second-order super-linear hyperbolic damped wave equation featuring a nonlinear source term that models viscoelasticity.

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p(x)-2}u, \quad (x,t) \in \Omega \times [0,T), \\ u = 0, \quad (x,t) \in \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \quad x \in \Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $T > 0$, and the initial value functions

$$u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega), \tag{1.2}$$

$$\omega \geq 0, \quad \mu > -\omega\lambda_1, \tag{1.3}$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions, where $-\Delta$ stands for the Laplacian concerning the spatial variables. We are interested in the finite time blow-up property, so employing the potential-well method and some inequality techniques, we prove the blow-up in a finite time of weak solutions and obtain a new blow-up criterion. Meanwhile, the lifespan

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2010 *Mathematics Subject Classification:* 35L05, 35L15, 35L70.
Submitted April 19, 2025. Published September 29, 2025

and upper- and lower-bound estimates of the blow-up time are also derived. The exponents $p(.)$ is a measurable function on Ω satisfying

$$2 < p_1 = ess \inf_{x\in\Omega} p\left(x\right) \leq p\left(x\right) \leq p_2 = ess \sup_{x\in\Omega} p\left(x\right) < \infty, \tag{1.4}$$

with

$$p_2 \leq \begin{cases} \dfrac{2n}{n-2}, & \text{for } \omega > 0, \\ \dfrac{2n-2}{n-2}, & \text{for } \omega = 0. \end{cases} \quad \text{if} \quad n \geq 3, \ x \in \Omega; \tag{1.5}$$
$$\text{or } 2 < p_1 \leq p\left(x\right) \leq p_2 < +\infty \quad \text{if} \quad n = 1, 2, \ x \in \Omega.$$

We also assume that $p\left(.\right)$ satisfies the following Zhikov–Fan uniform local continuity condition. There exists a constant $M > 0$ such that for all points x, y in Ω with $0 < |x-y| < \frac{1}{2}$, we have the inequality

$$\left|p\left(x\right) - p\left(y\right)\right| \leq \frac{M}{\left|\log\left|x-y\right|\right|}, \tag{1.6}$$

where $M(r)$ satisfies

$$\lim_{r\to 0^+}\sup M(r) \ln\left(\frac{1}{r}\right) = c < \infty.$$

The dynamic properties of viscoelastic materials are of great importance and interest as they appear in many applications in the natural sciences, for example, models of flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media [20,21,22], and the processing of digital images [23,24,25], and can all be associated with problem (1.1), more details on the subject can be found in [6,8,26] and the other references contained therein. In the classical case of constant exponent ($p(x) = \text{constant} = p$), this equation has its origin in the nonlinear vibration of an elastic string, where the source term $u^{p-2}u$ forces the negative-energy solutions to explode in finite time. It's known that several authors have looked at problem (1.1) concerning the results of the global existence and blow-up of solutions, and a powerful technique for treating it is the "potential well method," which was founded by the first author Sattinger [5] in 1968 and later enhanced by Liu and Zhao [2] by introducing the so-called family of potential wells which later became a significant method for the study of nonlinear evolution equations and has also provided many interesting results, in [17], [9] and [10], in which the authors discussed in a bounded domain of $\mathbb{R}^n$ the global existence and the explosion results when the initial data are at different energy levels $E(u_0) \leq d$, and $E(u_0) > d$ respectively. In particular, they obtained the blow-up phenomena of solutions and a lower bound for the blow-up time when the blow-up occurs if p and the initial value satisfies some conditions for $\omega \geq 0$ and $\mu > -\omega\lambda_1$, where λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions. Additional papers on those damped wave equations can be found throughout the literature, we're forwarding readers interested in [11,16,12,18,13,14] and the references it contains. We mention that the technique we adopted here for this equation is one generalized in the classical case, and the results obtained add new information about the upper and lower bounds of the blow-up time.

2. Preliminaries

Let $p : \Omega \to [1, \infty]$ be a measurable function. $L^{p(.)}(\Omega)$ denotes the set of the real measurable functions u on Ω such that

$$\int_\Omega \left|\lambda u\left(x\right)\right|^{p(x)} \mathrm{d}x < \infty \text{ for some } \lambda > 0.$$

The variable-exponent space $L^{p(.)}\left(\Omega\right)$ equipped with the Luxemburg-type norm

$$\left\|u\right\|_{p(.)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u\left(x\right)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1\right\},$$

is a Banach space. Throughout the paper, we use $\|.\|_q$ to indicate the L^q-norm for $1 \leq q \leq +\infty$. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the following norm:

$$\|u\|_{H_0^1(\Omega)} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

For Ω be a bounded domain of $\mathbb{R}^n$, $p(.) \subset (1, \infty)$ is a measurable function on $\overline{\Omega}$ satisfies (1.5) and (1.6), we have

Lemma 1 (Sobolev-Poincaré inequality) *([1])For all $u \in H_0^1(\Omega)$, then the following embedding*

$$H_0^1(\Omega) \hookrightarrow L^{p_2}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \hookrightarrow L^{p_1}(\Omega) \hookrightarrow L^2(\Omega),$$

are continuous, and we have

$$\begin{aligned} &\|u(t)\|_{p(.)} \leq B\|\nabla u(t)\|_2, \quad \|u\|_{\frac{2n}{n-2}} \leq \bar{B}\|\nabla u(t)\|_2 \quad (n \geq 3), \\ &\|u(t)\|_{p_2} \leq \hat{B}\|\nabla u(t)\|_2, \end{aligned} \tag{2.1}$$

where B, $\bar{B}$, and $\hat{B}$ are the optimal constant of the Sobolev embedding and $\|.\|_{p(.)}$ denotes the norm of $L^{p(\cdot)}(\Omega)$, having the following propriety

$$\begin{aligned} &\min\left(\|u(t)\|_{p(.)}^{p_1}, \|u(t)\|_{p(.)}^{p_2}\right) \\ &\leq \varrho(u) = \int_\Omega |u(t)|^{p(x)}\,\mathrm{d}x \leq \max\left(\|u(t)\|_{p(.)}^{p_1}, \|u(t)\|_{p(.)}^{p_2}\right), \end{aligned}$$

for any $u \in L^{p(\cdot)}(\Omega)$.

We denote $\|.\|_q$ and $\|.\|_{H^1(\Omega)}$ to the usual $L^q(\Omega)$ norm and $H^1(\Omega)$ norm, respectively.

2.1. Modified Potential Wells

We define the functional

$$J(u) = \frac{1}{2}\|\nabla u\|^2 - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\mathrm{d}x,$$

$$I(u) = \|\nabla u\|^2 - \int_\Omega |u|^{p(x)}\mathrm{d}x.$$

We then have the subsequent lemma.

Lemma 2 *For $p(x)$ be (1.5)-(1.6) and $u \in H_0^1(\Omega)\backslash\{0\}$. Let $F : [0, +\infty) \to \mathbb{R}$ the Euler functional defined by*

$$F(\lambda) = \frac{\lambda^2}{2}\|\nabla u\|^2 - \int_\Omega \frac{\lambda^{p(x)}}{p(x)}|u|^{p(x)}\mathrm{d}x,$$

then, F keeps the following properties:

(i) $\lim_{\lambda\to 0^+} F(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} F(\lambda) = -\infty$.

(ii) *There is at least one solution to the equation $F'(\lambda) = 0$ on the interval $[\lambda_1, \lambda_2]$, where*

$$\lambda_1 = \min\left\{\rho(u)^{\frac{1}{2-\mathrm{p}_1}}, \rho(u)^{\frac{1}{2-\mathrm{p}_2}}\right\}, \ \lambda_2 = \max\left\{\rho(u)^{\frac{1}{2-\mathrm{p}_1}}, \rho(u)^{\frac{1}{2-\mathrm{p}_2}}\right\}, \tag{2.2}$$

and

$$\rho(u) := \frac{\int_\Omega |\nabla u(x)|^2\mathrm{d}x}{\int_\Omega |u(x)|^{\mathrm{p}(x)}\mathrm{d}x}.$$

(iii) *There exists a $\lambda^* = \lambda^*(u) > 0$ such that $F(\lambda)$ gets its maximum at $\lambda = \lambda^*$. Furthermore, we have that $0 < \lambda^* < 1$, $\lambda^* = 1$ and $\lambda^* > 1$ provided $I(u) < 0$, $I(u) = 0$ and $I(u) > 0$, respectively.*

Beweis. Since $p(x) \in C_+(\bar{\Omega}) = \left\{ p \in C(\bar{\Omega}) : \inf_{x \in \bar{\Omega}} p(x) > 2 \right\}$, the assertion (i) is shown by the following:

$$F(\lambda) \leq \frac{\lambda^2}{2} \int_\Omega |\nabla u(x)|^2 \mathrm{d}x - \min \{\lambda^{p_1}, \lambda^{p_2}\} \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} \mathrm{d}x,$$

and

$$F(\lambda) \geq \frac{\lambda^2}{2} \int_\Omega |\nabla u(x)|^2 \mathrm{d}x - \max \{\lambda^{p_1}, \lambda^{p_2}\} \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} \mathrm{d}x.$$

For (ii). We have

$$F'(\lambda) = \lambda \int_\Omega |\nabla u(x)|^2 \mathrm{d}x - \int_\Omega \lambda^{p(x)-1} |u(x)|^{p(x)} \mathrm{d}x,$$

which implies that $F'(\lambda)$ lies in the following two inequalities

$$F'(\lambda) \geq \lambda \int_\Omega |\nabla u(x)|^2 \mathrm{d}x - \max \left\{\lambda^{p_1 - 1}, \lambda^{p_2 - 1}\right\} \int_\Omega |u(x)|^{\mathrm{p}(x)} \mathrm{d}x,$$

and

$$F'(\lambda) \leq \lambda \int_\Omega |\nabla u(x)|^2 \mathrm{d}x - \min \left\{\lambda^{p_1 - 1}, \lambda^{p_2 - 1}\right\} \int_\Omega |u(x)|^{\mathrm{p}(x)} \mathrm{d}x.$$

Since $p_1 > 2$, we signify that $F'(\lambda)$ has at least one zero point λ satisfying (2.2). So we get (ii). The definition of λ^* and the relation $I(\lambda u) = \lambda F'(\lambda)$ and

$$F'(\lambda) \leq \left(\lambda - \lambda^{p_2 - 1}\right) \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \lambda^{p_2 - 1} I(u), \text{ for } \lambda \in (0, 1),$$

and

$$F'(\lambda) \geq \left(\lambda - \lambda^{p_2 - 1}\right) \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \lambda^{p_2 - 1} I(u), \text{ for } \lambda \in (1, \infty),$$

lead to the last claim (iii). Completeness of the proof. ■

2.2. Assumptions and main results

As J is the Fréchet-differentiable functional with derivative J', let suppose that $u \neq 0$ is a critical point of J, i.e., $J'(u) = 0$. Then necessarily u is contained in the set

$$\mathcal{N} = \left\{ u \in H_0^1(\Omega) \backslash \{0\} : I(u) = \langle J'(u), u \rangle = 0 \right\},$$

so $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of J, $\mathcal{N}$ is called the Nehari manifold associated with the energy functional J. By Lemma 2 we know that $\mathcal{N}$ is not empty set. It is clear that $J(u)$ is coercive on $\mathcal{N}$. Set

$$d = \inf_{u \in \mathcal{N}} J(u). \tag{2.3}$$

Under the appropriate conditions, we have d is a positive finite number and is therefore well-defined. For E_d is a constant given by

$$\mathrm{E}_d = \frac{p_1 - 2}{p_1} \frac{p_2}{p_2 - 2} d \leq d, \tag{2.4}$$

we define the modified stable and unstable sets as follows

$$\mathcal{W} = \left\{ u \in H_0^1(\Omega) : J(u) < \mathrm{E}_d,\ I(u) > 0 \right\} \cup \{0\},$$
$$\mathcal{U} = \left\{ u \in H_0^1(\Omega) : J(u) < \mathrm{E}_d,\ I(u) < 0 \right\},$$

Let consider the energy functional $E : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \mathrm{E(t)} &= \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \|\nabla u(t)\|_2^2 - \int_\Omega \frac{1}{p(x)} |u(t)|^{p(x)} \mathrm{d}x, \\ \mathrm{E}(u, u_t)(\mathrm{t}) &= \frac{1}{2} \|u_t(t)\|_2^2 + J(u(t)), \end{aligned} \tag{2.5}$$

then we have

$$\mathrm{E}(t)=\mathrm{E}\left(u, u_t\right)(t),$$

and testing (1.1) by u_{t} we have $E(t)$ is nonincreasing, i.e.,

$$\frac{d}{dt}\mathrm{E(t)}=-\omega\left\|\nabla u_t(t)\right\|_2^2-\mu\left\|u_t(t)\right\|_2^2 \leq 0, \tag{2.6}$$

and

$$\mathrm{E}(t)+\int_0^t\left(\omega\left\|\nabla u_t(s)\right\|_2^2+\mu\left\|u_t(t)\right\|_2^2\right) \mathrm{d}s \leq \mathrm{E}(0). \tag{2.7}$$

3. Blow-up and bounds of blow-up time

In this section, we establish new bounds for the blow-up time of problem(1.1), considering two sets of given data that satisfy specific conditions related to the variable exponent $p(.)$ and the initial data u_0. Before stating our main results, without proof, we preferably give the following theorem of existence and uniqueness, as well as the regularity:

Definition 3 *A function $u(x,t)$ is said to be a weak solution of problem (1.1) defined on the time interval $[0,T]$, provide that $u(x,t) \in C^0\left([0,T], H_0^1(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right) \cap C^2\left([0,T], H^{-1}(\Omega)\right)$, with $u_t \in L^2\left([0,T], H_0^1(\Omega)\right)$ whenever $\omega>0$, if for every test-function $\eta \in H_0^1(\Omega)$ and a.e. $t \in [0,T]$, the following identity holds:*

$$\begin{aligned}
\langle u_{tt}(t), \eta\rangle+\int_{\Omega} \nabla u(t).\nabla \eta \mathrm{d}x+\omega \int_{\Omega} \nabla u_t(t).\nabla \eta \mathrm{d}x \\
+\mu \int_{\Omega} u_t(t) \eta \mathrm{d}x=\int_{\Omega}|u(t)|^{p(x)-2} u(t) \eta \mathrm{d}x.
\end{aligned} \tag{3.1}$$

Let us denote by $T_{\max}>0$ is the maximal existence time of $u(t)$:

$$T_{\max}=\sup\left\{T>0: u\left(., t\right) \text{ exist on } [0,T]\right\}<\infty.$$

We present the local existence of a solution to (1.1) without proof, which can be obtained using the Faedo-Galerkin methods in conjunction with the fixed point theorem in Banach spaces.

Theorem 4 *Assume that (1.4)-(1.6) hold. Then the problem (1.1) for given $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ admits a unique local solution*

$$u \in C\left([0, T_{\max}); H_0^1(\Omega)\right), \ u_t \in C\left([0, T_{\max}); L^2(\Omega)\right) \cap L^2\left([0, T_{\max}); H_0^1(\Omega)\right).$$

The proof of the main result necessitates the following lemma.

Establishing the following lemma is straightforward, so we will omit its proof here.

Lemma 5 *Allow (1.2)-(1.6) to apply. Let $u(t):=u(x,t)$ be a local solution to problem (1.1). Then the following assertions hold:*

(i) *If there is a time $t_0 \in [0, T_{\max})$ such that $u\left(t_0\right) \in \mathcal{W}$ and $\mathrm{E}\left(t_0\right)<d$, then $u(t)$ stays within the set $\mathcal{W}$ for all $t \in [t_0, T_{\max})$.*

(ii) *If there is a time $t_0 \in [0, T_{\max})$ such that $u\left(t_0\right) \in \mathcal{U}$ and $\mathrm{E}\left(t_0\right)<d$, then $u(t)$ stays within the set $\mathcal{U}$ for all $t \in [t_0, T_{\max})$.*

3.1. Blow-up under the first condition

Now, we will present our main theorems.

Let

$$
\begin{aligned}
c_1 &= \bar{B}^2\left(1+|\Omega|^{\frac{2(p_1-n(p_1-2))}{np_1}}\right),\\
c_2 &= 2^{q-1}\frac{p_2+p_1-2}{p_2-1}(p_2+2)^q c_1, \quad q=q_i,\ i=1,2,\\
c_3 &= 2^{q-1}\frac{p_2+p_1-2}{p_2-1}(2\mathrm{E}(0)+|\Omega|)^q c_1+\frac{p_2-p_1}{p_2-1}|\Omega|+2\mathrm{E}(0),
\end{aligned}
$$

and

$$
\begin{aligned}
q_1 &= \frac{3p_2-4}{p_2}, \quad 2<p_1\leq p(x)\leq p_2\leq\frac{2n}{n-1},\ n\geq 2,\\
q_2 &= p_2-1,\ \frac{2n}{n-1}<p_1\leq p(x)\leq p_2\leq\frac{2n-2}{n-2},\ n>3.
\end{aligned}
$$

and for any $\alpha: 1<\alpha<2$,

$$
\begin{aligned}
c_5 &= 2^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}}2^{\frac{\alpha}{\alpha-1}}\alpha^{-\frac{1}{\alpha-1}}\frac{\alpha-1}{\alpha}p_2^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}},\\
c_6 &= 2^{\frac{\alpha}{\alpha-1}}\alpha^{-\frac{1}{\alpha-1}}\frac{\alpha-1}{\alpha}c_*^{\frac{(p_2-1)\alpha}{\alpha-1}}2^{\frac{(p_2-1)\alpha}{\alpha-1}},\\
c_9 &= 2^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}}2^{\frac{\alpha}{\alpha-1}}\alpha^{-\frac{1}{\alpha-1}}\frac{\alpha-1}{\alpha}|\Omega|^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}}+2^{\frac{2}{2-\alpha}}c_*^{\frac{2\alpha}{2-\alpha}}\left(\frac{\alpha}{2\omega}\right)^{\frac{\alpha}{2-\alpha}}\frac{2-\alpha}{2},
\end{aligned}
$$

are some positive constants that will appear later in the proof.

Let us start with a simple proposition which defines the maximal solution and existence time

Proposition 6 *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega$. Assume that problem (1.1) possesses for each $u_0\in H_0^1(\Omega)$ a unique (classical $H_0^1(\Omega)$-) solution u on the interval $[0,T]$, where $T=T(u_0)>0$. Then there exists $T_{\max}=T_{\max}(u_0)\in(T,\infty]$ with the following properties.*

i *The solution u can be continued (in a unique way) to a classical $H_0^1(\Omega)$-solution on the interval $[0,T_{\max})$.*

ii *If $T_{\max}<\infty$, then u cannot be continued to a classical $H_0^1(\Omega)$-solution on $[0,\tau)$ for any $\tau>T_{\max}$. We call u the maximal (classical $H_0^1(\Omega)$-) solution starting from u_0 and $T_{\max}$ its maximal existence time.*

iii *Assume further that T can be chosen uniform for u_0 in bounded sets of $H_0^1(\Omega)$. Then*

$$\text{either } T_{\max}=+\infty \text{ or } \lim_{t\to T_{\max}}\|u(t)\|_{H_0^1(\Omega)}=+\infty.$$

Definition 7 *[?, Definition 16.3b.]Assume that problem (1.1) is well posed in $H_0^1(\Omega)$ and that $u_0\in H_0^1(\Omega)$. We say that the solution of (1.1) is global if $T_{\max}=\infty$. We say that blow-up occurs for problem (1.1) (or, more precisely, finite time blow-up in the $H_0^1(\Omega)$-norm) if*

$$T_{\max}<\infty \quad \text{and} \lim_{t\to T_{\max}}\|u(t)\|_{H_0^1(\Omega)}=\infty.$$

Note that, under the assumptions of proposition 6(iii), blow-up is equivalent to the condition $T_{\max}<\infty$.

Our blow-up result for the first condition is stated as follows

Theorem 8 *Assume (1.2)-(1.6) apply. Suppose further that $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ with $u_0\in\mathcal{U}$, $\int_\Omega u_0u_1dx>0$ and $\mathrm{E}(0)<\mathrm{E}_d$, where $\mathrm{E}(0)$ is given by (2.4). Then the solution $u(t)$ to (1.1) blows up in finite time in the measure $\varphi(t)$ and the following estimates hold:*

(i)

$$T_{\max} \geq \int_{\varphi(0)}^{\infty} \frac{\mathrm{d}\eta}{c_2\eta^q + 2\eta + c_3}, \text{ where } \varphi(0) = \int_\Omega \frac{1}{p(x)} |u_0|^{p(x)} \, \mathrm{d}x,$$

and

$$2 < p_1 \leq p(x) \leq p_2 \leq \frac{2n-2}{n-2}, n > 3.$$

(ii)

$$T_{\max} \geq \int_{\varphi(0)}^{\infty} \frac{\mathrm{d}\eta}{c_5\eta^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}} + c_6\eta^{\frac{(p_2-1)\alpha}{2(\alpha-1)}} + c_9},$$

where

$$\varphi(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 + \int_\Omega \frac{1}{p(x)} |u_0|^{p(x)} \, \mathrm{d}x,$$

and

$$2 < p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}, \; n \geq 3.$$

Remark 9 *It is noticed that if $p(.)$ is a constant function, then $\mathrm{E}_d = d$ and this theorem becomes more general of [9, Theorem 2.1.][11, Theorem 2.1.] [10, Theorem 1.2.].*

3.2. Proof of Theorem 8

The following two lemmas are essential for proving the first main statement.

Lemma 10 *[4]Suppose that a positive, twice-differentiable function $\varphi(t)$ satisfies on $t \geq 0$ the inequality*

$$\varphi''\varphi - (1+\alpha)(\varphi')^2 \geq 0, \; \alpha > 0.$$

If

$$\varphi(0) > 0, \; \text{and } \varphi'(0) > 0,$$

then, then there exists $t_1 \in \left(0, \frac{\varphi(0)}{\alpha\varphi'(0)}\right)$ such that

$$\varphi(t) \to \infty \text{ as } t \to t_1.$$

Lemma 11 *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with $u_0 \in \mathcal{U}$. Then we have*

$$d \leq \left(\frac{1}{2} - \frac{1}{p_2}\right) \|\nabla u(t)\|_2^2, \quad \text{for } t \in [0, T_{\max}).$$

Beweis. Because of $u_0 \in \mathcal{U}$, according to Lemma 5 $u(t) \in \mathcal{U}$ for $t \in [0, T_{\mathrm{maz}})$ and thus $I(u(t)) < 0$. By Lemma 2 there exists $\lambda^* \in (0,1)$ such that $I(\lambda^* u) = 0$, i.e.

$$\int_\Omega (\lambda^*)^{P(x)} |u(t)|^{P(x)} \mathrm{d}x = (\lambda^*)^2 \|\nabla u(t)\|_2^2.$$

Thanks to $\lambda^* < 1$ we derive from the definition of d that

$$\begin{aligned}
d \leq J(\lambda^* u(t)) &= \frac{(\lambda^*)^2}{2}\|\nabla u(t)\|_2^2 - \int_\Omega \frac{(\lambda^*)^{p(x)}}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \\
&\leq \frac{(\lambda^*)^2}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p_2}\int_\Omega (\lambda^*)^{p(x)} |u(t)|^{p(x)}\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{p_2}\right)(\lambda^*)^2 \|\nabla u(t)\|_2^2 + \frac{1}{p_2} I(\lambda^* u(t)) \\
&\leq \left(\frac{1}{2} - \frac{1}{p_2}\right)\|\nabla u(t)\|_2^2.
\end{aligned}$$

This finishes the proof. ■

Proof of Theorem 8. By contradiction we suppose that $u(t)$ exists globally and define the function

$$\begin{aligned}\theta(t) =&\|u(t)\|_2^2+\omega\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s\\&+\mu\int_0^t\|u(s)\|_2^2\mathrm{d}s+\mu(T-t)\|u_0\|_2^2+\omega(T-t)\|\nabla u_0\|_2^2,\ t\in[0,T].\end{aligned}\tag{3.2}$$

Then we have

$$\begin{aligned}\theta'(t)=&2\int_\Omega u(t)u_t(t)\mathrm{d}x+\omega\|\nabla u(t)\|_2^2-\omega\|\nabla u_0\|_2^2+\mu\|u(s)\|_2^2-\mu\|u_0\|_2^2\\=&2\int_\Omega u(t)u_t(t)\mathrm{d}x+\omega\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\|\nabla u(s)\|_2^2\mathrm{d}s+\mu\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|_2^2\mathrm{d}s\\=&2\int_\Omega u(t)u_t(t)\mathrm{d}x+2\omega\int_0^t\int_\Omega\nabla u_t(s)\nabla u(s)\mathrm{d}x\mathrm{d}s+2\mu\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s,\end{aligned}\tag{3.3}$$

and

$$\theta''(t)=2\int_\Omega u(t)u_{tt}(t)\mathrm{d}x+2\|u_t(t)\|^2+2\omega\int_\Omega\nabla u_t(t)\nabla u(t)\mathrm{d}x+2\mu\int_\Omega u_t(t)u(t)\mathrm{d}x.\tag{3.4}$$

By using (1.1), we deduce from (3.4) that

$$\theta''(t)=2\|u_t(t)\|_2^2-2\|\nabla u(t)\|_2^2+2\int_\Omega|u(t)|^{p(x)}\mathrm{d}x.\tag{3.5}$$

It follows from (3.2), (3.3) and (3.5) that

$$\begin{aligned}\theta''(t)\theta(t)-&\frac{p_1+2}{4}(\theta'(t))^2\\&=2\theta(t)\left[\|u_t(t)\|_2^2-\|\nabla u(t)\|_2^2+\int_\Omega|u(t)|^{p(x)}\mathrm{d}x\right]\\&-\frac{p_1+2}{4}\left(2\int_\Omega u(t)u_t(t)\mathrm{d}x+2\omega\int_0^t\int_\Omega\nabla u_t(s)\nabla u(s)\mathrm{d}x\mathrm{d}s\right.\\&\left.+2\mu\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)^2\\&=2\theta(t)\left[\|u_t(t)\|_2^2-\|\nabla u(t)\|_2^2+\int_\Omega|u(t)|^{p(x)}\mathrm{d}x\right]\\&+(p_1+2)\left[\eta(t)-\left(\theta(t)-\omega(T-t)\|\nabla u_0\|^2-\mu(T-t)\|u_0\|_2^2\right)\right.\\&\left.\left(\|u_t(t)\|_2^2+\omega\int_0^t\|\nabla u_t(s)\|_2^2\mathrm{d}s+\mu\int_0^t\|u_t(s)\|_2^2\mathrm{d}s\right)\right],\end{aligned}\tag{3.6}$$

where $\eta:[0,T]\to\mathbb{R}$ is the function given by

$$\begin{aligned}\eta(t)=&\left(\|u(t)\|_2^2+\omega\int_0^t\|\nabla u(s)\|_2^2\mathrm{d}s+\mu\int_0^t\|u(s)\|_2^2\mathrm{d}s\right)\left(\|u_t(t)\|_2^2+\omega\int_0^t\|\nabla u_t(s)\|_2^2\mathrm{d}s\right.\\&\left.+\mu\int_0^t\|u_t(s)\|_2^2\mathrm{d}s\right)\\&-\left(\int_\Omega u(t)u_t(t)\mathrm{d}x+\omega\int_0^t\int_\Omega\nabla u_t(s)\nabla u(s)\mathrm{d}x\mathrm{d}s+\mu\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)^2.\end{aligned}\tag{3.7}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left(\int_\Omega u(t)u_t(t)\mathrm{d}x\right)^2 &\le \|u(t)\|_2^2\,\|u_t(t)\|_2^2,\\ \left(\omega\int_0^t\int_\Omega \nabla u_t(s)\nabla u(s)\mathrm{d}x\mathrm{d}s\right)^2 &\le \omega^2\int_0^t \|\nabla u(s)\|_2^2\mathrm{d}s\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s,\\ \left(\mu\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)^2 &\le \mu^2\int_0^t \|u(s)\|_2^2\mathrm{d}s\int_0^t \|u_t(s)\|_2^2\,\mathrm{d}s,\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}&\omega\int_\Omega u(t)u_t(t)\mathrm{d}x\left(\int_0^t\int_\Omega \nabla u_t(s)\nabla u(s)\mathrm{d}x\mathrm{d}s\right)\\ &\le \omega\|u(t)\|_2\left(\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s\right)^{\frac12}\|u_t(t)\|_2\left(\int_0^t \|\nabla u(s)\|_2^2\mathrm{d}s\right)^{\frac12}\\ &\le \frac12\omega\|u(t)\|_2^2\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s+\frac12\omega\,\|u_t(t)\|_2^2\int_0^t \|\nabla u(s)\|_2^2\mathrm{d}s,\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}&\mu\int_\Omega u(t)u_t(t)\mathrm{d}x\left(\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)\\ &\le \mu\|u(t)\|_2\left(\int_0^t \|u_t(s)\|_2^2\,\mathrm{d}s\right)^{\frac12}\|u_t(t)\|_2\left(\int_0^t \|u(s)\|_2^2\mathrm{d}s\right)^{\frac12}\\ &\le \frac12\mu\|u(t)\|_2^2\int_0^t \|u_t(s)\|_2^2\,\mathrm{d}s+\frac12\mu\,\|u_t(t)\|_2^2\int_0^t \|u(s)\|_2^2\mathrm{d}s,\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}&\left(\omega\int_0^t\int_\Omega \nabla u_t(s)\nabla u(s)\mathrm{d}x\mathrm{d}s\right)\left(\mu\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)\\ &\le \mu\omega\left(\int_0^t \|u_t(s)\|_2^2\,\mathrm{d}s\right)^{\frac12}\left(\int_0^t \|u(s)\|_2^2\mathrm{d}s\right)^{\frac12}\left(\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s\right)^{\frac12}\left(\int_0^t \|\nabla u(s)\|_2^2\mathrm{d}s\right)^{\frac12}\\ &\le \frac12\mu\omega\int_0^t \|u(s)\|_2^2\mathrm{d}s\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s+\frac12\mu\omega\int_0^t \|u_t(s)\|_2^2\,\mathrm{d}s\int_0^t \|\nabla u(s)\|_2^2\mathrm{d}s.\end{aligned}\tag{3.11}$$

By (3.7)-(3.9), we obtain

$$\eta(t)\ge 0,\quad \forall t\in[0,T].\tag{3.12}$$

From (3.6) and (3.12) we find that

$$\theta''(t)\theta(t)-\frac{p_1+2}{4}\left(\theta'(t)\right)^2\ge\theta(t)\zeta(t),\tag{3.13}$$

where $\zeta(t)$ is given by

$$\begin{aligned}\zeta(t)=&-p_1\,\|u_t(t)\|_2^2-2\|\nabla u(t)\|_2^2+2\int_\Omega |u(t)|^{p(x)}\mathrm{d}x\\ &-(p_1+2)\,\omega\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s-(p_1+2)\,\mu\int_0^t \|u_t(s)\|_2^2\mathrm{d}s.\end{aligned}\tag{3.14}$$

Next we estimate $\zeta(t)$, (3.14) yields

$$\begin{aligned}\zeta(t) \geq & -p_1 \|u_t(t)\|_2^2 - 2\|\nabla u(t)\|_2^2 + 2\int_\Omega |u(t)|^{p(x)}\mathrm{d}x \\ & -(p_1+2)\,\omega\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s - (p_1+2)\,\mu\int_0^t \|u_{\underset{\cdot\cdot}{t}}(s)\|_2^2\mathrm{d}s \\ \geq & -2p_1\mathrm{E}(t) + p_1\|\nabla u(t)\|_2^2 - 2p_1\int_\Omega \frac{1}{p(x)}|u(t)|^{p(z)}\mathrm{d}x \\ & -2\|\nabla u(t)\|_2^2 + 2\int_\Omega |u(t)|^{p(x)}\mathrm{d}x - (p_1+2)\,\omega\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s \\ & -(p_1+2)\,\mu\int_0^t \|u_{\underset{\cdot\cdot}{t}}(s)\|_2^2\mathrm{d}s.\end{aligned} \tag{3.15}$$

From (2.7) we have that

$$\begin{aligned}\zeta(t) \geq & -2p_1\mathrm{E}(0) + 2p_1\int_0^t \left(\omega\,\|\nabla u_t(s)\|_2^2 + \mu\,\|u_t(t)\|_2^2\right)\mathrm{d}s \\ & -(p_1+2)\,\omega\int_0^t \|\nabla u_t(s)\|_2^2\,\mathrm{d}s - (p_1+2)\,\mu\int_0^t \|u_{\underset{\cdot\cdot}{t}}(s)\|_2^2\mathrm{d}s \\ & +2\int_\Omega\left(1-\frac{p_1}{p(x)}\right)|u(t)|^{p(x)}\mathrm{d}x + (p_1-2)\,\|\nabla u(t)\|_2^2 \\ \geq & \, 2p_1\left[\left(\frac{1}{2}-\frac{1}{p_1}\right)\|\nabla u(t)\|_2^2 - \mathrm{E}(0)\right] \\ & +(p_1-2)\int_0^t\left(\omega\,\|\nabla u_t(s)\|_2^2 + \mu\,\|u_t(t)\|_2^2\right)\mathrm{d}s + 2\int_\Omega\left(1-\frac{p_1}{p(x)}\right)|u(t)|^{p(x)}\mathrm{d}x.\end{aligned} \tag{3.16}$$

On the other hand, since $u_0 \in \mathcal{U}$, by virtue of Lemma 11 we have

$$d \leq \left(\frac{1}{2}-\frac{1}{p_2}\right)\|\nabla u(t)\|_2^2. \tag{3.17}$$

And by assuming $\mathrm{E}(0) < \mathrm{E}_d$ we get

$$\mathrm{E}(0) < \frac{\frac{1}{2}-\frac{1}{p_1}}{\frac{1}{2}-\frac{1}{p_2}}d \leq \left(\frac{1}{2}-\frac{1}{p_1}\right)\|\nabla u(t)\|_2^2. \tag{3.18}$$

If we combine (3.16) and (3.18) we get

$$\zeta(t) > \rho > 0. \tag{3.19}$$

From (3.13) and (3.19), we arrive at

$$\theta''(t)\theta(t) - \frac{p_1+2}{4}\left(\theta'(t)\right)^2 \geq \rho\theta(t). \tag{3.20}$$

From the continuity of θ and (3.17) we imply that there is a positive constant c such that $\theta(t) \geq c$ for $t \in [0,T]$. Thus, (3.20) yields

$$\theta''(t)\theta(t) - \frac{p_1+2}{4}\left(\theta'(t)\right)^2 \geq c\rho. \tag{3.21}$$

Now, in this case we show that T cannot be infinite, and therefore there is no weak solution all the time. From Lemma 10, it follows that there exists a $0 < t_1 < +\infty$ such that $\theta(t) \to \infty$ as $t \to t_1$, where

$$0 < t_1 < \frac{\|u_0\|_2^2 + \mu T\,\|u_0\|_2^2 + \omega T\,\|\nabla u_0\|_2^2}{(p_1-2)\int_\Omega u_0u_1\mathrm{d}x} < +\infty,$$

that there is $T^* < t_1$ such that

$$\lim_{t\to T^*} \|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 \mathrm{d}s + \int_0^t \|u(s)\|_2^2 \mathrm{d}s = +\infty.$$

This contradicts to our assumption. With this the proof is complete. ■

3.3. Lower bound of the blow-up time

1. $2 < p_1 \leq p(x) \leq p_2 \leq \frac{2n-2}{n-2}$, $n > 3$.

   Let

$$\varphi(t) = \int_\Omega \frac{1}{p(x)} |u(t)|^{p(x)} \mathrm{d}x,$$

   Then, applying Young's inequality

$$\begin{aligned}
\varphi'(t) = \int_\Omega |u(t)|^{p(x)-2} u u_t \mathrm{d}x &\leq \int_\Omega |u(t)|^{2p_1-2} \mathrm{d}x + \int_\Omega |u(t)|^{2p_2-2} \mathrm{d}x + \int_\Omega |u_t(t)|^2 \mathrm{d}x \\
&\leq \frac{p_2 - p_1}{p_2 - 1} |\Omega| + \frac{p_1 - 1}{p_2 - 1} \int_\Omega |u(t)|^{2p_2-2} \mathrm{d}x \\
&+ \int_\Omega |u(t)|^{2p_2-2} \mathrm{d}x + \int_\Omega |u_t(t)|^2 \mathrm{d}x \\
&= \frac{p_2 - p_1}{p_2 - 1} |\Omega| + \frac{p_2 + p_1 - 2}{p_2 - 1} \int_\Omega |u(t)|^{2p_2-2} \mathrm{d}x + \int_\Omega |u_t(t)|^2 \mathrm{d}x
\end{aligned} \tag{3.22}$$

   Let u be the solution of (1.1). Then,

$$\begin{aligned}
\int_\Omega |u(t)|^{p(x)} \mathrm{d}x &= \int_{\Omega_1} |u(t)|^{p(x)} \mathrm{d}x + \int_{\Omega_2} |u(t)|^{p(x)} \mathrm{d}x \\
&\geq \int_{\Omega_2} |u(t)|^{p_1} \mathrm{d}x + \int_{\Omega_1} |u(t)|^{p_2} \mathrm{d}x \geq \int_{\Omega_2} |u(t)|^{p_1} \mathrm{d}x \\
&= \int_\Omega |u(t)|^{p_1} \mathrm{d}x - \int_{\Omega_1} |u(t)|^{p_2} \mathrm{d}x \geq \|u(t)\|_{p_1}^{p_1} - |\Omega|,
\end{aligned} \tag{3.23}$$

   where

$$\Omega_1 = \{x \in \Omega / |u(x,t)| < 1\},\ \Omega_2 = \{x \in \Omega / |u(x,t)| \geq 1\}.$$

   We distinguish two cases to estimate the second term on the right side of the inequality (3.22) is as follows:

   Case $1: 2 < p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-1}, \quad n \geq 2.$

   Let $\alpha = 2p_2 - 2$, $\mu = n(p_1 - 2)$ and applying Hölder's inequality and the embedding inequality (2.1), we have

$$\begin{aligned}
\int_\Omega |u(t)|^\alpha \mathrm{d}x &= \int_\Omega |u(t)|^{\alpha\theta} |u(t)|^{\alpha(1-\theta)} \mathrm{d}x \\
&\leq \left(\int_\Omega |u(t)|^\mu \mathrm{d}x\right)^{\frac{\alpha\theta}{\mu}} \left(\int_\Omega |u(t)|^{\frac{2n}{n-2}} \mathrm{d}x\right)^{\frac{\alpha(1-\theta)(n-2)}{2n}},
\end{aligned}$$

   for $\theta = \frac{\mu\left(\frac{2n}{n-2} - \alpha\right)}{\alpha\left(\frac{2n}{n-2} - \mu\right)}$ what that satisfies

$$\frac{\alpha\theta}{\mu} + \frac{\alpha(1-\theta)(n-2)}{2n} = 1.$$

A simple calculation shows

$$\theta=\frac{1-\frac{\alpha(n-2)}{2n}}{\frac{\alpha}{\mu}-\frac{\alpha(n-2)}{2n}}=\frac{\mu\left(\frac{2n}{n-2}-\alpha\right)}{\alpha\left(\frac{2n}{n-2}-\mu\right)},$$
$$\frac{\theta\alpha}{\mu}=\frac{\frac{2n}{n-2}-\alpha}{\frac{2n}{n-2}-\mu}=\frac{2}{n},\quad \alpha\left(1-\theta\right)\frac{n-2}{2n}=1-\frac{2}{n},$$

and then we have

$$\begin{aligned}\|u\left(t\right)\|_{\alpha}^{\alpha} &\leq \|u\left(t\right)\|_{\mu}^{\alpha\theta}\|u\left(t\right)\|_{\frac{2n}{n-2}}^{\alpha(1-\theta)}=\|u\left(t\right)\|_{\mu}^{\frac{2\mu}{n}}\|u\left(t\right)\|_{\frac{2n}{n-2}}^{2}\\ &\leq \bar{B}^{2}\left(1+|\Omega|^{\frac{2(p_1-\mu)}{np_1}}\right)\|u\left(t\right)\|_{p_1}^{\frac{2\mu}{n}}\|\nabla u\left(t\right)\|_2^2\\ &\leq c_1\left(\|u\left(t\right)\|_{p_1}^{p_1}+\|\nabla u\left(t\right)\|_2^2\right)^{q_1},\\ &\leq c_1\left(\int_\Omega|u\left(t\right)|^{p(x)}\mathrm{d}x+|\Omega|+\|\nabla u\left(t\right)\|_2^2\right)^{q_1},\end{aligned}\tag{3.24}$$

with $q_1=\frac{3p_1-4}{p_1}$.

Case 2. $\frac{2n}{n-1}<p_1\leq p\left(x\right)\leq p_2\leq\frac{2n-2}{n-2},\quad n>3$. According to the lines of the inequality in proof (3.24), we deduce

$$\|u\left(t\right)\|_{\alpha}^{\alpha}\leq\max\left(c_1^{2p_1-3},c_1^{2p_2-3}\right)\left(\int_\Omega|u\left(t\right)|^{p(x)}\mathrm{d}x+|\Omega|+\|\nabla u\left(t\right)\|_2^2\right)^{q_2},\tag{3.25}$$

with $q_2=p_1-1$. Referring to (2.5) and definitions $E(t)$, we have

$$\|\nabla u\left(t\right)\|_2^2+\|u_t\left(t\right)\|_2^2\leq 2\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x+2\mathrm{E}(0)=2\varphi(t)+2\mathrm{E}(0).\tag{3.26}$$

If we combine (3.22)-(3.26), we get

$$\begin{aligned}\varphi'(t)\leq&\frac{p_2-p_1}{p_2-1}|\Omega|+\frac{p_2+p_1-2}{p_2-1}\int_\Omega|u\left(t\right)|^{2p_2-2}\mathrm{d}x+\int_\Omega|u_t\left(t\right)|^2\mathrm{d}x\\ \leq&\frac{p_2+p_1-2}{p_2-1}\left(\left(p_2+2\right)\varphi(t)+2\mathrm{E}(0)+|\Omega|\right)^q+\frac{p_2-p_1}{p_2-1}|\Omega|+2\varphi(t)+2\mathrm{E}(0)\end{aligned}$$

Thus

$$\begin{aligned}\varphi'(t)\leq&2^{q-1}\frac{p_2+p_1-2}{p_2-1}c_1\left(p_2+2\right)^q\varphi(t)^q+2\varphi(t)\\ &+2^{q-1}\frac{p_2+p_1-2}{p_2-1}c_1\left(2\mathrm{E}(0)+|\Omega|\right)^q+\frac{p_2-p_1}{p_2-1}|\Omega|+2\mathrm{E}(0),\end{aligned}\tag{3.27}$$

Therefore

$$\varphi'(t)\leq c_2\varphi(t)^q+2\varphi(t)+c_3,$$

Applying the fact that

$$\lim_{t\to T^*}\int_\Omega|u\left(t\right)|^{p_2}\mathrm{d}x=+\infty.\tag{3.28}$$

According to (3.27) and (3.28), we obtain

$$\int_{\varphi(0)}^{\infty}\frac{\mathrm{d}\eta}{c_2\eta^q+2\eta+c_3}\leq T^*.$$

Thus, we obtain a lower bound for the blow-up time.

2. $2 < p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}$, $n > 3$

Define

$$\varphi(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\,\mathrm{d}x.$$

Then,

$$\varphi'(t) = \int_\Omega \nabla u(t)\nabla u_t(t)\mathrm{d}x + \int_\Omega u_t(t)u_{tt}(t)\mathrm{d}x + \int_\Omega |u(t)|^{p(x)-2}u(t)u_t(t)\mathrm{d}x. \tag{3.29}$$

Multiply Eq. (1.1) by u_t and integrating over Ω, we have

$$\begin{aligned}\int_\Omega u_t(t)u_{tt}(t)\mathrm{d}x &= \int_\Omega u_t(t)\Delta u(t)\mathrm{d}x + \omega\int_\Omega u_t(t)\Delta u_t(t)\mathrm{d}x - \mu\int_\Omega |u_t(t)|^2\,\mathrm{d}x \\ &\quad + \int_\Omega |u(t)|^{p(x)-2}u(t)u_t(t)\mathrm{d}x \\ &= -\int_\Omega \nabla u(t).\nabla u_t(t)\mathrm{d}x - \omega\int_\Omega |\nabla u_t(t)|^2\,\mathrm{d}x - \mu\int_\Omega |u_t(t)|^2\,\mathrm{d}x \\ &\quad + \int_\Omega |u(t)|^{p(x)-2}u(t)u_t(t)\mathrm{d}x.\end{aligned} \tag{3.30}$$

Combining (3.30) with (3.29) gives

$$\varphi'(t) \leq -\omega\int_\Omega |\nabla u_t(t)|^2\,\mathrm{d}x + \int_\Omega |u(t)|^{p(x)-2}u(t)u_t(t)\mathrm{d}x. \tag{3.31}$$

Making use of (3.23), from the Hölder and Sobolev inequalities and the fact that

$$\max\left(\|u(t)\|_{p_1}, \|u(t)\|_{p_2}\right) \leq c_*\|\nabla u(t)\|_2,$$

from $1 < \alpha < 2$, the second term on the right-hand side of (3.31) has been estimated as follows

$$\begin{aligned}2\int_\Omega |u(t)|^{p(x)-1}|u_t(t)|\,\mathrm{d}x \leq& 2\int_\Omega \left(|u(t)|^{p_1-1} + |u(t)|^{p_2-1}\right)|u_t(t)|\,\mathrm{d}x \\ &\leq c_4\|u(t)\|_{p_1}^{\frac{(p_1-1)\alpha}{\alpha-1}} + c_4\|u(t)\|_{p_2}^{\frac{(p_2-1)\alpha}{\alpha-1}} + \|u_t(t)\|_{p_2}^\alpha + \|u_t(t)\|_{p_1}^\alpha \\ &\leq c_4\left(|\Omega| + \int_\Omega |u(t)|^{p(x)}\mathrm{d}x\right)^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}} + c_4c_*^{\frac{(p_2-1)\alpha}{\alpha-1}}\|\nabla u(t)\|_2^{\frac{(p_2-1)\alpha}{\alpha-1}} \\ &\quad + \|u_t(t)\|_{p_2}^\alpha + \|u_t(t)\|_{p_1}^\alpha \\ &\leq c_5\varphi(t)^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}} +_6 \varphi(t)^{\frac{(p_2-1)\alpha}{2(\alpha-1)}} + \|u_t\|_{p_2}^\alpha + \|u_t\|_{p_1}^\alpha + c_7,\end{aligned} \tag{3.32}$$

where $c_4 = 2^{\frac{\alpha}{\alpha-1}}\alpha^{-\frac{1}{\alpha-1}}\frac{\alpha-1}{\alpha}$. Also the Sobolev inequality applied in the second term on the right side of (3.32) to get

$$\|u_t(t)\|_{p_2}^\alpha + \|u_t(t)\|_{p_1}^\alpha \leq 2c_*^\alpha\|\nabla u_t(t)\|_2^\alpha,$$

where $c_* = c(\Omega, n)$ is the optimal constant. Using Young inequality, we have

$$\|u_t(t)\|_{p_2}^\alpha + \|u_t(t)\|_{p_1}^\alpha \leq \omega\|\nabla u_t(t)\|_2^2 + c_8.$$

Inserting this inequality in (3.32) yields

$$2\int_\Omega |u(t)|^{p(x)-1}|u_t(t)|\,\mathrm{d}x \leq \omega\|\nabla u_t(t)\|_2^2 + c_5\varphi(t)^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}} + c_6\varphi(t)^{\frac{(p_2-1)\alpha}{2(\alpha-1)}} + c_7 + c_8.$$

This inequality together with (3.31) implies that

$$\varphi'(t) \leq c_5\varphi(t)^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}} + c_6\varphi(t)^{\frac{(p_2-1)\alpha}{2(\alpha-1)}} + c_9, \tag{3.33}$$

where $c_9 = c_7 + c_8$. Then,

$$\frac{\mathrm{d}\varphi}{c_5\varphi(t)^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}} + c_6\varphi(t)^{\frac{(p_2-1)\alpha}{2(\alpha-1)}} + c_9} \le \mathrm{d}t.$$

Taking into account the fact that

$$\lim_{t\to T^*} \int_\Omega |u(t)|^{p(x)}\mathrm{d}x = +\infty. \tag{3.34}$$

And then, from (3.33) and (3.34), we obtain

$$\int_{\varphi(0)}^{\infty} \frac{\mathrm{d}\eta}{c_5\eta^{\frac{(p_1-1)\alpha}{p_1(\alpha-1)}} + c_6\eta^{\frac{(p_2-1)\alpha}{2(\alpha-1)}} + c_9} \le T^*.$$

3.4. Blow-up under the second condition

For our result, we need to consider the following functions

$$\alpha(t) = \|\nabla u(t)\|_2, \tag{3.35}$$

and for ε (positive small) and N is precise positive constants to be picked later,

$$\mathrm{A}(t) := H^{1-\beta}(t) + \varepsilon \int_\Omega u(x,t)u_t(x,t)\mathrm{d}x + \frac{1}{2}\varepsilon\omega \int_\Omega |\nabla u(x,t)|^2\mathrm{d}x + \varepsilon N\mathrm{E}_1 t, \quad t \in [0,T), \tag{3.36}$$

and

$$\varphi(t) = \frac{1}{p_2}\int_\Omega |u(t)|^{p_2}\mathrm{d}x. \tag{3.37}$$

Let B, α_1, α_0, c_* and E_1 be positive auxiliary constants satisfying

$$\begin{gathered} c_* = \max\left((2B)^{p_1}, (2B)^{p_2}\right),\ B = c_*^{\frac{-1}{p_2}} B_1,\ \alpha_1 = \left(\frac{p_1}{p_2} B_1^{-P_2}\right)^{\frac{1}{P_2-2}} \\ \alpha(0) = \alpha_0 = \|\nabla u_0\|_2, \quad \mathrm{E}_1 = \left(\frac{1}{2} - \frac{1}{p_2}\right)\alpha_1^2. \end{gathered} \tag{3.38}$$

Our blow-up result for the second condition is stated as follows

Theorem 12 *Assuming that $p(.)$ satisfy conditions (1.4)-(1.6). Then the local solution of problem (1.1) under boundary conditions satisfying $E(0) < E_1$, $\|\nabla u_0\| > \alpha_1$ blows up in finite time T^*, which provide the following estimates*

$$\int_{\varphi(0)}^{+\infty} \frac{\mathrm{d}\eta}{\mathrm{c}_7\left(\eta^\delta + \eta^{\delta\frac{p_1}{p_2}} + \eta + \eta^{\frac{p_1}{p_2}} + 1\right)} \le T^* \le \frac{1-\beta}{\beta\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\beta}{1-\beta}}(0)},$$

where

$$0 < \beta \le \frac{p_1 - 2}{2p_1}, \tag{3.39}$$

and δ, δ_1, δ_2 are defined in (3.71), (3.62), (3.67), respectively.

3.5. Proof of Theorem 12

The proof relies on the following lemmas.

Lemma 13 *[19]Let $h : [0, +\infty) \to \mathbb{R}$ be defined by*

$$h(t) := h(\alpha) = \frac{1}{2}\alpha^2 - \frac{B_1^{p_2}}{P_1}\alpha^{p_2}, \tag{3.40}$$

then h has the following properties:

(i) *h is increasing for $0 < \alpha \le \alpha_1$ and decreasing for $\alpha \ge \alpha_1$,*

(ii) $\lim_{\alpha \to +\infty} h(\alpha) = -\infty$ *and* $h(\alpha_1) = E_1$,

iii $E(t) \ge h(\alpha(t))$,

where $\alpha(t)$ is given in (3.35), α_1 and E_1 are given in (3.38).

Beweis. $h(\alpha)$ is continuous and differentiable in $[0, +\infty)$,

$$h'(\alpha) = \alpha\left(1 - B_1^{p_2}\alpha^{p_2-2}(t)\right)\begin{cases} > 0, & \alpha \in (0, \alpha_1) \\ < 0, & \alpha \in (\alpha_1, +\infty), \end{cases}$$

which means that

$$\begin{aligned} &h(\alpha) \text{ is strictly increasing in } (0, \alpha_1), \\ &h(\alpha) \text{ is strictly decreasing in } (\alpha_1, +\infty). \end{aligned} \tag{3.41}$$

Then **(i)** follows. Since $p_2 - 2 > 0$, we have $\lim_{\alpha \to +\infty} h(\alpha) = -\infty$. A simple computation yields to $h(\alpha_1) = E_1$. Then **(ii)** holds valid.

By Lemma 1

$$\begin{aligned} \int_\Omega |u(t)|^{p(x)} \mathrm{d}x &\le \max\left\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right\} \\ \le c_* \max\left(\left(\int_{\{\|\nabla u\|_2 \ge 1\}} |\nabla u(t)|^2 \mathrm{d}x\right)^{p_1}, \left(\int_{\{\|\nabla u\|_2 \ge 1\}} |\nabla u(t)|^2 \mathrm{d}x\right)^{p_2}\right) \\ = c_* \left(\int_{\{\|\nabla u\|_2 \ge 1\}} |\nabla u(t)|^2 \mathrm{d}x\right)^{p_2} &\le c_* \left(\int_\Omega |\nabla u(t)|^2 \mathrm{d}x\right)^{p_2}. \end{aligned} \tag{3.42}$$

Using (2.5) and Lemma 1, we have

$$\begin{aligned} \mathrm{E}(t) &\ge \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p_1}\int_\Omega |u(t)|^{p(x)} \mathrm{d}x \\ &\ge \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p_1} B^{p_2} c_* \left(\int_\Omega |\nabla u(t)|^2 \mathrm{d}x\right)^{p_2} \\ &\ge \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{B_1^{p_2}}{p_1}\|\nabla u(t)\|_2^{p_2} \\ &= \frac{1}{2}\alpha^2(t) - \frac{B_1^{p_2}}{p_1}\alpha^{p_2}(t) = h(\alpha(t)). \end{aligned}$$

Then **(iii)** holds true. ■

Lemma 14 *Assuming the conditions in Theorem 12 are fulfilled, there is a positive constant $\alpha_2 > \alpha_1$ such that*

$$\alpha(t) \ge \alpha_2 > \alpha_1, \ t \ge 0; \tag{3.43}$$

$$\varrho(u) \ge B_1^{p_2}\alpha_2^{p_2}, \tag{3.44}$$

where α_1, B_1 and E_1 are given in (3.38).

Beweis. Since $E(0) < E_1$ and $h(\alpha)$ is a continuous function, there exist α_2' and α_2 with $\alpha_2' < \alpha_1 < \alpha_2$ such that $h(\alpha_2') = h(\alpha_2) = E(0)$ which join with Lemma 13 give

$$h(\alpha_0) \le \mathrm{E}(0) = h(\alpha_2). \tag{3.45}$$

From Lemma 13**(i)**, we infer that

$$\alpha_0 \geq \alpha_2, \tag{3.46}$$

so (3.43) holds for $t = 0$.

Now we prove (3.43), we proceed by contradiction and assume there exists $t^* > 0$ such that $\alpha(t^*) < \alpha_2$, then we distinguish two cases,

Case 1. If $\alpha_2' < \alpha(t^*) < \alpha_2$, we know through Lemma 13 and (3.41) that

$$h(\alpha(t^*)) > \mathrm{E}(0) \geq \mathrm{E}(t^*),$$

which contradicts Lemma 13**(iii)**.

Case 2. If $\alpha(t^*) \leq \alpha_2'$, then $\alpha(t^*) \leq \alpha_2' < \alpha_2$. Setting $\lambda(t) = \alpha(t) - \frac{\alpha_2 + \alpha_2'}{2}$, then $\lambda(t)$ is a continuous function, $\lambda(t^*) < 0$ and by applying (3.46) $\lambda(0) > 0$. Hence, there exists $t_0 \in (0, t^*)$ such that $\lambda(t_0) = 0$, that means $\alpha(t_0) = \frac{\alpha_2 + \alpha_2'}{2}$, which signifies

$$h(\alpha(t_0)) > \mathrm{E}(0) \geq \mathrm{E}(t_0).$$

This contradicts to Lemma 13**(iii)**, hence (3.43) follows.

By (2.5), we have

$$\frac{1}{2}\|\nabla u(t)\|_2^2 \leq \mathrm{E}(t) + \frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x,$$

which give

$$\begin{aligned}\frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x \geq \frac{1}{2}\|\nabla u(t)\|_2^2 - \mathrm{E}(t) \geq& \frac{1}{2}\|\nabla u(t)\|_2^2 - \mathrm{E}(0) \\ \geq& \frac{1}{2}\alpha_2^2 - h(\alpha_2) = \frac{B_1^{p_2}}{P_1}\alpha_2^{p_2},\end{aligned}$$

then the second inequality in (3.44) holds. ■

Let

$$\mathrm{H}(t) = \mathrm{E}_1 - \mathrm{E}(t) \text{ for } t \geq 0. \tag{3.47}$$

The following lemma hold.

Lemma 15 *[7]Under the assumptions of Theorem 12, if* $0 \leq E(0) < E_1$, *the functional* $\mathrm{H}(t)$ *defined in* (3.47) *satisfies the following estimates:*

$$0 < \mathrm{H}(0) \leq \mathrm{H}(t) \leq \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \leq \frac{1}{p_1}\varrho(u),\ \ t \geq 0. \tag{3.48}$$

Beweis. Lemma 1 guarantees that $\mathrm{H}(t)$ does not decrease in t. Thus

$$\mathrm{H}(t) \geq \mathrm{H}(0) = \mathrm{E}_1 - \mathrm{E}(0) > 0, \quad t \geq 0. \tag{3.49}$$

By (3.38) and Lemma 14, we have

$$\begin{aligned}&\mathrm{E}_1 - \left[\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2\right] \\ &= \mathrm{E}_1 - \left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\alpha^2(t)\right] \\ &\leq \mathrm{E}_1 - \frac{1}{2}\alpha^2(t) \leq \mathrm{E}_1 - \frac{1}{2}\alpha_1^2 = -\frac{1}{p_2}\alpha_1^2 < 0,\end{aligned}$$

for all $t \in [0, T)$, which gives

$$\begin{aligned}\mathrm{H}(t) =& \mathrm{E}_1 - \left(\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2\right) \\ &+ \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \leq \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \leq \frac{1}{p_1}\varrho(u).\end{aligned} \tag{3.50}$$

(3.48) follows from (3.49) and (3.50). ■

Lemma 16 *[27]Assume that the conditions in Theorem 12 hold, then there exists a positive constant C such that*

$$\|\nabla u(t)\|_2^2 \leq \mathrm{c}\varrho(u\,(t)). \tag{3.51}$$

for all $t \in [0, T)$.

Beweis. By Lemma 14 and $\alpha_2 > \alpha_1$, we have

$$\varrho(u) \geq B_1 \alpha_2^{p_2} > B_1 \alpha_1^{p_2-2} \alpha_1^2 = \frac{p_1}{p_2} B_1^{1-P_2} \alpha_1^2,$$

which combining with (3.38) imply

$$\mathrm{E}_1 \leq B_1^{1-P_2} \frac{p_2}{p_1} \left(\frac{1}{2} - \frac{1}{p_2} \right) \varrho(u). \tag{3.52}$$

combining (3.47), (3.52) and the definition of $\mathrm{H}(t)$, we have

$$\begin{aligned}
\frac{1}{2}\|\nabla u(t)\|_2^2 &= \mathrm{E}(t) - \frac{1}{2}\|u_t\|_2^2 + \int_\Omega \frac{1}{p\,(x)} |u(t)|^{p(x)} \mathrm{d}x \\
&\leq B_1^{1-P_2} \frac{p_2}{p_1} \left(\frac{1}{2} - \frac{1}{p_2} \right) \varrho(u) - \mathrm{H}(t) - \frac{1}{2}\|u_t\|_2^2 + \frac{1}{p_1}\varrho(u) \\
&= \left(B_1^{1-P_2} \frac{p_2}{p_1} \left(\frac{1}{2} - \frac{1}{p_2} \right) + \frac{1}{p_1} \right) \varrho(u) - \mathrm{H}(t) - \frac{1}{2}\|u_t\|_2^2 \\
&\leq \left(B_1^{1-P_2} \frac{p_2}{p_1} \left(\frac{1}{2} - \frac{1}{p_2} \right) + \frac{1}{p_1} \right) \varrho(u).
\end{aligned} \tag{3.53}$$

Then the desired result, with $\mathrm{c} = \left(B_1^{1-P_2} \frac{p_2}{p_1} \left(1 - \frac{2}{p_2} \right) + \frac{2}{p_1} \right)$. ■

Based on the above Lemmas, the proof of Theorem 12 is shown as follows

Proof of Theorem 12 . Case 1. If $0 \leq E(0) < E_1$, then by differentiating (3.36), we get

$$\mathrm{A}'(t) = (1-\beta)\mathrm{H}^{-\beta}(t)\mathrm{H}'(t) + \varepsilon\omega \int_\Omega \nabla u_t(s) \nabla u(t) \mathrm{d}x + \varepsilon \int_\Omega \left(u_t^2 + u u_{tt} \right) \mathrm{d}x + N\mathrm{E}_1.$$

Integrating by parts on Ω, recalling Eq (1.1), we obtain

$$\begin{aligned}
\mathrm{A}'(t) &= (1-\beta)\mathrm{H}^{-\beta}(t)\mathrm{H}'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u(t)\|_2^2 \\
&\quad - \varepsilon\mu \int_\Omega u(t) u_t(t) \mathrm{d}x + \varepsilon \int_\Omega |u(t)|^{p(x)} \mathrm{d}x + \varepsilon N \mathrm{E}_1
\end{aligned} \tag{3.54}$$

By combining (3.47) and (2.5) in (3.54)

$$\begin{aligned}
\mathrm{A}'(t) &\geq (1-\beta)\mathrm{H}^{-\beta}(t)\mathrm{H}'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|\nabla u(t)\|_2^2 \\
&\quad - \varepsilon\mu \int_\Omega u_t u \mathrm{d}x + \varepsilon p_1 (\mathrm{H}(t) - \mathrm{E}_1) + \frac{p_1}{2}\varepsilon\|u_t\|_2^2 + \frac{p_1}{2}\varepsilon\|\nabla u(t)\|_2^2 + \varepsilon N \mathrm{E}_1 \\
&\geq (1-\beta)\mathrm{H}^{-\beta}(t)\mathrm{H}'(t) + \varepsilon\left(\frac{p_1}{2} + 1\right)\|u_t\|_2^2 + \left(\frac{p_1}{2} - 1\right)\varepsilon\|\nabla u(t)\|_2^2 \\
&\quad + \varepsilon(N - p_1)\mathrm{E}_1 + \varepsilon p_1 \mathrm{H}(t) - \varepsilon\mu \int_\Omega u_t u \mathrm{d}x
\end{aligned} \tag{3.55}$$

From Hölder inequality, for a large enough constant $\beta > 1$ which to be determined later, the last term on the right-hand side of (3.55), by using (2.6) can be estimated as follows

$$\begin{aligned}
\mu \int_\Omega |u_t(t)|\,|u(t)| \mathrm{d}x &\leq \frac{\mu}{\lambda^2} \int_\Omega \mathrm{H}^\beta(t) |u(t)|^2 \mathrm{d}x + \mu\lambda^2 \mathrm{H}^{-\beta}(t) \int_\Omega |u_t(t)|^2 \mathrm{d}x \\
&\leq \frac{\mu}{\lambda^2} \int_\Omega \mathrm{H}^\beta(t) |u(t)|^2 \mathrm{d}x + \lambda^2 \mathrm{H}^{-\beta}(t)\mathrm{H}'(t).
\end{aligned} \tag{3.56}$$

Joining (3.55) with (3.56) gives

$$\begin{aligned} \mathrm{A}'(t) \geq & \left[(1-\beta) - \varepsilon\lambda^2\right] \mathrm{H}^{-\beta}(t)\mathrm{H}'(t) + \varepsilon\left(\frac{p_1}{2}+1\right)\|u_t(t)\|_2^2 \\ & + \left(\frac{p_1}{2}-1\right)\varepsilon\|\nabla u(t)\|_2^2 + \varepsilon(N-p_1)\mathrm{E}_1 + \varepsilon p_1 \mathrm{H}(t) - \varepsilon\lambda^{-2}\mathrm{H}^{\beta}(t)\int_\Omega |u(t)|^2 \mathrm{d}x. \end{aligned} \tag{3.57}$$

When $0 < \mathrm{H}(t) \leq 1$, according to (3.50), we have

$$\begin{aligned} \int_\Omega |u|^2 \mathrm{H}^\beta(t)\mathrm{d}x \leq \int_\Omega |u|^2\mathrm{d}x = \|u\|_2^2 \leq c_1 \|u\|_{p(.)}^2 \\ \leq & \mathrm{c}_1 \max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_2}}, \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_1}}\right) \\ \leq & \mathrm{c}_1 \max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2-p_2}{p_2}}, \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2-p_1}{p_1}}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x \\ \leq & \mathrm{c}_1 \max\left((p_1\mathrm{H}(0))^{\frac{2-p_2}{p_2}}, (p_1\mathrm{H}(0))^{\frac{2-p_1}{p_1}}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x = \mathrm{c}_2\int_\Omega |u|^{p(x)}\mathrm{d}x. \end{aligned}$$

When $\mathrm{H}(t) > 1$, we have

$$\begin{aligned} & \mathrm{H}^\beta(t)\int_\Omega |u|^2\mathrm{d}x \\ & \leq \mathrm{c}_1\mathrm{H}^\beta(t)\max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_1}}, \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_1}}\right) \\ & \leq \mathrm{c}_1\left(\frac{1}{p_1}\right)^\beta\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^\beta \max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_2}}, \right. \\ & \left.\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{p_1}}\right) \\ & \leq \mathrm{c}_1\left(\frac{1}{p_1}\right)^\beta \max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2-p_2}{p_2}+\beta}, \right. \\ & \left.\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{2-p_1}{p_1}+\beta}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x \\ & \leq \mathrm{c}_2\mathrm{H}^\beta(0)\int_\Omega |u|^{p(x)}\mathrm{d}x. \end{aligned}$$

Combine the two cases, we have

$$\mathrm{H}^\beta(t)\int_\Omega |u|^2\mathrm{d}x \leq \mathrm{c}_3\int_\Omega |u|^{p(x)}\mathrm{d}x, \tag{3.58}$$

where

$$\begin{aligned} \mathrm{c}_1 &= \left(1 + |\Omega|^{\frac{p_2-2}{p_2}}\right), \\ \mathrm{c}_2 &= \mathrm{c}_1 \max\left((p_1\mathrm{H}(0))^{\frac{2-p_2}{p_2}}, (p_1\mathrm{H}(0))^{\frac{2-p_1}{p_1}}\right), \\ \mathrm{c}_3 &= \mathrm{c}_2\left(1 + \mathrm{H}^\beta(0)\right). \end{aligned}$$

Combining (3.57) and (3.58) yields

$$\begin{aligned} \mathrm{A}'(t) \geq & \left[(1-\beta) - \varepsilon\lambda^2\right]\mathrm{H}^{-\beta}(t)\mathrm{H}'(t) + \varepsilon\left(\frac{p_1}{2}+1\right)\|u_t\|_2^2 \\ & + \varepsilon\left(\frac{p_1}{2}-1\right)\|\nabla u(t)\|_2^2 + \varepsilon(N-p_1)\mathrm{E}_1 + \varepsilon p_1\mathrm{H}(t) - \varepsilon\lambda_3^{-2}\mathrm{c}_3\int_\Omega |u|^2\mathrm{d}x, \end{aligned} \tag{3.59}$$

clearly, we have

$$2a\left(\mathrm{H}(t)-\mathrm{E}_1\right)+a\|\nabla u(t)\|_2^2+a\left\|u_t\right\|_2^2 \geq \frac{a}{p_2}\varrho(u),\ \ a>0. \tag{3.60}$$

For $0<a<\frac{p_1}{2}-1$,by (3.60) we rewriting (3.59) as follows

$$\begin{aligned}
\mathrm{A}'(t) \geq \left[(1-\beta)-\varepsilon\lambda^2\right]\mathrm{H}^{-\beta}(t)\mathrm{H}'(t)+\varepsilon\left(\frac{p_1}{2}+1-a\right)\left\|u_t\right\|_2^2 \\
+\varepsilon\left(\frac{p_1}{2}-1-a\right)\|\nabla u(t)\|_2^2 \\
+\varepsilon(N-(p_1-2a))\mathrm{E}_1+\varepsilon\left(p_1-2a\right)\mathrm{H}(t) \\
+\varepsilon\left(\frac{2}{p_2}a-\lambda_3^{-2}\mathrm{c}_3\right)\varrho(u).
\end{aligned}$$

In this matter, we pick λ and N large enough so that

$$\begin{aligned}
\gamma_1=\frac{2}{p_2}a-\lambda_3^{-2}\mathrm{c}_3>0, \\
N-(p_1-2a)>0.
\end{aligned}$$

Once N and λ are fixed (i.e. γ_1), since $\mathrm{H}(0)+\frac{1}{2}\varepsilon\omega\int_\Omega|\nabla u_0|^2dx>0$, we choose ε small enough so that

$$(1-\beta)-\varepsilon\lambda^2>0,\ \text{and}\ \mathrm{A}(0)=\mathrm{H}^{1-\beta}(0)+\varepsilon\int_\Omega u_0u_1\mathrm{d}x+\frac{1}{2}\varepsilon\omega\int_\Omega|\nabla u_0|^2\mathrm{d}x>0. \tag{3.61}$$

Then, there is a constant δ_1 satisfying

$$0<\delta_1\leq\min\left\{\frac{p_1}{2}-1-a,\gamma_1,p_1-2a\right\}, \tag{3.62}$$

and

$$\mathrm{A}'(t)\geq\delta_1\varepsilon\left[\|u_t\|_2^2+\|\nabla u(t)\|_2^2+\mathrm{H}(t)+\varrho(u)\right], \tag{3.63}$$

which derived by combining with (3.61)

$$\mathrm{A}(t)\geq\mathrm{A}(0)>0,\ \ \forall t\in[0,T).$$

Now, picking $\varepsilon>0$ such that $\varepsilon<\frac{1}{T(N\mathrm{E}_1)^\beta}\left(\frac{\alpha_2}{\alpha_1}\right)^{p_2(1-\beta)}$, recalling Lemma (14) and then, we have

$$|\varepsilon N\mathrm{E}_1T|^{\frac{1}{1-\beta}}\leq\left(\frac{\alpha_2}{\alpha_1}\right)^{p_2}N\mathrm{E}_1\leq\frac{N\mathrm{E}_1}{B_1\alpha_1^{p_2}}\varrho(u). \tag{3.64}$$

Applying Hölder's and Young's inequalities, using the embedding $L^{p(\cdot)}(\Omega)\hookrightarrow L^2(\Omega)$, we see that

$$\begin{aligned}
\left|\int_\Omega uu_t\mathrm{d}x\right|^{\frac{1}{1-\beta}} &\leq\|u\|_2^{\frac{1}{1-\beta}}\|u_t\|_2^{\frac{1}{1-\beta}} \\
&\leq(1+|\Omega|)^{\frac{p_1-2}{p_1(1-\beta)}}\|u\|_{p(x)}^{\frac{1}{1-\beta}}\|u_t\|_2^{\frac{1}{1-\beta}} \\
&\leq \mathrm{c}_4\left(\|u_t\|_2^2+\|u\|_{p(x)}^{\frac{2}{1-2\beta}}\right) \\
&\leq \mathrm{c}_4\|u_t\|_2^2+c_4\max\left\{\left(\int_\Omega|u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{(1-2\beta)p_1}},\left(\int_\Omega|u|^{p(x)}\mathrm{d}x\right)^{\frac{2}{(1-2\beta)p_2}}\right\} \\
&\leq \mathrm{c}_4\|u_t\|_2^2+\mathrm{c}_5\int_\Omega|u|^{p(x)}\mathrm{d}x,
\end{aligned} \tag{3.65}$$

where

$$c_4 = (1+|\Omega|)^{\frac{p_1-2}{p_1(1-\beta)}}$$

$$c_5 = c_4 \max\left\{(p_1\mathrm{H}(0))^{\frac{2}{(1-2\beta)p_1}-1}, (p_1\mathrm{H}(0))^{\frac{2}{(1-2\beta)p_2}-1}\right\},$$

Employing (3.36), (3.64), (3.65), and Cauchy-Schwarz's inequality,

$$\begin{aligned} \mathrm{A}^{\frac{1}{1-\beta}}(t) \leq & 2^{1/(1-\beta)+1}\left(\mathrm{H}(t)+\varepsilon^{\frac{1}{1-\beta}}\left|\int_\Omega uu_t(x,t)\mathrm{d}x\right|^{\frac{1}{1-\beta}}+\varepsilon^{\frac{1}{1-\beta}}(N\mathrm{E}_1T)^{\frac{1}{1-\beta}}\right) \\ \leq & \delta_2\left[\mathrm{H}(t)+\|u_t\|_2^2+\varrho(u)\right], \end{aligned} \tag{3.66}$$

where δ_2 is a positive constant, such that

$$\delta_2 = 2^{1/(1-\beta)+1}\max\left(1, \varepsilon^{\frac{1}{1-\beta}}, c_4, c_5+\frac{N\mathrm{E}_1}{B_1\lambda_1^{p_2}}\right). \tag{3.67}$$

Combine (3.64), (3.65), with (3.63), income

$$\mathrm{A}'(t) \geq \frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{1}{1-\beta}}(t), \text{ for all } t\geq 0, \tag{3.68}$$

We conclude with a simple integration of (3.68) over $(0,t)$ that

$$\mathrm{A}^{\frac{\beta}{1-\beta}}(t) \geq \frac{1}{\mathrm{A}^{\frac{\beta}{1-\beta}}(0)-\frac{\beta}{1-\beta}\frac{\delta_1}{\delta_2}t}. \tag{3.69}$$

Hence, $\mathrm{A}(t)$ blows up in a finite time $\widehat{T}$,

$$\widehat{T} \leq \frac{1-\beta}{\beta\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\beta}{1-\beta}}(0)}.$$

Since $\mathrm{A}(0)>0$, (3.69) shows that $\lim_{t\to T^*}\mathrm{A}(t)=\infty$, where $T^* = \frac{1-\beta}{\beta\frac{\delta_1}{\delta_2}\mathrm{A}^{\frac{\beta}{1-\beta}}(0)}$.

Case 2. In the case $\mathrm{E}(0)<0$. If you set $\mathrm{H}(t)=-\mathrm{E}(t)$ in Lemma (16), we get a similar result as in Lemma (16). Before $0<-E(0)=\mathrm{H}(0)\leq\mathrm{H}(t)$ and $\mathrm{H}(t)\leq\frac{1}{p_1}\varrho(u)$. If we take $N=0$ in (3.36) and apply the same reasoning as in part **Case 1**, we can get our result. ∎

3.6. Lower bound of the blow-up time

Beweis. It comes from (3.37)

$$\varphi'(t) = \int_\Omega |u(t)|^{p_2-2}uu_t\mathrm{d}x \leq \int_\Omega |u(t)|^{2p_2-2}\mathrm{d}x + \int_\Omega |u_t(t)|^2\,\mathrm{d}x. \tag{3.70}$$

We need to estimate the first term on the right-hand side of the above inequality. for that, we examine the following five cases.

Case.1. $n<3$. Exploiting the embedding inequality, we have

$$\int_\Omega |u(t)|^{2p_2-2}\mathrm{d}x \leq \hat{B}^{2p_2-2}\|\nabla u(t)\|_2^{2(p_2-1)} \leq \hat{B}^{2p_2-2}\left(\|\nabla u(t)\|_2^2+\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right)^{p_2-1}.$$

Case.2. $n\geq 3$,

a $2 < p_2 < \frac{2n}{n-1}$,. Exploiting Hölder's and embedding inequalities, we have

$$\begin{aligned}\int_\Omega |u(t)|^{2p_2-2}\mathrm{d}x &= \int_\Omega |u(t)|^{2p_2-4}u^2\mathrm{d}x \\ &\le \left(\int_\Omega |u(t)|^{(p_2-2)n}\mathrm{d}x\right)^{\frac{2}{n}}\left(\int_\Omega |u(t)|^{\frac{2n}{n-2}}\mathrm{d}x\right)^{1-\frac{2}{n}} \\ &\le |\Omega|^{\frac{2}{n}-\frac{2(p_2-2)}{p_2}}\|u(t)\|^2_{\frac{2n}{n-2}}\left(\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right)^{\frac{2(p_2-2)}{p_2}} \\ &\le B_1^2|\Omega|^{\frac{2}{n}-\frac{2(p_2-2)}{p_2}}\|\nabla u(t)\|_2^2\left(\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right)^{\frac{2(p_2-2)}{p_2}} \\ &\le B_1^2|\Omega|^{\frac{2}{n}-\frac{2(p_2-2)}{p_2}}\left(\|\nabla u(t)\|_2^2+\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right)^{\frac{3p_2-4}{p_2}}.\end{aligned}$$

b $\frac{2n}{n-1} \le p_2 < \frac{2n-2}{n-2}$, $n \ge 3$. Through emulating **a**, we have

$$\int_\Omega |u(t)|^{2p_2-2}\mathrm{d}x \le B^2|\Omega|^{-\frac{2n(p_2-2)}{2n-np_2+2p_2+2}}\left(\|\nabla u(t)\|_2^2+\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right)^{\frac{2n-np_2+4p_2}{2n-np_2+2p_2+2}}.$$

c $\frac{2n-2}{n-2} \le p_2 \le \frac{2n}{n-2}$, $n \ge 3$. Hölder's and embedding inequalities have allowed us to

$$\begin{aligned}\int_\Omega |u(t)|^{2p_2-2}\mathrm{d}x &= \int_\Omega |u(t)|^{2(p_2-1)}\mathrm{d}x \\ &\le |\Omega|^{\frac{n-(p_2-1)(n-2)}{n}}\left(\int_\Omega |u(t)|^{2\frac{n}{n-2}}\mathrm{d}x\right)^{\frac{(p_2-1)(n-2)}{n}} \\ &\le |\Omega|^{\frac{n-(p_2-1)(n-2)}{n}}\|u(t)\|^{2(p_2-1)}_{2\frac{n}{n-2}} \\ &\le B_1^{2(p_2-1)}|\Omega|^{\frac{n-(p_2-1)(n-2)}{n}}\|\nabla u(t)\|_2^{2(p_2-1)} \\ &\le B_1^{2(p_2-1)}|\Omega|^{\frac{n-(p_2-1)(n-2)}{n}}\left(\|\nabla u(t)\|_2^2+\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right)^{p_2-1}.\end{aligned}$$

Therefore, in every case, we find

$$\int_\Omega |u(t)|^{2p_2-2}\mathrm{d}x \le c^*\left(\int_\Omega |u(t)|^{p_2}\mathrm{d}x+\int_\Omega |\nabla u(t)|^2\mathrm{d}x\right)^{\delta}. \tag{3.71}$$

where, for the above mentioned five cases $\delta > 1$ equals p_2-1, $\frac{3p_2-4}{p_2}$, $\frac{2n-np_2+4p_2}{2n-np_2+2p_2+2}$. Using $E(t)$ definition's we can observe that

$$\begin{aligned}\frac{1}{2}\int_\Omega |u_t(t)|^2\mathrm{d}x+\frac{1}{2}\int_\Omega |\nabla u(t)|^2\mathrm{d}x &\le \mathrm{E}(0)+\frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x \\ &\le \mathrm{E}(0)+\frac{1}{p_1}\left(\int_\Omega |u(t)|^{p_1}\mathrm{d}x+\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right) \\ &\le \mathrm{E}(0)+\frac{c_6}{p_1}\left(\left(\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right)^{\frac{p_1}{p_2}}+\int_\Omega |u(t)|^{p_2}\mathrm{d}x\right) \\ &= \mathrm{E}(0)+\frac{c_6}{p_1}\left(\varphi^{\frac{p_1}{p_2}}(t)+\varphi(t)\right)\end{aligned} \tag{3.72}$$

where $c_6 = (1+|\Omega|)^{\frac{p_2-p_1}{p_1}}$. If we join (3.70)-(3.72), we get

$$\begin{aligned}\varphi'(t) &\le c^* \left[2E(0) + \left(1+\frac{2c_6}{p_1}\right)\varphi(t) + \frac{2c_6}{p_1}\varphi^{\frac{p_1}{p_2}}(t)\right]^{\delta} \\ &\quad + \left(2E(0) + \frac{2c_6}{p_1}\varphi(t) + \frac{2c_6}{p_1}\varphi^{\frac{p_1}{p_2}}(t)\right) \\ &\le c_7\left(\varphi^{\delta}(t) + \varphi^{\frac{p_1}{p_2}\delta}(t) + \varphi(t) + \varphi^{\frac{p_1}{p_2}}(t) + 1\right),\end{aligned}$$

where

$$c_7 = \max\left(4^{\delta-1}c^*\left(1+\frac{2c_6}{p_1}\right)^{\delta}, \frac{2^{3\delta-2}cc_6^{\delta}}{p_1^{\delta}}, \frac{2c_6}{p_1}, 2^{3\delta-2}cE^{\delta}(0) + 2E(0)\right). \tag{3.73}$$

According to the definition of T^*.

$$\lim_{t\to T^*}\int_{\Omega}|u(t)|^{p_2}dx = +\infty,$$

We achieve that

$$\int_{\varphi(0)}^{+\infty}\frac{d\eta}{c_7\left(\eta^{\delta} + \eta^{\delta\frac{p_1}{p_2}} + \eta + \eta^{\frac{p_1}{p_2}} + 1\right)} \le T^*.$$

The proof is finished. ∎

Acknowledgements.

The authors would like to thank the anonymous referees and the handling editor for their reading and relevant remarks/suggestions.

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