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Blow-up phenomena for a reaction-diffusion equation with a singular coefficient

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ABSTRACT: In this paper, we consider an initial boundary value problem for a p-Laplacian parabolic reaction-diffusion equation with a singular coefficient. We address this problem at three different initial energy levels. For sub-critical initial energy, we obtain blow-up results and estimate the lower and upper bounds of the blow-up time. For critical initial energy, we demonstrate global existence, asymptotic behavior, finite-time blow-up, and the lower bound of the blow-up time. For super-critical initial energy, we prove finite-time blow-up and estimate the lower and upper bounds of the blow-up time. This investigation generalizes and enhances previous literature outcomes.

Key Words: Blow-up, blow-up time, global existence, Reaction diffusion equation, singular coefficient.

Contents

1. Introduction

The law of conservation states that diffusion processes involving a reaction are described by a well-known equation [1]

$$u_t - \operatorname{div}(D(u,\nabla u)\nabla u) = f(x,t,u,\nabla u),\ u(x,0) = u_0(x). \tag{1.1}$$

Where the function $u(x,t)$ represents the mass concentration in chemical reaction processes or temperature in heat conduction, at a specific position x in the diffusion medium and time t, the function D is known as the diffusion coefficient or the thermal diffusivity. The term $\operatorname{div}(D(u,\nabla u)\nabla u)$ represents the rate of change due to diffusion while $f(x,t,u,\nabla u)$ represents the rate of change due to reaction. Actually, despite being an uncommon occurrence, the diffusion and reaction process is quite significant because it describes the behavior of a wide variety of chemical systems in which the diffusion of a material competes with the synthesis of that substance through a chemical reaction. The two most prevalent are diffusion, which causes the substances to disperse over a surface in space, and local chemical reactions, which modify the concentration of one or more chemical substances over time and space. Nevertheless, dynamical processes of a non-chemical nature can also be described by the system. Biology, geology, physics (neutron diffusion theory), and ecology are among the fields that provide examples. In this paper, we consider the following semilinear pseudo-parabolic equation that involves a singular coefficient $D = |x|^{-s}$

$$\begin{cases} \frac{u_t}{|x|^s} - \Delta u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \end{cases} \tag{1.2}$$

Assuming Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ that contains the origin 0 with a smooth boundary $\Gamma = \partial\Omega$, we assume that $u_0 \in W_0^{1,p}(\Omega)$, $0 \leq s \leq 2$ and $u_0(x) \not\equiv 0$, where $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ for

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all $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$, $-\text{div}(|\nabla u|^{p-2}\nabla u)$ called the p-Laplace operator. We also assume that p is a constant

$$2\le p<\infty, \tag{1.3}$$

that satisfies condition (H).

H The given constants (p,q) satisfying

$$\begin{aligned} 1 &< q<\infty \text{ if } n\le p,\\ p &< q+1<\frac{np}{n-p} \text{ if } n>p. \end{aligned}$$

Determining whether the answers to evolution problems result in a blow-up and, if so, when this event occurs is crucial in many real-life situations. Nevertheless, it is typically impossible to pinpoint the precise moment T of the blow-up.Setting the lower and upper bounds for T is therefore crucial, and we simply direct the reader to [2,18,3,20], and the references it contains. The blow-up features of solutions to nonlinear evolution equations and systems have been extensively studied in the literature. A unifying method to solving highly intriguing linked problems, such as extinction, is provided by the general perspective of treating blow-up problems as singularities. Time derivatives in the highest order term, which characterize a number of important physical processes, are a feature of pseudo-parabolic equations. Among these is the seepage of uniform fluids through guaranteed rock [7], two temperature systems and heat conduction [8], nonlinear, dispersive long waves propagating unidirectionally [9,10], flow of fluids in guaranteed porous mediums [11], two-phase flow with dynamic capillary pressure in porous media [12,13], and the aggregation of populations [14]. Using Neumann boundary conditions for the pseudo-parabolic equation that follows.

$$\begin{cases} u_t-\Delta u_t-\Delta u=|u|^{p-1}u-\frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u dx, & (x,t)\in\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega,\\ \int_\Omega u_0 dx=0, & x\in\Omega,\\ \frac{\partial u}{\partial\nu}=0, & (x,t)\in\partial\Omega\times(0,T), \end{cases}$$

where $u_0\in H^1(\Omega)$, $T\in(0,\infty]$ and p satisfies

$$1<p<+\infty \text{ if } n=1,2,\;\; 1<p<\frac{n+2}{n-2} \text{ if } n\ge 3.$$

The initial boundary value problem of a semi-near pseudo-parabolic equation with Neumann boundary conditions was studied by Wang et al. [16]. They proved the blow-up phenomena for solutions with sub-critical initial energy and the existence, uniqueness, and asymptotic behavior of the global solution. Cases with essential starting energy have also been included in the extension of these findings. Additionally, it has been demonstrated that solutions with supercritical starting energy can blow out. The following generalized semilinear pseudo parabolic p-Laplacian equation was further examined by Na et al. [15]

$$\begin{cases} u_t-\Delta u_t-\text{div}(|\nabla u|^{p-2}\nabla u)=|u|^{q-1}u-\frac{1}{|\Omega|}\int_\Omega |u|^{q-1}u dx, & (x,t)\in\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega,\\ \int_\Omega u_0 dx=0, & x\in\Omega,\\ \frac{\partial u}{\partial\nu}=0, & (x,t)\in\partial\Omega\times(0,T), \end{cases}$$

where p satisfies

$$2<p<q+1<\begin{cases} p\left(1+\frac{2}{n}\right), \text{ if } n>p\\ \infty, \text{ if } n\le p \end{cases},$$

$$p=2,\;\; 2<q+1<\begin{cases} \frac{2n}{n-2}, \text{ if } n>2\\ \infty, \text{ if } n\le 2. \end{cases}$$

They looked into the existence, uniqueness, and asymptotic behavior of the global solution under various initial energy circumstances. They also looked into the weak solution's blow-up phenomenon. Additionally, they determined lower and upper bound estimations for the weak solution's rate and blow-up time under various beginning energy situations. As mentioned above, a great deal of research has been done on the blow-up phenomena for Equation (1.1) in both limited domains and the entire space when $D = 1$. The blow-up features of solutions to problem (1.2), however, have received a lot less attention. In [5], Han investigated the finite-time blow-up of local solutions to the reaction diffusion equation using a specific diffusion process, as well as the presence and asymptotic estimates of global solutions.

$$\begin{cases} \frac{u_t}{|x|^2} - \Delta u = k\left(t\right)|u|^{q-1}u, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \end{cases} \tag{1.4}$$

He illustrates the finite time blow-up result with negative initial energy using the potential well method, Hardy inequality, and Philippin's [6], first-order differential inequality approach.We refer the reader to articles [19,17] and [18] for further details on the problems associated with special medium vacuum. We will examine the blow-up phenomenon related to problem (1.2) and examine how the exponents p and q affect the blow-up conditions and the blow-up time of solutions to problem (1.2), building on the previously cited works. To clarify our approach, we will demonstrate that, under assumption (H) concerning p and q, the solutions to problem (1.2) will experience blow-up in finite time if any of the following three conditions are met:

- **(i)** The initial Nehari energy is positive and the initial energy is smaller than the potential well depth, i.e. $\mathrm{I}\left(u_0\right) > 0$ and $\mathrm{J}(u_0) < d$.
- **(ii)** The initial Nehari energy is positive and the initial energy is equal the potential well depth, i.e. $\mathrm{I}\left(u_0\right) \geq 0$ and $\mathrm{J}(u_0) = d$.
- **(iii)** High (sup-critical) initial energy, i.e. $\mathrm{J}\left(u_0\right) > 0$.

Furthermore, we use Gagliardo Nirenberg's inequality to show the blow-up time's global existence, asymptotic behavior, and upper and lower bounds. This equation is very difficult to evaluate because of the singular coefficient $D(x) = |x|^{-s}$; in order to prove instances (i) and (ii), we must take into account the relationship between $\mathrm{I}(u)$, $\int_\Omega |\nabla u(x)|^2 dx$, and the depth of the potential well d. To address the difficulty caused by $|x|^{-s}$, we utilize the Hardy inequality when addressing case (iii). This paper's remainder is organized as follows: Additional lemmas and explanations will be included as preliminary information in Section 2. The blow-up time upper bounds for each instance will be established in Section 3, which will also offer three adequate conditions for the answers to problem (1.2) pertaining to the blow-up in finite time. In Section 4, the blow-up time lower bound will be provided.

2. Preliminaries and Main result

We will start by going over some fundamental findings that are necessary for the sections that follow. Although we have given references to the pertinent papers, the results are frequently published without supporting proof. In addition, some of our notation conventions are introduced. First, we denote $\|.\|_q$ to the usual $L^q(\Omega)$ norm for $1 \leq q \leq \infty$, and $\|\nabla.\|_k$ the Dirichlet norm in $W_0^{1,k}(\Omega)$. Additionally, going forward, C stands for various positive constants that vary according on the known numbers and could change with each advent.

In this section, we introduce sets and functionals.

$$\mathrm{J}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{1}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x.$$

$$\begin{aligned} \mathrm{H}(t) &= \mathrm{J}(u_0) - \mathrm{J}(u) \\ &= \mathrm{J}(u_0) - \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x. \end{aligned}$$

$$
\begin{aligned}
\mathrm{I}(u) &= \int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega |u|^{q+1}\,\mathrm{d}x,\\
W &= \left\{u \in W_0^{1,p}(\Omega) \mid \mathrm{I}(u)>0,\ \mathrm{J}(u)<d\right\} \cup \{0\},\\
V &= \left\{u \in W_0^{1,p}(\Omega) \mid \mathrm{I}(u)<0,\ \mathrm{J}(u)<d\right\},
\end{aligned}
$$

and the depth of potential well d is defined by

$$0<d=\inf_{u\in\mathcal{N}} \mathrm{J}(u), \quad t\geq 0,$$

where the Nehari manifold is defined by

$$\mathcal{N}=\left\{u \in W_0^{1,p}(\Omega) \mid \mathrm{I}(u)=0,\ \int_\Omega |\nabla u|^2\,\mathrm{d}x \neq 0\right\},$$

and $\mathcal{N}$ separates two unbounded sets

$$\mathcal{N}_+=\left\{u \in W_0^{1,p}(\Omega) \mid \mathrm{I}(u)>0\right\} \cup \{0\},$$

and

$$\mathcal{N}_-=\left\{u \in W_0^{1,p}(\Omega) \mid \mathrm{I}(u)<0\right\}.$$

To address the singularity present in the model equation, we introduce the following functional sequences $\lambda_n(x)$.

$$\lambda_n(x)=\min\left\{|x|^{-s}, n\right\}, \text{ with } n\in\mathbb{N}. \tag{2.1}$$

Further we denote

$$
\begin{aligned}
(u,v)_* &= \textstyle\int_\Omega \nabla u.\nabla v dx + \int_\Omega \lambda_n u.v dx,\\
\|v\|_{H_0^1(\Omega)} &= \left(\|\nabla v\|^2+\|v\|^2\right)^{\frac{1}{2}},\\
\|v\|_* &= \left(\|\nabla v\|^2+\textstyle\int_\Omega \lambda_n v^2 dx\right)^{\frac{1}{2}},
\end{aligned} \tag{2.2}
$$

A weak solution of problem (1.2) can be defined as follows. Let denote by

$$
\begin{aligned}
\tilde{W} \ :\ &= \left\{u : u\in L^2\left(0,T;L^2(\Omega)\right) \cap L^\infty\left(0,T;W_0^{1,p}(\Omega)\right) \cap L^{q+1}(\Omega\times(0,T))\right.\\
&\text{with } \left.\nabla u \in L^p(\Omega\times(0,T))\right\}.
\end{aligned}
$$

Definition 1 (Weak solution) *A solution to the problem (1.2) is a function $u=u(x,t)\in\tilde{W}$ with $\frac{u}{|x|^s}\in L^2\left(0,T;L^2(\Omega)\right)$ such that*

(i)

$$
\begin{aligned}
&\textstyle\int_0^t\int_\Omega \left(-\frac{u}{|x|^s}\frac{\mathrm{d}\varphi}{\mathrm{d}t} + |\nabla u|^{p-2}\nabla u\nabla\varphi + \nabla u_t\nabla\varphi - |u|^{q-1}u\varphi\right)\mathrm{d}x\mathrm{d}\tau\\
&\qquad = -\textstyle\int_\Omega \frac{u}{|x|^s}\varphi \mathrm{d}x\Big|_0^t,\ t\in(0,T),
\end{aligned} \tag{2.3}
$$

holds for any $t\leq T$ and all $\varphi\in\tilde{W}$ with $\frac{\mathrm{d}\varphi}{\mathrm{d}t}\in\tilde{W}^*$, where $\tilde{W}^*$ is the dual space of $\tilde{W}$.

(ii) *$u(x,0)=u_0(x)$ in $W_0^{1,p}(\Omega)$.*

(iii) *for $0\leq t<T$*

$$\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau+\mathrm{J}(u)\leq \mathrm{J}(u_0). \tag{2.4}$$

Before presenting our main result, we first state the following theorem regarding existence, uniqueness, and regularity. It is important to note that by employing the Faedo-Galerkin method in conjunction with the fixed point theorem, we can readily establish the well-posedness of the solution to the problem (1.2).

Theorem 2 (Local solution) *Let $u_0 \in W_0^{1,p}(\Omega)\backslash\{0\}$ and q satisfy (H). Then there exist a $T_{\max} > 0$ and a unique weak solution u of (1.2) satisfying $u \in C\left(0, T; W_0^{1,p}(\Omega)\right)$, and the energy inequality*

$$\int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_\tau\|_2^2 \right) \mathrm{d}\tau + \mathrm{J}(u(t)) \leq \mathrm{J}\left(u_0\right), \quad 0 \leq t \leq T_{\max},$$

where $T_{\max}$ is the maximum existence time of solution $u(t)$. Moreover,

(i) *If $T_{\max} < \infty$, then*

$$\lim_{t \to T} \|u\|_q = \infty \text{ for all } q > 1 \text{ such that } q > \frac{n(p_1 - 2)}{2}.$$

(ii) *If $T_{\max} = \infty$, then $u(t)$ is a global solution of problem (1.2).*

The qualitative analysis of $\mathrm{J}(u)$ and $\mathrm{I}(u)$ is presented here. The next lemma is as follows.

Lemma 3 *For p satisfy (1.3), (H) and $u \in W_0^{1,p}(\Omega)\backslash\{0\}$. Let $F : [0, +\infty) \to \mathbb{R}$ the Euler functional defined by*

$$F(\lambda) = \frac{\lambda^p}{p} \int_\Omega |\nabla u|^p \mathrm{d}x - \frac{\lambda^{q+1}}{q+1} \int_\Omega |u|^{q+1} \mathrm{d}x, \tag{2.5}$$

then, F keeps the following properties:

(i) $\lim_{\lambda \to 0^+} F(\lambda) = 0$ *and* $\lim_{\lambda \to +\infty} F(\lambda) = -\infty$.

(ii) *There is at least one solution $\lambda_* = \lambda_*(u)$ to the equation $F'(\lambda) = 0$ on the interval $(0, +\infty)$.*

(iii) *$F(\lambda)$ gets its maximum at $\lambda = \lambda^*$, and is increasing on $[0, \lambda_*]$ and decreasing on $[\lambda_*, +\infty)$. In addition, we have that $0 < \lambda_* < 1$, $\lambda_* = 1$ and $\lambda_* > 1$ provided $I(u) < 0$, $I(u) = 0$ and $I(u) > 0$, respectively*

Beweis. (2.5) clearly illustrates the claim (i).

For (ii). From

$$\begin{aligned} F'(\lambda) &= \lambda^{p-1} \int_\Omega |\nabla u(x)|^p \mathrm{d}x - \lambda^q \int_\Omega |u(x)|^{q+1} \mathrm{d}x \\ &= \lambda^{p-1} \int_\Omega \left[|\nabla u(x)|^p - \lambda^{q-p+1} |u(x)|^{q+1} \right] \mathrm{d}x, \end{aligned} \tag{2.6}$$

brings about the conclusion.

The definition of λ_* and the relation $\mathrm{I}(\lambda u) = \lambda F'(\lambda)$ and

$$F'(\lambda) = \left(\lambda^{p-1} - \lambda^q\right) \int_\Omega |\nabla u(x)|^p \mathrm{d}x + \lambda^q \mathrm{I}(u), \quad \lambda \in (0, +\infty),$$

lead to the final assertion in (iii). the proof's completeness. ■

A ball with a radius of $\int_\Omega |\nabla u(x)|^p dx$ is identified in the $W_0^{1,p}(\Omega)$ space in the following lemma. This enhances our understanding of the relationship between $\mathrm{I}(u)$, $\int_\Omega |\nabla u(x)|^p dx$, and the depth of the potential well d. We start by recalling a classical result primarily attributed to Hardy-Sobolev.

Lemma 4 *For any $u \in W_0^{1,p}(\Omega)$, assume that $n \geq 3$. Then $\frac{u}{|x|^{\frac{s}{2}}} \in L^2(\Omega)$, and*

$$\left(c_e \int_\Omega \frac{|u|^2}{|x|^s} \mathrm{d}x \right)^{\frac{1}{2}} \leq \left(\int_\Omega |\nabla u|^p \mathrm{d}x \right)^{\frac{1}{p}}, \tag{2.7}$$

where c_e is a constant depending on p, s, and n.

From (2.7) we observe that $\int_\Omega \lambda_n v^2 \mathrm{d}x$ is bounded. Thus, $\|v\|_*$ is equivalent to $\|v\|_{H_0^1(\Omega)}$.

Lemma 5 *Let $u \in W_0^{1,p}(\Omega)$ and assume that (1.3), (H) and $\mathrm{J}(u) \leq d$ hold.*

(i) *If $0 < \|\nabla u\|_p < r$, then $\mathrm{I}(u) > 0$ and*

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x < \frac{p(q+1)}{q-p+1} d.$$

(ii) *If*

$$\int_\Omega |\nabla u|^p \mathrm{d}x > \frac{p(q+1)}{q-p+1} d.$$

then $\mathrm{I}(u) < 0$ and $\|\nabla u\|_p > r$.

(iii) *If $\mathrm{I}(u) = 0$, then $\|\nabla u\|_p = 0$ or*

$$r^p \leq \|\nabla u\|_p^p \leq \frac{p(q+1)}{q-p+1} d,$$

where $r = \left(\frac{1}{C_*^{q+1}} \right)^{\frac{1}{q-p+1}}$ and C_* is the best embedding constant from $W_0^{1,p}(\Omega)$ to $L^{q+1}(\Omega)$.

Beweis. (i) From (H), (1.3) and $0 < \|\nabla u\|_p < r$, we have

$$\begin{aligned} \int_\Omega |u|^{q+1} \, \mathrm{d}x &\leq C_*^{q+1} \|\nabla u\|_p^{q-p+1} \|\nabla u\|_p^p \\ &< C_*^{q+1} r^{q-p+1} \|\nabla u\|_p^p \leq \|\nabla u\|_p^p, \end{aligned} \tag{2.8}$$

which gives $\mathrm{I}(u) > 0$. According to (1.3), $\mathrm{I}(u) > 0$, and the definition of $\mathrm{J}(u)$, we check that

$$\begin{aligned} \mathrm{J}(u) &= \frac{1}{p} \int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{1}{q+1} \int_\Omega |u|^{q+1} \, \mathrm{d}x \\ &\geq \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega |\nabla u|^p \, \mathrm{d}x \\ &\quad + \frac{1}{q+1} \left(\int_\Omega |\nabla u|^p \, \mathrm{d}x - \int_\Omega |u|^{q+1} \, \mathrm{d}x \right) \\ &= \left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega |\nabla u|^p \, \mathrm{d}x + \frac{1}{q+1} \mathrm{I}(u) \\ &> \left(\frac{1}{p} - \frac{1}{q+1} \right) \|\nabla u\|_p^p, \end{aligned} \tag{2.9}$$

since $\mathrm{J}(u) \leq d$ gives that

$$\left(\frac{1}{p} - \frac{1}{q+1} \right) \int_\Omega |\nabla u|^p \, \mathrm{d}x < d,$$

i.e.

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x < \frac{p(q+1)}{q-p+1} d.$$

(ii) By (2.9) and $\int_\Omega |\nabla u|^p dx > \frac{p(q+1)}{q-p+1} d$, we have

$$\mathrm{J}(u) \geq \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q+1}\mathrm{I}(u) > d + \frac{1}{q+1}\mathrm{I}(u),$$

then $\mathrm{J}(u) \leq d$ gives

$$\mathrm{I}(u) < 0,$$

As a result of the Sobolev inequality, $\int_\Omega |\nabla u|^p \,\mathrm{d}x \neq 0$. Consequently, $\mathrm{I}(u) < 0$ provides

$$\|\nabla u\|_p^p < \int_\Omega |u|^{q+1} \,\mathrm{d}x \leq C_*^{q+1} \|\nabla u\|_p^{q+p-1} \|\nabla u\|_p^p,$$

so that $\|\nabla u\|_p > r$.

(iii) As $\mathrm{I}(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega |u|^{q+1} \,\mathrm{d}x = 0$. If $\|\nabla u\|_p \neq 0$, Afterward, by

$$\|\nabla u\|_p^p < \int_\Omega |u|^{q+1} \,\mathrm{d}x \leq C_*^{q+1} \|\nabla u\|_p^{q+p-1} \|\nabla u\|_p^p,$$

We obtain $\|\nabla u\|_p \geq r$. According to (2.9) with $\mathrm{I}(u) = 0$, we deduce

$$\mathrm{J}(u) \geq \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_\Omega |\nabla u|^p \,\mathrm{d}x \geq \frac{q-p+1}{p(q+1)} \|\nabla u\|_p^p.$$

Adding $\mathrm{J}(u) \leq d$ results in

$$\|\nabla u\|_p^p \leq \frac{p(q+1)}{q-p+1} d.$$

■

This lemma establishes a connection between $\mathrm{J}(u)$, $\mathrm{I}(u)$, and d, gives the expression of d in terms of r, and demonstrates that $\mathrm{J}(u)$ is not increasing.

Lemma 6 **(i)** *If r is defined as in Lemma (5), then we can conclude that*

$$d = \frac{q-p+1}{p(q+1)} r^p. \tag{2.10}$$

(ii) *The functional energy* $\mathrm{J(u)}$ *is nonincreasing.*

(iii) *Let $u \in W_0^{1,p}(\Omega)$ with $\mathrm{I}(u) < 0$. We have*

$$\mathrm{I(u)} < (q+1)(\mathrm{J(u)} - d). \tag{2.11}$$

Beweis. (i) Let $u \in N$, we may be certain that $\|\nabla u\|_p \geq r$ based on 5, (iii). Combining this with (2.9) yields the following outcome

$$\begin{aligned} \mathrm{J}(u) &\geq \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q+1}\mathrm{I}(u) \\ &\geq \frac{q-p+1}{2(p_1+1)} \|\nabla u\|_p^p \geq \frac{q-p+1}{p(q+1)} r^p. \end{aligned}$$

The definition of d allows us to obtain equation (2.10).

(ii) Let $\nu = u_t$ in (2.3), we obtain

$$\int_\Omega \frac{|u_t|^2}{|x|^s} \mathrm{d}x + \int_\Omega |\nabla u_t|^2 \,\mathrm{d}x + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega |\nabla u|^p \,\mathrm{d}x = \frac{1}{q+1}\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega |u|^{q+1} \,\mathrm{d}x,$$

it indicates that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{p}\int_{\Omega}|\nabla u|^{p}\,\mathrm{d}x-\frac{1}{q+1}\int_{\Omega}|u|^{q+1}\,\mathrm{d}x\right)=-\int_{\Omega}\frac{|u_t|^2}{|x|^s}\mathrm{d}x-\int_{\Omega}|\nabla u_t|^2\,\mathrm{d}x,$$

or

$$\mathrm{J}'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{J}(u)=-\int_{\Omega}\frac{|u_t|^2}{|x|^s}\mathrm{d}x-\int_{\Omega}|\nabla u_t|^2\,\mathrm{d}x\leq 0.$$

(iii) Lemma 3, (iii) and I(u) <0 implies existence of $\lambda_*\in(0,1)$ such that $\mathrm{I}(\lambda_* u)=0$. Let

$$h(\lambda):=(q+1)\,\mathrm{J}(\lambda u)-\mathrm{I}(\lambda u),\quad \lambda>0.$$

From Lemma 5, by the definition $\mathrm{J}(u)$, $\mathrm{I}(u)$, (1.3) and (ii), we obtain

$$\begin{aligned}
h'(\lambda) &= (q+1)\frac{\mathrm{dJ}(\lambda u)}{\mathrm{d}\lambda}-\frac{\mathrm{dI}(\lambda u)}{\mathrm{d}\lambda}\\
&= (q+1)\left(\int_{\Omega}\lambda^{p-1}|\nabla u(x)|^{p}\mathrm{d}x-\lambda^{q}\int_{\Omega}|u(x)|^{q+1}\mathrm{d}x\right)\\
&\quad -p\int_{\Omega}\lambda^{p-1}|\nabla u(x)|^{p}\mathrm{d}x+(q+1)\,\lambda^{q}\int_{\Omega}|u(x)|^{q+1}\mathrm{d}x\\
&= (q-p+1)\lambda^{p-1}\int_{\Omega}|\nabla u(x)|^{p}\mathrm{d}x\\
&> (q-p+1)\lambda^{p-1}r^{p}>0.
\end{aligned}$$

For $\lambda>0$, $h(\lambda)$ is hence strictly increasing. Consequently, for $\lambda_*\in(0,1)$, $h(1)>h(\lambda_*)$. Knowing that $\mathrm{I}(\lambda_* u)=0$ and applying the definition of d, we get

$$\begin{aligned}
(q+1)\,\mathrm{J}(u)-\mathrm{I}(u) &> (q+1)\,\mathrm{J}(\lambda_* u)-\mathrm{I}(\lambda_* u)\\
&= (q+1)\,\mathrm{J}(\lambda_* u)\geq(q+1)\,d,
\end{aligned}$$

so (2.11). ■

3. Global existence, asymptotic behavior, and blow-up in finite time with $\mathrm{J}(u_0)<d$

In this section, we compute the upper and lower bounds of the explosion time when $\mathrm{J}(u_0)<d$ and discover that the solution to problem (1.2) produces an explosion in limited time. The global existence and asymptotic behavior of the solution to problem (1.2) with $\mathrm{J}(u_0)<d$ and $\mathrm{I}(u_0)>0$ can similarly obtaining it using reference [21]. The proof has been ignored, and we only mention it to illustrate the routine results.

Theorem 7 *[21, Theorem 4. Eq.14]Assuming that $J(u_0)<d$ and $\mathrm{I}(u_0)>0$, let $p(.)$ satisfy (H) and $u_0\in W_0^{1,p}(\Omega)$. Then problem (1.2) admits a global weak solution $u(t)\in L^{\infty}\left(0,\infty;W_0^{1,p}(\Omega)\right)$. Furthermore, $u_{\mathrm{t}}\in L^2\left(0,\infty;L^2(\Omega)\right)$ and $u(t)\in W$ for $0\leq t<\infty$. Additionally, there exists a constant $\kappa>0$ such that*

$$\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u\|_2^2\leq\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)e^{-2\kappa t}.$$

We have to provide the invariant set V in the following Lemma in the manner described below.

Lemma 8 (Invariant set for $J(u_0)<d$.) *Assuming (H) holds for p, and $u_0\in W_0^{1,p}(\Omega)$, let $T_{\max}$ be the maximal existence time. Then, for $\mathrm{J}(u_0)<d$, the weak solution u of problem (1.2) is belongs to V for $0\leq t<T_{\max}$, as long as $\mathrm{I}(u_0)<0$.*

Beweis. We know that $u_0 \in V$ because $\mathrm{J}(u_0) < d$ and $\mathrm{I}(u_0) < 0$. Our goal is to prove that $u(t) \in V$ for $0 < t < T_{\max}$. Let's assume the contrary and suppose that there exists $\mathrm{t}_0 \in (0, T_{\max})$ such that $\mathrm{J}(u(\mathrm{t}_0)) = d$ or $\mathrm{I}(u(\mathrm{t}_0)) = 0$ and $\|\nabla u(\mathrm{t}_0)\|_p \neq 0$. Since $\mathrm{J}(u)$ and $\mathrm{I}(u)$ are continuous in t, we can assume that t_0 is the first time such that $\mathrm{J}(u(\mathrm{t}_0)) = d$ or $\mathrm{I}(u(\mathrm{t}_0)) = 0$ and $\|\nabla u(\mathrm{t}_0)\|_p \neq 0$. By Definition 1 (iii) and the fact that $\mathrm{J}(u_0) < d$, we have

$$\int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x \mathrm{d}\tau + \mathrm{J}(u) \leq \mathrm{J}(u_0) < d, \quad 0 \leq t < T_{\max}. \tag{3.1}$$

Therefore, $\mathrm{J}(u(\mathrm{t}_0)) \neq d$. If $\mathrm{I}(u(\mathrm{t}_0)) = 0$ and $\|\nabla u(\mathrm{t}_0)\|_p \neq 0$, then by the definition of d, we have $\mathrm{J}(u(\mathrm{t}_0)) \geq d$, which contradicts (3.1). Thus, we have proven that $u(t) \in V$ for $0 < t < T_{\max}$. The proof has been achieved. ∎

By adding a simple auxiliary function, we establish the blow-up in finite time of solution and give an adequate condition in the following Theorem 9.

Theorem 9 (Blow -up for $J(u_0) < d$.) *Assuming that p satisfies (H) and $u_0 \in W_0^{1,p}(\Omega)$, if $\mathrm{J}(u_0) < d$ and $\mathrm{I}(u_0) < 0$, then the weak solution $u(t)$ of problem (1.2) blows up in a finite time.*

Beweis. Problem (1.2) has a single local weak solution $u \in C\left(0, T; W_0^{1,p}(\Omega)\right)$, according to Theorem 2, where $T_{\max}$ is the maximum existence time of $u(t)$. We can demonstrate that time has a finite existence. We do this by using contradiction and assuming that the time of existence $T_{\max} = +\infty$. Next, we define

$$\mathrm{M}(t) := \int_0^t \int_\Omega \left(\frac{|u|^2}{|x|^s} + |\nabla u|^2 \right) \mathrm{d}x \mathrm{d}\tau, \quad t \in [0, +\infty), \tag{3.2}$$

then

$$\mathrm{M}'(t) = \int_\Omega \left(\frac{|u|^2}{|x|^s} + |\nabla u|^2 \right) \mathrm{d}x - \int_\Omega \left(\frac{|u_0|^2}{|x|^s} + |\nabla u_0|^2 \right) \mathrm{d}x.$$

When (2.4) and (2.9) are combined, we obtain

$$\begin{aligned} \mathrm{J}(u_0) &\geq \mathrm{J}(u) + \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x \mathrm{d}\tau \\ &\geq \frac{q-p+1}{p(q+1)} \|\nabla u\|_p^p + \frac{1}{q+1} \mathrm{I}(u) + \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x \mathrm{d}\tau, \end{aligned}$$

which implies

$$\frac{1}{q+1} \mathrm{I}(u) \leq \mathrm{J}(u_0) - \frac{q-p+1}{p(q+1)} \|\nabla u\|_p^p - \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x \mathrm{d}\tau,$$

i.e.,

$$\begin{aligned} \mathrm{I}(u) \leq\ & (q+1)\mathrm{J}(u_0) - \frac{q-p+1}{p} \|\nabla u\|_p^p \\ & -(q+1) \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x \mathrm{d}\tau. \end{aligned} \tag{3.3}$$

By substituting (3.3) into (3.21), we can derive

$$\begin{aligned} \mathrm{M}''(t) \geq & -2(q+1)\mathrm{J}(u_0) + 2\frac{q-p+1}{p} \|\nabla u\|_p^p \\ & +2(q+1) \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x \mathrm{d}\tau. \end{aligned} \tag{3.4}$$

From

$$\int_0^t \left(\left(\frac{u_\tau}{|x|^s}, u \right) + (\nabla u_\tau, \nabla u) \right) \mathrm{d}\tau = \frac{1}{2} \left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 - \frac{1}{2} \left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2} \|\nabla u_0\|_2^2,$$

we derive

$$\begin{gathered} 4 \left(\int_0^t \left(\left(\frac{u_\tau}{|x|^s}, u \right) + (\nabla u_\tau, \nabla u) \right) \mathrm{d}\tau \right)^2 \\ = \left(\left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u\|_2^2 \right) - \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \right)^2 \\ \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u\|_2^2 \right)^2 + \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)^2 \\ = \left((\mathrm{M}'(t))^2 - 2 \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \mathrm{M}'(t) + \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)^2 \right), \end{gathered}$$

then

$$\begin{aligned} (\mathrm{M}'(t))^2 &= 4 \left(\int_0^t (u, \frac{u_\tau}{|x|^s}) d\tau + \int_0^t (\nabla u, \nabla u_\tau) d\tau \right)^2 \\ &\quad + 2 \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \mathrm{M}'(t) - \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)^2 . \end{aligned} \tag{3.5}$$

Therefore, by combining (3.4) and (3.5) we deduce

$$\begin{gathered} \mathrm{M}(t)\mathrm{M}''(t) - \frac{q+1}{2} (\mathrm{M}'(t))^2 \\ \geq \mathrm{M}(t) \begin{pmatrix} -2(q+1)\,\mathrm{J}(u_0) + 2\left(\frac{q-p+1}{p}\right) \|\nabla u\|_p^p \\ +2(q+1) \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x\mathrm{d}\tau \end{pmatrix} \\ -\frac{q+1}{2} \begin{pmatrix} 4 \left(\int_0^t (u, \frac{u_\tau}{|x|^s}) d\tau + \int_0^t (\nabla u, \nabla u_\tau) d\tau \right)^2 \\ -2 \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \mathrm{M}'(t) + \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)^2 \end{pmatrix} \\ \geq -2(q+1)\,\mathrm{J}(u_0)\,\mathrm{M}(t) + 2\left(\frac{q-p+1}{p}\right) \|\nabla u\|_p^p \,\mathrm{M}(t) \\ +2(q+1) \begin{pmatrix} \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x\mathrm{d}\tau \int_0^t \int_\Omega \left(\frac{|u|^2}{|x|^s} + |\nabla u|^2 \right) \mathrm{d}x\mathrm{d}\tau \\ - \left(\int_0^t (u, \frac{u_\tau}{|x|^s}) d\tau + \int_0^t (\nabla u, \nabla u_\tau) d\tau \right)^2 \end{pmatrix} \\ -(q+1) \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \mathrm{M}'(t) + \frac{q+1}{2} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)^2 \\ > -2(q+1)\,\mathrm{J}(u_0)\,\mathrm{M}(t) + 2\left(\frac{q-p+1}{p}\right) \|\nabla u\|_p^p \,\mathrm{M}(t) \\ +2(q+1) \left(\int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x\mathrm{d}\tau - \left(\int_0^t (u, \frac{u_\tau}{|x|^s}) d\tau + \int_0^t (\nabla u, \nabla u_\tau) d\tau \right)^2 \right) \\ -(q+1) \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \mathrm{M}'(t) \end{gathered} \tag{3.6}$$

The Cauchy-Schwarz inequality is used to get

$$\begin{aligned}&\left(\int_0^t (u, \frac{u_\tau}{|x|^s})d\tau + \int_0^t (\nabla u, \nabla u_\tau)d\tau\right)^2\\ \leq\ & \int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2\right) \mathrm{d}x\mathrm{d}\tau \int_0^t \int_\Omega \left(\frac{|u|^2}{|x|^s} + |\nabla u|^2\right) \mathrm{d}x\mathrm{d}\tau,\end{aligned} \tag{3.7}$$

which drives together (3.6)

$$\begin{aligned}&\mathrm{M}(t)\mathrm{M}''(t) - \frac{q+1}{2}\left(\mathrm{M}'(t)\right)^2 > 2\left(\frac{q-p+1}{p}\right)\|\nabla u\|_p^p\,\mathrm{M}(t)\\ &-2\left(q+1\right)\mathrm{J}\left(u_0\right)\mathrm{M}(t) - \left(q+1\right)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\mathrm{M}'(t).\end{aligned} \tag{3.8}$$

In other hand for suitable $\varepsilon > 0$, we have

$$\begin{aligned}2\left(\frac{q-p+1}{p}\right)\|\nabla u\|_p^p &\geq 2c\left(\varepsilon\right)\|\nabla u\|_p^2 - \left(q+1\right)\varepsilon\\ &\geq c\left(\varepsilon\right)\|\nabla u\|_p^2 + c_p c\left(\varepsilon\right)\|\nabla u\|_2^2 - \left(q+1\right)\varepsilon.\end{aligned} \tag{3.9}$$

It can be inferred from (1.3), (H), and (3.9) that

$$2\left(\frac{q-p+1}{p}\right)\|\nabla u\|_p^p\,\mathrm{M}(t) \geq c\left(\varepsilon\right)\min\left(\mathrm{c}_e, c_p\right)\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right)\mathrm{M}(t) - \left(q+1\right)\varepsilon\mathrm{M}(t).$$

Consequently, (3.8) becomes

$$\begin{aligned}&\mathrm{M}(t)\mathrm{M}''(t) - \frac{q+1}{2}\left(\mathrm{M}'(t)\right)^2\\ >\ & c\left(\varepsilon\right)\min\left(\mathrm{c}_e, c_p\right)\mathrm{M}(t)\mathrm{M}'(t) - \left(q+1\right)\left(\varepsilon + 2\mathrm{J}\left(u_0\right)\right)\mathrm{M}(t)\\ &-\left(q+1\right)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\mathrm{M}'(t).\end{aligned} \tag{3.10}$$

Now, we distinguish the following two issues for the level, i.e. $\mathrm{J}\left(u_0\right) \leq 0$ and $0 < \mathrm{J}\left(u_0\right) < d$.

(i) If $\mathrm{J}\left(u_0\right) \leq 0$, choosing $\varepsilon = -\mathrm{J}\left(u_0\right) > 0$, from (3.10) we derive for $\mathrm{c}{=}c\left(\varepsilon\right)\min\left(\mathrm{c}_e, c_p\right)$,

$$\mathrm{M}(t)\mathrm{M}''(t) - \frac{q+1}{2}\left(\mathrm{M}'(t)\right)^2 > \mathrm{c}\mathrm{M}(t)\mathrm{M}'(t) - \left(q+1\right)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\mathrm{M}'(t). \tag{3.11}$$

By combining (2.3), (2.9) and $\mathrm{J}\left(u_0\right) \leq 0$, we arrive at

$$0 \geq \mathrm{J}\left(u_0\right) > \mathrm{J}(u) \geq \left(\frac{1}{p} - \frac{1}{q+1}\right)\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{1}{q+1}\mathrm{I}(u),$$

which means $\mathrm{I(u)} < 0$. Using this and (3.21), we deduce that $\mathrm{M}''(t) > 0$ for $\mathrm{t} \geq 0$, this implies that $\mathrm{M}'(t)$ is increasing with $t \in (0, \infty)$. Since $\mathrm{M}'(0) > 0$ and $\mathrm{M}''(t) > 0$, we conclude that $\mathrm{M}'(t) > \mathrm{M}'(0) > 0$ for $t > 0$. This means $\mathrm{M}(t)$ is increasing over $[0, \infty)$, which leads to $\mathrm{M}(t) > \mathrm{M}(0) = 0$. Consequently, we can say that

$$\mathrm{M}(t) - \mathrm{M}(0) = \int_0^t \mathrm{M}'(\tau)\mathrm{d}\tau > \int_0^t \mathrm{M}'(0)\mathrm{d}\tau = \mathrm{M}'(0)t,$$

that is

$$\mathrm{M}(t)>\mathrm{M}'(0)t,\quad t>0.$$

Therefore, selecting a large enough t, for $\mathrm{M}'(t)>\mathrm{M}'(0)>0$, we find

$$\mathrm{cM}(t)>(q+1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right),$$

which (3.11) have the shape

$$\mathrm{M}(t)\mathrm{M}''(t)-\frac{q+1}{2}\left(\mathrm{M}'(t)\right)^2>\mathrm{M}'(t)\left(\mathrm{cM}(t)-(q+1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)>0.$$

(ii) If $0<\mathrm{J}(u_0)<d$, then Lemma 8 implies $u(t)\in V$ for $t\geq 0$. By (2.11), (2.4) and $0<\mathrm{J}(u_0)<d$, (3.21) becomes

$$\begin{aligned}\mathrm{M}''(t)&=-2\mathrm{I}(\mathrm{u})\\&>2(q+1)(d-\mathrm{J}(\mathrm{u}))\\&\geq 2(q+1)\left(d-\mathrm{J}(u_0)+2(q+1)\int_0^t\int_\Omega\left(\frac{|u_\tau|^2}{|x|^s}+|\nabla u_\tau|^2\right)\mathrm{d}x\mathrm{d}\tau\right)\\&>2(q+1)(d-\mathrm{J}(u_0))=:\mathrm{C}>0.\end{aligned}\tag{3.12}$$

Therefore, (3.12) and $\mathrm{M}'(0)>0$ gives

$$\mathrm{M}'(t)-\mathrm{M}'(0)=\int_0^t\mathrm{M}''(\tau)\mathrm{d}\tau>\mathrm{C}t,\quad 0<t<\infty,$$

that is

$$\mathrm{M}'(\mathrm{t})>\mathrm{C}t+\mathrm{M}'(0)>\mathrm{C}t.\tag{3.13}$$

By the same manner, since $\mathrm{M}''(t)>0$, $\mathrm{M}(0)=0$ and (3.13), for $t\in(0,\infty)$ we conclude

$$\mathrm{M}(t)-\mathrm{M}(0)=\int_0^t\mathrm{M}'(\tau)\mathrm{d}\tau>\int_0^t\mathrm{C}\tau\mathrm{d}\tau=\frac{1}{2}\mathrm{C}t^2,$$

i.e.,

$$\mathrm{M}(t)>\frac{1}{2}\mathrm{C}t^2+\mathrm{M}(0)=\frac{1}{2}\mathrm{C}t^2.\tag{3.14}$$

Thus, the fact $\mathrm{M}'(t)>\mathrm{M}'(0)>0$, (3.13) and (3.14), for sufficiently large t, gives

$$\frac{1}{2}\mathrm{cM}(t)>(q+1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right),\tag{3.15}$$

and

$$\frac{1}{2}\mathrm{cM}'(t)>(\varepsilon+2\mathrm{J}(u_0))(q+1)\mathrm{J}(u_0).\tag{3.16}$$

Therefore, using (3.15) and (3.16), (3.10), for sufficiently large t, becomes

$$\begin{aligned}&\mathrm{M}(t)\mathrm{M}''(t)-\frac{q+1}{2}\left(\mathrm{M}'(t)\right)^2\\&\geq\left(\frac{1}{2}\mathrm{cM}(t)-(q+1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\mathrm{M}'(t)\\&+\left(\frac{1}{2}\mathrm{cM}'(t)-(q+1)(\varepsilon+2\mathrm{J}(u_0))\mathrm{J}(u_0)\right)\mathrm{M}(t)>0.\end{aligned}\tag{3.17}$$

Because of $\mathrm{M}(t)$, $\mathrm{M}'(t)$ and $\mathrm{M}''(t)$ are all positive for sufficiently large t_*, then (3.17) leading to

$$\frac{\mathrm{M}''(t)}{\mathrm{M}'(t)} > \frac{(q+1)\,\mathrm{M}'(t)}{2\mathrm{M}(t)},\ t \in [t_*, \infty).$$

When we integrate the aforementioned inequality on (t_*, t), we obtain

$$\int_{t_*}^{t} \frac{\mathrm{dM}'(\tau)}{\mathrm{M}'(\tau)} > \frac{q+1}{2} \int_{t_*}^{t} \frac{\mathrm{dM}(\tau)}{\mathrm{M}(\tau)},$$

which give

$$\frac{\mathrm{M}'(t)}{(\mathrm{M}(t))^{\frac{q+1}{2}}} > \frac{\mathrm{M}'(t_*)}{(\mathrm{M}(t_*))^{\frac{q+1}{2}}}.$$

Reintegrating on (t_*, t) yields

$$\mathrm{M}(t)^{-\frac{q-1}{2}}(\mathrm{t}) > \mathrm{M}(t_*)^{-\frac{q-1}{2}} \left(1 - \frac{(q-1)\mathrm{M}'(t_*)}{2\mathrm{M}(t_*)}(t - t_*)\right),$$

i.e.,

$$\mathrm{M}(t) > \mathrm{M}(t_*) \left(1 - \frac{(q-1)\mathrm{M}'(t_*)}{2\mathrm{M}(t_*)}(t - t_*)\right)^{-\frac{2}{q-1}}. \tag{3.18}$$

Note that, for the duration $\bar{t}$

$$0 < \bar{t} \le t_* + \frac{2\mathrm{M}(t_*)}{(q-1)\mathrm{M}'(t_*)},$$

we have

$$\lim_{t \to \bar{t}} \mathrm{M}(t) = +\infty,$$

this contradicts the assumption $T_{\max} = +\infty$. ■

To estimate the upper bound of the blow-up time, we need to consider the following lemmas

Lemma 10 *Suppose that a positive, twice-differentiable function $\varphi(t)$ satisfies the inequality*

$$\varphi''(t)\varphi(t) - (1+\theta)\left(\varphi'(t)\right)^2 \ge 0, \quad t > 0,$$

where $\theta > 0$ is some constant. If $\varphi(0) > 0$ and $\varphi'(0) > 0$, then there exists $0 < \mathrm{t}_1 \le \frac{\varphi(0)}{\theta\varphi'(0)}$ such that $\varphi(t)$ tends to infinity as $t \to t_1$.

To illustrate the finite-time blow-up indicated by Theorem 9, we utilize an alternative auxiliary function. Furthermore, we evaluate the upper bound of the blow-up time, noting that two separate proofs leading to the same conclusion regarding finite-time blow-up results are provided in 9 and 11.

Theorem 11 *Assuming (p, q) satisfy (H) and $u_0 \in W_0^{1,p}(\Omega)$ such that $\mathrm{J}(u_0) < d$ and $\mathrm{I}(u_0) < 0$. Then the weak solution $u(t)$ of problem (1.2) blows up in finite time. The upper bound of the blow-up time estimated as follows.*

$$T \le \frac{2\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)}{(q-1)^2\beta},$$

where $0 < \beta \le \frac{(q+1)(d - \mathrm{J}(u_0))}{q}$ is a constant.

Beweis. We shall use contradiction to demonstrate that time is finite. Assume that $T_{\max} = +\infty$. The positive function is defined for a sufficient $T_0 > 0$

$$\mathrm{F}(t) := \frac{1}{2}\int_0^t \left(\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u\|_2^2 \right) \mathrm{d}\tau$$

$$+\frac{1}{2}(T_0 - t)\left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) + \frac{1}{2}\beta(t+t_0)^2 \text{ for } t \in [0, T_0]\,,\ t_0 > 0, \tag{3.19}$$

where t_0, and T_0 are positive constants to be determined later.Applying $\mathrm{J}(u)$, $\mathrm{I}(u)$ and (2.9), definitions, we obtain

$$\begin{aligned} \mathrm{J}(u) &\geq \left(\frac{1}{p} - \frac{1}{q+1}\right)\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{q+1}\mathrm{I}(u) \\ &= \frac{1}{q+1}\mathrm{I}(u) + \frac{q-p+1}{p\,(q+1)}\|\nabla u\|_p^p\,, \end{aligned}$$

that is

$$\mathrm{I}(u) \leq (q+1)\,\mathrm{J}(u) - \frac{q-p+1}{p}\|\nabla u\|_p^p\,. \tag{3.20}$$

When we substitute $\nu = u_{\mathrm{s}}$ in equation (2) for $\nu = u$ in (2.3), we obtain

$$\begin{aligned} \mathrm{F}''(\mathrm{t}) &= 2\left(u, \frac{u_t}{|x|^s}\right) + 2\,(\nabla u, \nabla u_t) \\ &= 2\left(|u|^{q-1}u, u\right) - 2\,\|\nabla u\|_p^p \\ &= -2\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega |u|^{q+1}\,\mathrm{d}x\right) = -2I(u), \end{aligned} \tag{3.21}$$

$$\left(u, \frac{u_t}{|x|^s}\right) + (\nabla u, \nabla u_t) = -\mathrm{I}(u). \tag{3.22}$$

For each $t \in [0, T)$, we may infer from (3.19)-(3.22) and (2.4) that

$$\mathrm{F}'(t) = \frac{1}{2}\left\| \frac{u}{|x|^{\frac{s}{2}}} \right\|_2^2 - \frac{1}{2}\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\|\nabla u_0\|_2^2 + \beta(t+t_0)$$

$$= \int_0^t \left(u, \frac{u_\tau}{|x|^s}\right)\mathrm{d}\tau + \int_0^t (\nabla u, \nabla u_\tau)\,\mathrm{d}\tau + \beta(t+t_0), \tag{3.23}$$

and

$$\begin{aligned} \mathrm{F}''(t) &= \left(u, \frac{u_t}{|x|^s}\right) + (\nabla u, \nabla u_\tau) + \beta = -I + \beta \\ &\geq \frac{q-p+1}{p}\|\nabla u\|_p^p - (q+1)\,\mathrm{J}\,(u_0) \\ &\quad + (q+1)\int_0^t \int_\Omega \left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2 \right) \mathrm{d}x\mathrm{d}\tau + \beta. \end{aligned} \tag{3.24}$$

Therefore, it may be inferred from (3.19) and (3.23) that

$$
\begin{aligned}
\mathrm{FF}'' - \alpha\left(\mathrm{F}'\right)^2 &\geq \mathrm{FF}'' - \alpha\left(\int_0^t\left(u, \frac{u_\tau}{|x|^s}\right)\mathrm{d}\tau + \int_0^t\left(\nabla u, \nabla u_\tau\right)\mathrm{d}\tau + \beta(t+t_0)\right)^2 \\
&= \mathrm{FF}'' - \alpha\left(\int_0^t\left(u, \frac{u_\tau}{|x|^s}\right)\mathrm{d}\tau + \int_0^t\left(\nabla u, \nabla u_\tau\right)\mathrm{d}\tau + \beta(t+t_0)\right)^2 \\
+\alpha&\left(\int_0^t\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau + \int_0^t\|\nabla u\|_2^2 d\tau + \beta(t+t_0)^2\right)\left(\int_0^t\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau + \int_0^t\|\nabla u_\tau\|_2^2 d\tau + \beta\right) \\
-\alpha&\left(2\mathrm{F}-(T_0-t)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\right)\left(\int_0^t\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau + \int_0^t\|\nabla u_\tau\|_2^2 d\tau + \beta\right).
\end{aligned} \tag{3.25}
$$

Young's inequality and (3.7) allow us to determine that for any $t \in [0, T)$,

$$
\begin{aligned}
&\left(\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right)d\tau + \beta(t+t_0)^2\right)\left(\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)d\tau + \beta\right) \\
&\qquad - \left(\int_0^t\left(\left(u, \frac{u_\tau}{|x|^s}\right) + \left(\nabla u, \nabla u_\tau\right)\right)\mathrm{d}\tau + \beta(t+t_0)\right)^2 \\
&= \int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right)d\tau \int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)d\tau \\
&\qquad - \left(\int_0^t\left(\left(u, \frac{u_\tau}{|x|^s}\right) + \left(\nabla u, \nabla u_\tau\right)\right)\mathrm{d}\tau\right)^2 \\
&\quad + \left(\begin{array}{c}\beta\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right)d\tau + \beta(t+t_0)^2\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)d\tau \\ -2\beta(t+t_0)\int_0^t\left(\left(u, \frac{u_\tau}{|x|^s}\right) + \left(\nabla u, \nabla u_\tau\right)\right)\mathrm{d}\tau\end{array}\right) \\
&\geq 2\beta(t+t_0)\left(\int_0^t\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right)d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)d\tau\right)^{\frac{1}{2}} \\
&\qquad -2\beta(t+t_0)\int_0^t\left(\left(u, \frac{u_\tau}{|x|^s}\right) + \left(\nabla u, \nabla u_\tau\right)\right)\mathrm{d}\tau \geq 0.
\end{aligned} \tag{3.26}
$$

After then, (3.26) and (3.24), (3.25) become

$$
\begin{aligned}
\mathrm{FF}'' - \alpha\left(\mathrm{F}'\right)^2 &\geq \mathrm{FF}'' - \alpha\left(2\mathrm{F}-(T_0-t)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\right) \\
&\quad\times\left(\int_0^t\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau + \int_0^t\|\nabla u_\tau\|_2^2 d\tau + \beta\right) \\
&\geq \mathrm{F}\left(\mathrm{F}'' - 2\alpha\left(\int_0^t\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 d\tau + \int_0^t\|\nabla u_\tau\|_2^2 d\tau + \beta\right)\right) \\
&\geq \mathrm{F}\left(\begin{array}{c}\frac{q-p+1}{p}\|\nabla u\|_p^p - (q+1)\,\mathrm{J}\left(u_0\right) \\ +\left((q+1)-2\alpha\right)\int_0^t\int_\Omega\left(\frac{|u_\tau|^2}{|x|^s} + |\nabla u_\tau|^2\right)\mathrm{d}x\mathrm{d}\tau - \beta\left(2\alpha-1\right)\end{array}\right).
\end{aligned} \tag{3.27}
$$

Let $\alpha = \frac{q+1}{2}$, (3.27) gives

$$
\mathrm{FF}'' - \alpha\left(\mathrm{F}'\right)^2 \geq \mathrm{F}\left(\frac{q-p+1}{p}\|\nabla u\|_p^p - (q+1)\,\mathrm{J}\left(u_0\right) - \beta q\right),\ t \in [0, T).
$$

Lemma 5 and (2.10) allow us to use property (ii) to infer that

$$\frac{q-p+1}{p}\|\nabla u\|_p^p > \frac{q-p+1}{p} r^p = (q+1)\, d,$$

picking $0<\beta<\frac{(q+1)(d-\mathrm{J}(u_0))}{q}$ to find

$$\mathrm{FF}'' - \alpha\left(\mathrm{F}'\right)^2 \geq \mathrm{F}\left((q+1)\left(d-\mathrm{J}\left(u_0\right)\right)-\beta q\right)>0,\ \ t\in[0,T_0], \tag{3.28}$$

Consequently, T cannot be infinite, indicating that there is never a weak solution, and

$$0<T\leq \frac{2\mathrm{F}(0)}{(q-1)\mathrm{F}'(0)} = \frac{T_0\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)+\beta {t_0}^2}{(q-1)\beta t_0}. \tag{3.29}$$

Let us select suitable values for t_0 and T_0. Any value that solely depends on p, q, β and u_0 can be used to set t_0.

$$t_0 > \frac{\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2}{(q-1)\beta}.$$

If t_0 remains constant, it can be selected T_0 as

$$T_0 = \frac{T_0\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)+\beta {t_0}^2}{(q-1)\beta t_0},$$

in order for

$$T_0 = \frac{\beta t_0^2}{(q-1)\beta t_0 - \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)}.$$

A specific value limits the lifespan of the solution $u(x,t)$ as

$$T_0 = \inf_{t\geq t_0}\frac{\beta t^2}{(q-1)\beta t - \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)} = \frac{2\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)}{(q-1)^2\beta}. \tag{3.30}$$

Because $T_0<T$ is arbitrary, it follows that

$$T\leq \frac{2\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)}{(q-1)^2\beta}.$$

■

The lower bound for the blow-up time will be estimated. The requirements outlined in Theorem 9, Theorem 11, and Theorem 12 will be utilized to achieve this.

Theorem 12 (Lower bound of blow-up time) *Let $p<q+1<p+\frac{2p}{n}$. Assume that $\mathrm{J}\left(u_0\right)<d$ and $\mathrm{I}\left(u_0\right)<0$. The lower bound of the blow-up time estimated as follows*

$$\bar{t} > \frac{\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2\right)^{2-\eta(q+1)}}{\left(\eta\left(q+1\right)-2\right)|\Omega|^{\eta(q+1)}\,\mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}}},$$

where c_g is the constant of Gagiiardo-Nirenbery's inequality

$$\|u\|_{q+1}\leq \mathrm{c}_g\,\|\nabla u\|_2^{\theta}\,\|u\|_2^{1-\theta},$$

$$\theta = \frac{(q-1)\,n}{(pn+2p-2n)\,(q+1)}\in(0,1)\,,\ \textit{and}\ \eta=\frac{p\,(1-\theta)}{p-(q+1)\,\theta}>1.$$

Beweis. Remember that the u solution to issue (1.2) blows up in finite time, as stated in Theorem 9, in the sense that $\lim_{t\to\bar{t}}\int_0^t \|\frac{u}{|x|^{\frac{s}{2}}}\|_2^2 \mathrm{d}\tau = +\infty$, i.e.

$$\lim_{t\to\bar{t}} \|u\|_2^2 = +\infty. \tag{3.31}$$

We have $\mathrm{I}(u) < 0$ by Lemma 8, (3.9) and (2.8) which means that $\|\nabla u\|_p^p < \|u\|_{q+1}^{q+1}$. After that, we combine Gagliardo-Niremberg's inequality to get

$$\|u\|_{q+1} \le \mathrm{c}_g \|\nabla u\|_p^{\theta} \|u\|_2^{(1-\theta)} < \mathrm{c}_g \|u\|_{q+1}^{\frac{q+1}{p}\theta} \|u\|_2^{(1-\theta)},$$

which arrives

$$\|u\|_{q+1} < \mathrm{c}_g^{\frac{p}{p-(q+1)\theta}} \|u\|_2^{\eta}, \tag{3.32}$$

where $\eta = \frac{p(1-\theta)}{p-(q+1)\theta} > 1$, $\theta = \frac{(q-1)n}{(pn+2p-2n)(q+1)} \in (0,1)$ and $\left(\frac{q+1}{p}\right)\theta < 1$ due to $p < q+1 < p + \frac{2p}{n}$. Substituting (3.32) into (3.21), yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) &\le -2\mathrm{I}(u) = 2\int_\Omega |u|^{q+1}\,\mathrm{d}x - 2\int_\Omega |\nabla u|^p\,\mathrm{d}x \\
&< 2\int_\Omega |u|^{q+1}\,\mathrm{d}x \le 2\mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}} \|u\|_2^{\eta(q+1)} \\
&\le 2\mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}} \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{\eta(q+1)}{2}} \\
&\le 2|\Omega|^{\eta(q+1)} \mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}} \left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{\eta(q+1)}{2}}.
\end{aligned}$$

Following the resolution of the aforementioned differential inequality, we have acquired

$$\begin{aligned}
&\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right)^{2-\eta(q+1)} + (\eta(q+1)-2)|\Omega|^{\eta(q+1)} \mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}} t \\
&\quad > \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^{2-\eta(q+1)}.
\end{aligned}$$

Since (3.31) and $(q+1)\eta > 2$, letting $t \to \bar{t}$, we have

$$\bar{t} > \frac{\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^{2-\eta(q+1)}}{(\eta(q+1)-2)|\Omega|^{\eta(q+1)} \mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}}} > 0.$$

■

4. Global existence, asymptotic behavior and blow-up in finite time with $\mathrm{J}(u_0) = d$

To extend the results from the subcritical initial energy $\mathrm{J}(u_0) < d$ to the critical initial energy $\mathrm{J}(u_0) = d$, we will only discuss the theories of global existence, asymptotic behavior, and finite-time blow-up. The proof is omitted and referenced in [22, Theorem 2.] and [23, Theorem 5.3.], with only minor modifications.

Theorem 13 (Global existence for $J(u_0)=d$) *Let (p,q) satisfy condition (H) and let $u_0 \in W_0^{1,p}(\Omega)$. Suppose that $\mathrm{J}(u_0)=d$ and $\mathrm{I}(u_0)\geq 0$. Then the problem (1.2) admits a global weak solution $u(t)\in L^\infty\left(0,\infty;W_0^{1,p}(\Omega)\right)$ with $u_t\in L^2\left(0,\infty;L^2(\Omega)\right)$.*

Theorem 14 (Asymptotic behavior of solution for $J(u_0)=d$) *Let (p,q) satisfy (H), $u_0\in W_0^{1,p}(\Omega)$. Assume that $\mathrm{J}(u_0)=d$ and $\mathrm{I}(u_0)>0$. Then for the global weak solution $u(x,t)$ of problem (1.2), there exist constants $t_1>0$ and $\kappa>0$ such that*

$$\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u\|_2^2\leq\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)e^{-2\kappa(t-t_1)}.$$

for $t\in(t_1,+\infty)$.

Theorem 15 (Blow-up for $J(u_0)=d$) *Assuming that q satisfies (H) and $u_0\in W_0^{1,p}(\Omega)$, if $\mathrm{J}(u_0)=d$ and $\mathrm{I}(u_0)<0$, then the weak solution for problem (1.2) blows up in a finite time.*

Beweis. Firstly, using (3.2)-(3.10) and $\mathrm{J}(u_0)=d$, we obtain

$$\begin{aligned}&\mathrm{M}(t)\mathrm{M}''(t)-\frac{q+1}{2}\left(\mathrm{M}'(t)\right)^2\\ &\geq\left(\frac{1}{2}\mathrm{c}\mathrm{M}(t)-(q+1)\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\mathrm{M}'(t)\\ &\quad+\left(\frac{1}{2}\mathrm{c}\mathrm{M}'(t)-(q+1)(\varepsilon+2d)\,\mathrm{J}(u_0)\right)\mathrm{M}(t).\end{aligned}$$

Since both $\mathrm{J}(u)$ and $\mathrm{I}(u)$ are continuous in t, it follows from $\mathrm{J}(u_0)=d>0$, $\mathrm{I}(u_0)<0$ that there exists a $t_1>0$ small enough that $\mathrm{J}(u(t_1))>0$ and $\mathrm{I}(u)<0$ for $t\in[0,t_1)$. Equation (3.22) combined with this gives us $\left(u,\frac{u_t}{|x|^2}\right)+(\nabla u,\nabla u_t)=-\mathrm{I}(u)>0$ for $t\in[0,t_1]$, indicating that $u_t\neq 0$. Equation (2.4) allows us to further deduce that

$$0<\mathrm{J}(u(t_1))\leq d-\int_0^{t_1}\int_\Omega\left(\frac{|u_\tau|^2}{|x|^s}+|\nabla u_\tau|^2\right)\mathrm{d}x\mathrm{d}\tau=d_1<d.$$

We obtain $u\in V$ for $0<t<\infty$ by setting $t=t_1$ as a new initial time. The remainder of the proof is comparable to Theorem 9. ∎

Theorem 16 *Assuming $p<q+1<p+\frac{2p}{n}$ and $\mathrm{J}(u_0)=d$, with $\mathrm{I}(u_0)<0$, we have a lower bound estimate for the blow-up time of the solution to problem (1.2)*

$$\bar{t}>\frac{\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)^{2-\eta(q+1)}}{(\eta(q+1)-2)\,|\Omega|^{\eta(q+1)}\,\mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}}},$$

where c_g, η, and θ as in in Theorem 12.

Beweis. The solution of issue (1.2) blows up in finite time $T>0$ and $\mathrm{I}(u)<0$ for $0<t<T$, as per Theorem 15. Theorem 12 is comparable to the continuing proof. ∎

5. Blow-up and blow-up time with High (sup-critical) initial energy $\mathrm{J}(u_0)>0$

We will demonstrate that the solution to problem (1.2) has a finite time blow-up in this section. We will employ the concave function method to establish the upper bound of the blow-up time for large initial energy. To support our main findings, we will utilize the following lemma.

Lemma 17 *Assuming that* u_0 *belongs to* $W_0^{1,p}(\Omega)$ *and satisfies*

$$\mathrm{J}\left(u_0\right) < \mathrm{A}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right), \tag{5.1}$$

where $\mathrm{A} = \frac{\mathrm{c}}{4p(q+1)}$. *Then* $u \in \mathcal{N}_- = \left\{u \in W_0^{1,p}(\Omega) \mid \mathrm{I}(u) < 0\right\}$.

Beweis. Let $u(t)$ represent issue (1.2)'s weak solution. By taking $\varepsilon = \frac{1}{(q+1)}\mathrm{I}(u_0)$ in (3.9) and applying the definition of $\mathrm{J}(u)$, (2.9), (2.7), and (3.9), we may infer

$$\begin{aligned} \mathrm{J}(u_0) &\geq \frac{1}{q+1}\mathrm{I}(u_0) + \frac{1}{2\,(q+1)}c\,(\varepsilon)\,\|\nabla u_0\|_p^2 - \frac{1}{2}\varepsilon \\ &\geq \frac{1}{q+1}\mathrm{I}(u_0) + \frac{\mathrm{c}}{2\,(q+1)}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right) - \frac{1}{2}\varepsilon \\ &\geq \frac{1}{2\,(q+1)}\mathrm{I}(u_0) + \mathrm{A}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right), \end{aligned}$$

(5.1) causes $\mathrm{I}\left(u_0\right) < 0$. Next, we demonstrate that for every $t \in [0, T)$, $u(t) \in \mathcal{N}_-$. Assuming that there is a $s \in (0, T)$ such that $u(t) \in \mathcal{N}_-$ for $0 \leq t < s$ and $u(s) \in \mathcal{N}$, by contradiction and utilizing the continuity of $\mathrm{I}(u)$ in t, (3.22) entails

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) = -2\mathrm{I}(u) > 0 \quad \text{for } t \in [0, s),$$

which give

$$\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2 < \left\|\frac{u\,(s)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\,(s)\|_2^2. \tag{5.2}$$

It is evident from Lemma $6_{(ii)}$ that

$$\mathrm{J}(u(s)) < \mathrm{J}\left(u_0\right). \tag{5.3}$$

From the definition of $\mathrm{J}(u)$, $u(s) \in \mathcal{N}$, (2.7), (3.9) and (5.2), we stem

$$\begin{aligned} \mathrm{J}(u(s)) &\geq \frac{1}{q+1}\mathrm{I}(u(s)) + \frac{q-p+1}{p\,(q+1)}\,\|\nabla u(s)\|_p^p \\ &= \frac{q-p+1}{p\,(q+1)}\,\|\nabla u(s)\|_p^p \\ &\geq \frac{\mathrm{c}}{2\,(q+1)}\left(\left\|\frac{u\,(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) \\ &\geq \frac{\mathrm{c}}{4p\,(q+1)}\left(\left\|\frac{u\,(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) \\ &= \mathrm{A}\left(\left\|\frac{u\,(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right), \end{aligned}$$

then further combining (5.1) and (5.3), we get

$$\mathrm{A}\left(\left\|\frac{u\,(s)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\,(s)\|_2^2\right) \leq \mathrm{J}(u(s)) < \mathrm{J}\left(u_0\right) < \mathrm{A}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right),$$

This runs counter to (5.2). ■

Next, we demonstrate that the solution blows up in finite time when $\mathrm{J}(u_0) > 0$. With the assistance of Lemma 10 and Theorem 12, we estimate the upper and lower bounds of the blow-up time.

Theorem 18 *Assuming $u_0 \in W_0^{1,p}(\Omega)$, and $\mathrm{J}(u_0) > 0$, let (p, q) satisfy (H) and (5.1) hold. Then, the solution $u(x, t)$ of problem (1.2) blows up in finite time. An upper bound estimate of the blow-up time is provided,*

$$0 < t_* \leq \frac{c}{(\alpha - 1)\varepsilon_2^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2},$$

where $1 < \alpha < \frac{\mathrm{A}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)}{\mathrm{J}(u_0)}$, *and* $c > \frac{1}{4}\varepsilon^{-2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2$.

Beweis. Two ideas from [4] and [5] were combined in this proof, and they were modified to deal with the singularity term.

Before demonstrating that this results in a contradiction, we first suppose that u exists in the classical sense on $\Omega \times [0, \infty)$ i.e., $T_{\max} = +\infty$ (The interval of existence of u is unbounded, or u is defined in the complete interval $(0, +\infty)$), and then, we demonstrate that this results in a contradiction with that condition (5.1). We display a $\varphi(t)$ with the following structure.

$$\varphi(t) := \int_0^t \left(\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) \mathrm{d}\tau, \quad \text{for } 0 < t < \infty,$$

then we have

$$\varphi'(t) = \left(\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) \quad \text{for all } t \in [0, \infty).$$

The definition of $\mathrm{J}(u)$, $\mathrm{I}(u)$, and (3.22) give us

$$\begin{aligned}
\varphi''(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) = -2\mathrm{I}(u) \\
&= -2\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x - \int_\Omega |u|^{q+1} \,\mathrm{d}x\right) \\
&\geq -p\left(\frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{1}{q+1}\int_\Omega |u|^{q+1} \,\mathrm{d}x\right) \\
&\quad + \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1} \,\mathrm{d}x \\
&\geq -p\mathrm{J}(u) + \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1} \,\mathrm{d}x. \qquad (5.4)
\end{aligned}$$

We differentiate between two situations:

Case 1 $\mathrm{J}(u) \geq 0$ for all $t > 0$. (5.1) allows us to select α as such.

$$1 < \alpha < \frac{\mathrm{A}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)}{\mathrm{J}(u_0)}. \qquad (5.5)$$

When we inject (2.4) into (5.4), we obtain

$$\begin{aligned}\varphi''(t) &= p(\alpha-1)\mathrm{J}(u) - p\alpha\mathrm{J}(u) \\ &\quad + \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x \\ &> -p\alpha\mathrm{J}(u) + \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x \\ &\geq -p\alpha\mathrm{J}(u_0) + p\alpha\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \\ &\quad + \frac{q-p+1}{q+1}\int_\Omega |u|^{q+1}\,\mathrm{d}x. \end{aligned} \tag{5.6}$$

By applying Lemma 17, (which states that $\mathrm{I}(u) < 0$), and (3.22), we arrive to

$$\varphi''(t) = -2\mathrm{I}(u) > 0. \tag{5.7}$$

turning (3.9) and (5.7), (5.6) into

$$\begin{aligned}\varphi''(t) &> -.p\alpha\mathrm{J}(u_0) + 2p\alpha\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \\ &\quad + \frac{q-p+1}{q+1}\|\nabla u\|_p^p \\ &\geq -p\alpha\mathrm{J}(u_0) + p\alpha\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \\ &\quad + \frac{p\mathrm{c}}{2(q+1)}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) - \frac{p}{2}\varepsilon \\ &\geq -p\alpha\mathrm{J}(u_0) + p\alpha\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \\ &\quad + 2p\mathrm{A}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) - \frac{p}{2}\varepsilon. \end{aligned} \tag{5.8}$$

By using $\varepsilon = 2\alpha\mathrm{J}(u_0)$, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) - 2p\mathrm{A}\left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2\right) > -2p\alpha\mathrm{J}(u_0).$$

This, when resolved, provides

$$\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u\|_2^2 > \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)e^{\mathrm{A}t} + \frac{\alpha}{\mathrm{A}}\mathrm{J}(u_0)\left(1 - e^{2p\mathrm{A}t}\right). \tag{5.9}$$

Putting (5.9) into (5.8) results in

$$\begin{aligned}\varphi''(t) &> -2p\alpha \mathrm{J}(u_0) + p\alpha \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_\tau\|_2^2 \right) \mathrm{d}\tau \\ &\quad +2p\mathrm{A} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) e^{2p\mathrm{A}t} \\ &\quad +2p\alpha \mathrm{J}(u_0)\left(1 - e^{2p\mathrm{A}t}\right) \\ &= 2p\left(\mathrm{A}\left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) - \alpha \mathrm{J}(u_0) \right) e^{\mathrm{A}t} \\ &\quad + p\alpha \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_\tau\|_2^2 \right) \mathrm{d}\tau. \end{aligned} \tag{5.10}$$

We can infer from (5.5) that $\varepsilon > 0$ so that

$$0 < \varepsilon < \frac{\mathrm{A}\left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) - \alpha \mathrm{J}(u_0)}{\alpha \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)},$$

which when (5.10) is combined, results in

$$\varphi''(t) > 2p\alpha\varepsilon \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) e^{\mathrm{A}t} + p\alpha \int_0^t \left(\left\| \frac{u_\tau}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_\tau\|_2^2 \right) \mathrm{d}\tau. \tag{5.11}$$

Assume that the auxiliary function ϕ is defined as

$$\phi(t) := \varphi^2(t) + \varepsilon^{-1} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \varphi(t) + \gamma,$$

and $\gamma > 0$ sufficiently large (if required), so that

$$4\varepsilon^2 \gamma > \left(\varphi'(0)\right)^2. \tag{5.12}$$

then

$$\phi'(t) = \left(2\varphi(t) + \varepsilon^{-1} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \right) \varphi'(t), \tag{5.13}$$

and

$$\phi''(t) = \left(2\varphi(t) + \varepsilon^{-1} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \right) \varphi''(t) + 2\left(\varphi'(t)\right)^2. \tag{5.14}$$

From (5.13), we derive

$$\begin{aligned}\left(\phi'(t)\right)^2 &= \left(2\varphi(t) + \varepsilon^{-1} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \right)^2 \left(\varphi'(t)\right)^2 \\ &= \begin{pmatrix} 4\varphi^2(t) + 4\varepsilon^{-1} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right) \varphi(t) \\ +\varepsilon^{-2} \left(\left\| \frac{u_0}{|x|^{\frac{s}{2}}} \right\|_2^2 + \|\nabla u_0\|_2^2 \right)^2 \end{pmatrix} \left(\varphi'(t)\right)^2. \end{aligned}$$

Let $\delta := 4\gamma - \varepsilon^{-2}\left(\varphi'(0)\right)^2 > 0$, then

$$(\phi'(t))^2 = \begin{pmatrix} 4\varphi^2(t) + 4\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\varphi(t) \\ +4\gamma - \delta \end{pmatrix}(\varphi'(t))^2 \\ = (4\phi(t) - \delta)(\varphi'(t))^2, \tag{5.15}$$

i.e.,

$$4\phi(t)(\varphi'(t))^2 = (\phi'(t))^2 + \delta(\varphi'(t))^2. \tag{5.16}$$

Observing that

$$\int_0^t \left(\left(\frac{u_t(s)}{|x|^s}, u\right) + (\nabla u_t(s), \nabla u)\right) \mathrm{d}s = \frac{1}{2}\int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}s}\left(\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u(s)\|_2^2\right)\right) \mathrm{d}s \\ = \frac{1}{2}\left(\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u(t)\|_2^2\right) - \frac{1}{2}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right).$$

Therefore,

$$\left(\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u(t)\|_2^2\right) = \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right) \\ +2\int_0^t \int_\Omega \left(\frac{u_t(s)}{|x|^s} u + \nabla u_t(s) \nabla u\right) \mathrm{d}x\mathrm{d}s.$$

Applying Hölder's and Young's inequalities provides

$$(\varphi'(t))^2 = \left(\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u(t)\|_2^2\right)^2$$

$$= \left(\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right) + 2\int_0^t \int_\Omega \left(\frac{u_t(s)}{|x|^s} u + \nabla u_t(s) \nabla u\right) \mathrm{d}x\mathrm{d}s\right)^2$$

$$\leq \begin{pmatrix} \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2 + 2\left(\int_0^t \left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \\ +2\left(\int_0^t \|\nabla u\|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u_\tau\|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \end{pmatrix}^2$$

$$\leq \begin{pmatrix} \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2 \\ +2\left(\int_0^t \left(\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2 + \int_0^t \|\nabla u\|_2^2\right) \mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right) \mathrm{d}\tau\right)^{\frac{1}{2}} \end{pmatrix}^2$$

$$= \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2$$

$$+4\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)(\varphi(t))^{\frac{1}{2}} \left(\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right) \mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$+4\varphi(t)\left(\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right) \mathrm{d}\tau\right)$$

$$\leq \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2 + 4\varphi(t)\int_0^t \left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right) \mathrm{d}\tau$$

$$+2\varepsilon\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\varphi(t)$$

$$+2\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_\tau\|_2^2\right)d\tau. \tag{5.17}$$

From (5.14) and (5.16), we get

$$\begin{aligned}
&2\phi(t)\phi''(t)\\
=\;&2\begin{pmatrix}\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)\\ +2\left(\varphi'(t)\right)^2\end{pmatrix}\phi(t)\\
=\;&2\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)\phi(t)\\
&+4\left(\varphi'(t)\right)^2\phi(t)\\
=\;&2\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)\phi(t)\\
&+\left(\phi'(t)\right)^2+\delta\left(\varphi'(t)\right)^2.
\end{aligned}\tag{5.18}$$

From (5.18), (5.15) and the value of δ, we now get

$$\begin{aligned}
&2\phi(t)\phi''(t)-(1+\alpha)\left(\phi'(t)\right)^2\\
&=2\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)(t)\phi(t)\\
&\quad+\left(\phi'(t)\right)^2+\delta\left(\varphi'(t)\right)^2-(1+\alpha)\left(\phi'(t)\right)^2\\
&=2\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)\phi(t)\\
&\quad-\alpha\left(\phi'(t)\right)^2+\delta\left(\varphi'(t)\right)^2\\
&=2\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)\phi(t)\\
&\quad-\alpha(4\phi(t)-\delta)\left(\varphi'(t)\right)^2+\delta\left(\varphi'(t)\right)^2\\
&=2\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)\phi(t)\\
&\quad-4\alpha\phi(t)\left(\varphi'(t)\right)^2+\delta(1+\alpha)\left(\varphi'(t)\right)^2\\
&>2\phi(t)\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)\\
&\quad-4\alpha\phi(t)\left(\varphi'(t)\right)^2\\
&=2\phi(t)\left(\left(2\varphi(t)+\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2+\|\nabla u_0\|_2^2\right)\right)\varphi''(t)-2\alpha\left(\varphi'(t)\right)^2\right).
\end{aligned}$$

Remembering (5.11), (5.17), (1.3) as well as the fact that $e^{\mathrm{A}t} > 1$and $p > 2$ leads to

$$\begin{aligned}
&\left(2\varphi(t) + \varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\right)\varphi''(t) - 2\alpha\left(\varphi'(t)\right)^2 \\
&> \left(2\varphi(t) + \varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\right)\left(\begin{array}{c} 2p\alpha\varepsilon\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)e^{\mathrm{A}t} \\ +p\alpha\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \end{array}\right) \\
&\qquad -2\alpha\left(\varphi'(t)\right)^2 \\
&> 2\alpha\left(2\varphi(t) + \varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\right)\left(\begin{array}{c} 2\varepsilon\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right) \\ +2\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \end{array}\right) \\
&\qquad -2\alpha\left(\varphi'(t)\right)^2 \\
&= 2\alpha\left(\begin{array}{c} 2\varepsilon\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\varphi(t) + \left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2 \\ +4\varphi(t)\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \\ +2\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)\int_0^t\left(\left\|\frac{u_\tau}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_\tau\|_2^2\right)\mathrm{d}\tau \end{array}\right) \\
&\qquad -2\alpha\left(\varphi'(t)\right)^2 \geq 0,
\end{aligned}$$

that is

$$\phi(t)\phi''(t) - \frac{1+\alpha}{2}\left(\phi'(t)\right)^2 > 0,$$

We now demonstrate that T cannot be infinite in this instance, proving that there is never a weak solution.

According to Lemma 10, there is a $0 < t_1 < +\infty$ such that $\phi(t) \to \infty$ as $t \to t_1$, for which

$$0 < t_1 \leq \frac{2\phi(0)}{(\alpha-1)\phi'(0)} = \frac{\gamma}{(\alpha-1)\varepsilon^{-1}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^2}.$$

Considering that $\varphi(t)$ is continuous with regard to $\phi(t)$, we deduce that there is a $T_1 \leq t_1$ such that $\lim\limits_{t\to T_1}\int_0^t\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u(t)\|_2^2\,ds = +\infty \Rightarrow \lim\limits_{t\to T_1}\sup\left\|\frac{u(t)}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u(t)\|_2^2 = +\infty$. Hence, $u(x,t)$ discontinuing at some finite time T_1. Taking into account the continuity of ϕ with respect to y, we can now conclude that $\varphi(t)$ tends to infinity at some finite time. This means that $\frac{u(x,t)}{|x|^{\frac{s}{2}}}$ does not exist for all time, i.e., $\frac{u(x,t)}{|x|^{\frac{s}{2}}}$ blows up at a time T_1, which will result in the nonexistence result stated in the theorem. Afterward, φ blows up at time T_1 in the $L^2(\Omega)$-norm, which contradicts. Therefore, each solution has a finite explosion time if the data fulfill (5.1).

Case 2 Assume that there exists $t_0 > 0$ such that $J(u(t_0)) < 0$, $\left(\frac{u(t_0)}{|x|^{\frac{s}{2}}} \neq 0\right)$. Given that $J(0) > 0$ and that $J(t)$ is continuous, we can conclude that there is a $t_1 \in (0, t_0)$ such that $J(t_1) = 0$. Furthermore, using $J(t)$'s monotonicity, we derive $J(t) \geq 0$, $0 < t \leq t_1$.

We can demonstrate that the solution to problem (1.2) breaks down before the time t_0, just like in **Case 1.**

In finite time, we complete the blow-up of the answer by combining **Cases 1 and 2**. Since $\mathrm{J}(u_0) < \mathrm{A}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)$ indicates $\mathrm{I}(u) < 0$, we can get the same lower bound of blow-up time as $\mathrm{J}(u_0) \leq d$. The proof is over now. ∎

Theorem 19 *Under the assumptions* $p < q < p + \frac{2p}{n}$, $d < \mathrm{J}(u_0) < \mathrm{A}\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)$. *The lower bounded of blow-up time of solution for problem* (1.2) *estimate by*

$$\bar{t} > \frac{\left(\left\|\frac{u_0}{|x|^{\frac{s}{2}}}\right\|_2^2 + \|\nabla u_0\|_2^2\right)^{2-\eta(q+1)}}{(\eta(q+1) - 2)\,|\Omega|^{\eta(q+1)}\,\mathrm{c}_g^{\frac{p(q+1)}{p-(q+1)\theta}}} > 0,$$

where c_g, η, *and* θ *are defined as in Theorem* 12.

Beweis. It is evident from Lemma 17 that $\mathrm{I}(u) < 0$. The remainder of the proof is then comparable to Theorem 12. ■

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